Hurstville Boys High School

Trial Higher School Certificate Examination 1988

4 Unit Mathematics

Question 1.

(i) Find these indefinite integrals:

(a) $\int \frac{x-5}{x^2-x-2}\,dx$

(b) $\int \frac{x+3}{x^2+4}\,dx$

(c) $\int te^{-t}\,dt$

(d) $\int \frac{dx}{x\log_e x}$

(ii) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\pi/2} \frac{dx}{1+\sin x}$

(iii) Use the substitution $x = 3\sin\theta$ to evaluate $\int_0^{3/\sqrt{2}} \sqrt{9-x^2}\,dx$

Question 2.

(i) Sketch the curve $y = 2 + \frac{1}{x^2-1} \quad (x \neq \pm 1)$ showing the location and nature of all stationary points and the equations of all asymptotes.

(ii) Prove that the condition that the line $ax + by + c = 0$ is a tangent to the circle $x^2 + y^2 = R^2$ is that $R^2(a^2 + b^2) = c^2$.

(iii) Let n be a positive integer, and let $I_n = \int_1^2 (\log_e x)^n\,dx$. Show that $I_n = 2(\log_e 2)^n - n.I_{n-1}$. Hence evaluate $\int_1^2 (\log_e x)^4\,dx$ a polynomial in $\log_e 2$.

Question 3.

The hyperbola H has cartesian (x, y) equation $\frac{x^2}{5} - \frac{y^2}{5} = 1$. Write down its eccentricity, the co-ordinates of its foci S and S', the equation of each directrix, and the equation of the asymptotes. Sketch the curve and indicate on your diagram the foci, directrices, and asymptotes. P is an arbitrary point $(\sqrt{5}\sec\theta, \sqrt{5}\tan\theta)$. Show that P lies on H and prove that the tangent to H at P has equation $\frac{x\sec\theta}{\sqrt{5}} - \frac{y\tan\theta}{\sqrt{5}} = 1$. This tangent cuts the asymptotes in L and M. Prove that $LP = PM$ and the area of ΔOLM is independent of the position of P on H. (O is the origin.)

Question 4.

(i) Express the roots of $z^2 - (1-i)z + 7i - 4 = 0$ in the form $x + iy$.

(ii) Express $z = \frac{1+2i}{1-i} + \frac{1}{i}$ in the form $r(\cos\theta + i\sin\theta)$ where $r = |z|$ and $\theta = \arg z$.

(iii) On the argand diagram show clearly the region defined by

(a) $|z - 3i| \leq 2$

(b) $|z - 2| < |z + 2|$

(iv) Find the complex cube roots of -1, expressing them in the form $r(\cos\theta + i\sin\theta)$. Show the roots on the argand diagram.

Question 5.

(i) **(a)** Find the first and second derivative of $y = e^{-\frac{1}{2}x^2}$

(b) Sketch the curve $y = e^{-\frac{1}{2}x^2}$, showing clearly any stationary points and points of inflexion.

(ii) The area between the curve $y = e^{-\frac{1}{2}x^2}$ the x and y axes and the line $x = \sqrt{2}$ is rotated about the y axis. Using the method of cylindrical shells (parallel to y axis), find the volume of the solid of revolution so fromed.

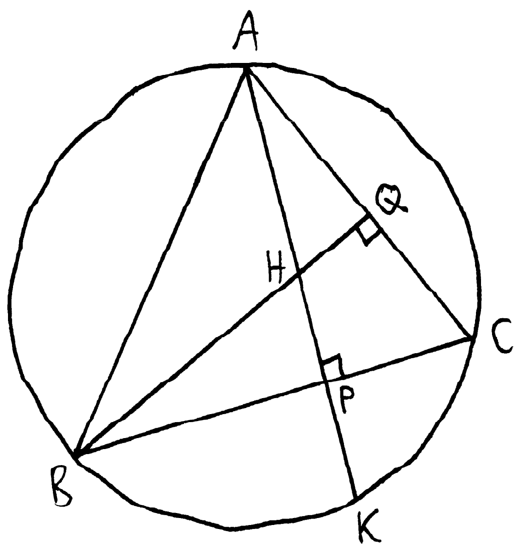
(iii) The base of a certain solid is the region between the x axis and the curve $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$. Each plane perpendicular to the x axis is an equilateral triangle with one side on the base of the solid. Find the volume of the solid.

Question 6.

(i) Solve $\cos 2x - \sin x = 1$ for $0 \leq x \leq 2\pi$.

(ii) As the result of an experiment, a curve of the form $y = f(x)$ is drawn. It is suspected that $f(x)$ is expressible in the form $f(x) = (1 + ax)^n$ where "a" is real and "n" is a positive integer. For values of x so small that third and higher powers of x can be neglected: $y = 1 - 6x + 16x^2$ is practically identical with the given curve. Assuming the curves coincide for these values of x, what are the values of "a" and "n"?

(iii) The altitudes AP and BQ of an acute angled triangle meet at H. AP produced cuts the circle through A, B, C at K. Prove that $HP = PK$.



Question 7.

(i) (a) State the factor theorem for polynomials.

(b) If $P(x) = x^3 + 5x^2 + 9x + 6$ resolve $P(x)$ into irreducible factors over the field of comlex numbers.

(ii) If α, β, γ are the roots of the cubic equation $x^3 + qx + r = 0$, find the value of (in terms of q, r)

(a) $\alpha + \beta + \gamma$

(b) $\alpha\beta + \alpha\gamma + \beta\gamma$

(c) $\alpha\beta\gamma$

Hence find the value of

$$(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2$$

(iv) Prove by mathematical induction that

$$3^n > 1 + 2n \quad \text{for} \quad n > 1.$$

Question 8.

(i) Sketch the following curves (without the use of calculus) for $-2\pi \leq x \leq 2\pi$:

(a) $y = |\sin x|$

(b) $y = \sin |x|$

(ii) A ball is projected from a point on the ground distance a units from the foot of a vertical wall of height b units. The ball is projected at an elevation with speed v. Find how high above the wall the ball will pass.
If the ball just clears the wall, prove that the greatest height reached is

$$\frac{1}{4}\left[\frac{a^2 \tan^2 \theta}{a \tan \theta - b}\right].$$