New South Wales

Higher School Certificate

Mathematics Level 1 (4 Unit)

Examinations 1967-1974

Typeset with $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

NSW HSC Mathematics Level 1 (4 Unit) Examination 1967

1. Given that $a > 1$, prove the theorems in the following sequence:

(i) If m is a positive integer, then

$$\frac{a^m + a^{m-1} + \cdots + 1}{m+1} > \frac{a^{m-1} + a^{m-2} + \cdots + 1}{m}.$$

(ii) If m and n are positive integers and $n > m$, then

$$\frac{1}{n}(a^n - 1) > \frac{1}{m}(a^m - 1).$$

(iii) If r and s are positive rationals and $r > s$, then

$$\frac{1}{r}(a^r - 1) > \frac{1}{s}(a^s - 1).$$

2. Express

$$\frac{1 - abx^2}{(1-ax)(1-bx)}$$

in the form $l + \frac{m}{1-ax} + \frac{n}{1-bx}$, where l, m, n are constants.
Given that $R_n(x)$ is a polynomial and that

$$1 - abx^2 \equiv (1-ax)(1-bx)(1 + u_1x + u_2x^2 + \cdots + u_rx^r + \cdots + u_nx^n) + x^{n+1}R_n(x)$$

find u_r and $R_n(x)$.

3. (i) A variable point P on the Argand diagram represents the complex number z. α and ρ are fixed complex numbers. Describe and illustrate with rough sketches the geomtrical relations between the point:

(a) P_1 representing $z + \alpha$, and the point P;

(b) P_2 representing ρz, and the point P;

(c) P_3 representing ρz, and the point P_2.

(ii) A sequence of points, $z_0, z_1, z_2, \ldots$ is defined from an arbitrary point z_0 by the transformations

$$z_n = c\bar{z}_{n-1} + c - 1,$$

where $c = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.

Determine whether the sequence consists of repetitions of a finite set of points, and if so how many points there are in the set.

4. In a plane, in relation to a given co-ordinate system with the origin O, P_i is the point represented by the column-vector $\mathbf{r}_i$. Let $\mathbf{r}' = \mathbf{M}\mathbf{r}$ be the equation of an affine transformation of the plane onto itself in which P_i is transformed into P_i'.

Prove that:

(i) if P_1, P_2, P_3 are collilear, then

(a) P_1', P_2', P_3' are collinear, and

(b) $*P_1'P_2'/*P_1'P_3' =*P_1P_2/*P_1P_3$;

(ii) area $\Delta OP_1'P_2' =$ (area ΔOP_1P_2). $\det \mathbf{M}$.

5. (i) Prove that if, for two 2×2 matrices, $\mathbf{M}, \mathbf{N}$, a matrix $\mathbf{C}$ exists such that

$$\mathbf{N} = \mathbf{C}\mathbf{M}\mathbf{C}^{-1},$$

then the two quadratic polynomials $\det(x\mathbf{1} - \mathbf{M})$ and $\det(x\mathbf{1} - \mathbf{N})$ are identical.

(ii) $\mathcal{D}$ is the operation of displacement through one unit parallel to the x-axis in a rectangular co-ordinate system, origin O; $\mathcal{S}$ is the operation of reflexion in the line $y = x$. Draw a sketch to show the points $\mathcal{D}O, \mathcal{D}^2O, \mathcal{S}\mathcal{D}O, \mathcal{D}\mathcal{S}\mathcal{D}O, \mathcal{D}\mathcal{S}\mathcal{D}\mathcal{S}\mathcal{D}O$.
Prove that, for any sequence of non-negative integers $r, s, t, \ldots,$

$$\ldots \mathcal{D}^t\mathcal{S}\mathcal{D}^s\mathcal{S}\mathcal{D}^rO = \mathcal{D}^m\mathcal{S}\mathcal{D}^nO,$$

where m, n are non-negative integers. Describe the set of points which corresponds to he set of ordered pairs (m, n).

6. Referred to a rectangular co-ordinate system in space, A is $(3, 11, -4)$, B is $(5, -3, 0)$, C is $(0, 2, -5)$.

Find the equations of

(i) the plane through the line OA and perpendicular to the plane OBC;

(ii) the plane through the line OA and parallel to the line BC;

(iii) the circular cylinder, with generators parallel to the y-axis, which passes through A, B, C.

7. (i) Prove that

$$\frac{1}{2}\Big(\frac{a}{x} + x\Big) - \sqrt{a} = \frac{(\sqrt{a} - x)^2}{2x}.$$

(ii) In a certain set of tables, in which the value of $\sqrt{a}$ is given to 4 significant figures, the value of $\sqrt{2}$ is given as $1 \cdot 414$. If $c \leqslant 9$ what is the largest integer n for which

$$\Big|\sqrt{2} - \frac{1}{2}\Big(\frac{2}{1 \cdot 414} + 1 \cdot 414\Big)\Big| < c10^{-n}.$$

Interpret this as a statement of the greatest number of significant figures in the value of $\sqrt{2}$ that can be computed in one application of formula (i) to the given approximation $1 \cdot 414$.

8. c is that part of the curve represented by the equation

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

which lies in the first quadrant ($x \geqslant 0, y \geqslant 0$).

(i) Find the equation of the tangent to c at (x_0, y_0).

(ii) Show that the segment intercepted on the tangent by the co-ordinate axes has constant length.

(iii) If s is the distance along c from some fixed point on c, prove that $\frac{ds}{dx} = \left(\frac{a}{x}\right)^{\frac{1}{3}}$.

(iv) Find the length of c.

9. (i) If $u_n = \int \frac{dx}{(x^2+1)^n}$
show that $u_n - u_{n-1}$ can be expressed in the form

$$\frac{Ax}{(x^2+1)^{n-1}} + Bu_{n-1}$$

where A and B are constants.

(ii) Find $\lim\limits_{k \to \infty} \int_0^k \frac{dx}{(x^2+1)^2}$.

10. A train of mass M, pulled by a locomotive which exerts a tractive force $P(v)$, is moving at speed v along a level track against a resistive force $R(v)$. Both $P(v)$ and $R(v)$ are functions of v.

(i) If $R(v) = a + bv^2$, and $P(v) = 2a$, where a, b are positive constants, prove that there is an upper bound to the speed that the train can attain, and find the value of this upper bound.

(ii) Show that, in accelerating from speed v_0 to speed v_1, the distance travelled by the train is $kM \int_{v_0}^{v_1} \frac{vdv}{P(v)-R(v)}$ where k is a constant.

What is the value of k?

NSW HSC Mathematics Level 1 (4 Unit) Examination 1968

1. Using Euclid's algorithm, or otherwise, find the highest common factor, $h(x)$ of x^7+1 and x^5+1.

By reversing the steps in the algorithm, or otherwise, find the polynomials, $f(x)$ and $g(x)$, of the lowest degree, such that

$$(x^7+1)f(x)+(x^5+1)g(x)\equiv h(x)$$

2. (i) The complex number w is such that w =2, $\arg w=\frac{1}{6}\pi$. Mark approximately on an Argand diagram (not on graph paper) the following points:

A corresponding to w, B to w^2, C to w^2-w, D to $\overline{w}$, E to $w\overline{w}$.

(ii) Sketch on an Argand diagram the locus of the point P which satisfies the equation

$$|z-1|=1.$$

(iii) From the diagram and using Euclidean properties of the locus, or otherwise, show that the point P of part (ii) also satisfies the condition

$$\arg(z-1)=\arg z^2.$$

What is the complete locus of a point satisfying this condition?

3. (i) By suitably grouping the terms in the series

$$1+\frac{1}{2^\alpha}+\frac{1}{3^\alpha}+\cdots+\frac{1}{n^\alpha}+\cdots,$$

or otherwise, prove that when $\alpha>1$ the series converges, and when $\alpha\leqslant 1$ the series diverges.
(You may assume that $1+g+g^2+\cdots+g^n+\cdots$ converges when $-1<g<1$.)

(ii) Sequences u_n, v_n are defined by

$$u_n=\frac{2n}{2n+100},\, v_n=\frac{2n-100}{2n}$$

Discuss the convergence of the two series:

(a) $\sum u_n$; and

(b) $\sum(u_n-v_n)$.

4. Prove that a necessary and sufficient condition that the three planes

$$\begin{aligned} lx + my + nz &= 0 \\ l'x + m'y + n'z &= 0 \\ l''x + m''y + n''z &= 0 \end{aligned}$$

should have a common line is

$$l''(mm' - m'n) + m''(nl' - n'l) + n''(lm' - l'm) = 0.$$

Show that for three and only three values of a the three planes

$$\begin{aligned} ax - y + z &= 0 \\ x - (a + \frac{1}{2})y &= 0 \\ x + (a - \frac{1}{2})z &= 0 \end{aligned}$$

have a common line.

Find the direction cosines of each of the three common lines and prove that these lines are mutually perpendicular.

5. (i) By transferring the origin to a suitable point (p, q) the equation

$$7x^2 + 12xy - 2y^2 - 2x - 16y - 12 = 0$$

is reduced to the form

$$ax^2 + 2hxy + by^2 = 1.$$

Find p, q, a, h and b.

(ii) $\mathcal{S}_a, \mathcal{S}_b, \mathcal{S}_c$ are the operations of reflection in the lines a, b, c, respectively, where a, b, c contain the sides of an equilateral triangle. Draw a diagram (not on graph paper) which shows the lines e and f, and a segment determining the displacement $\mathcal{D}$, which are such that:

(a) $\mathcal{S}_a\mathcal{S}_b\mathcal{S}_a = \mathcal{S}_e$.

(b) $\mathcal{S}_b\mathcal{S}_a\mathcal{S}_b = \mathcal{S}_f$.

(c) $\mathcal{S}_c\mathcal{S}_b\mathcal{S}_a = \mathcal{D}\mathcal{S}_b$.

6. S is a set of elements $\{A, B, \dots\}$ and $A \times B$ is an object defined from A and B by the operation $\times$.
State sufficient conditions for the set S under the operation $\times$ to form a group.
The matrices $\mathbf{J}, \mathbf{K}$ are

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \text{ matrix } i^2 = -1.$$

Prove that the set of eight matrices $\{\pm\mathbf{1}, \pm\mathbf{J}, \pm\mathbf{K}, \pm\mathbf{JK}\}$ forms a group under the operation of matrix multiplication.

7. Prove that the curve

$$\left(\frac{x}{u}\right)^{1/2} + \left(\frac{y}{v}\right)^{1/2} = 1, u > 0, v > 0,$$

touches the axes Ox and Oy, say at U and V respectively.

Find the area bounded by the axes and the arc of the curve between U and V.

Prove that the curve represented by the rationalized form of the equation above is a parabola. What part of th curve is represented by the printed (irrational) form of the equation?

8. A function $K(x)$ is defined, over the domain $x \geqslant 0$, by the relations

$$\frac{dK(x)}{dx} = -\frac{1}{x} \text{ and } K(1) = 0.$$

(i) Prove that for any positive number a and any number c, and for any positive value of x:

(a) $\frac{dK(ax)}{dx} = -\frac{1}{x}$;

(b) $K(a) + K(x) = K(ax)$;

(c) $K(x^c) = cK(x)$.

(ii) Prove that there is one and only one value h such that $K(h) = 1$, and that h has the property $h^{K(x)} = x$.

9. A, B are the two points where a line in $z = 0$ parallel to Oy meets the parabolic cylinder $y^2 = x$. A, B, C, D are the vertices, in order, of a square in a plane perpendicular to Ox, and the z-coordinates of C and D are positive. A solid block has as its base the region in $z = 0$ bounded by the parabola $y = x^2$ and the line $x = 1$; its sections by planes perpendicular to Ox are squares such as $ABCD$.

(i) Find the equation of the section of this block by the plane $y = 0$.

(ii) The block has two plane faces, one in $z = 0$ and one in $x = 1$. Prove that the remaining part of the surface of the block consists of regions on two papabolic cylinders.

(iii) Find the volume of the block.

10. A particle of unit mass is projected vertically upwards against a constant gravitational force g and a resistance v/c, where v is the velocity of the particle and c is

a constant. s is the distance travelled in time t; at $t = 0, s = 0$, and $v = c(h - g)$ where h is a constant. Write down the equation of motion of the particle.

Find the time taken by the particle to reach its highest point, and find the height of that point.

The particle falls to its original position under gravity and under the same law of resistance. Will the time of descent be greater or less than the time of ascent? Give reasons for your answer.

NSW HSC Mathematics Level 1 (4 Unit) Examination 1969

1. (i) Use Euclid's algorithm to show that 299 and 323 are relatively prime (i.e. their greatest common divisor is 1).

(ii) If a, b, k are integers such that a, k are relatively prime and also b, k are relatively prime prove that ab, k are relatively prime.

2. (i) If m, n are non-negative integers prove that

$$\int_0^\pi \cos mx \cos nx \; dx = 0 \text{ if } m \neq n$$

and calculate the value of this integral when $m = n$.

(ii) By applying Euclid's algorithm to $(x^2 + 4)$ and $(x + 1)$ obtain a decomposition of

$$\frac{40}{(x+1)(x^2+4)}$$

into partial fractions and hence show that

$$\int_0^2 \frac{40 \; dx}{(x+1)(x^2+1)} = \pi - 4\log 2 + 8 \log 3.$$

3. (i) Find the length of the curve

$$2y = e^x + e^{-x}$$

between $x = 0$ and $x = \log 2$.

(ii) Find the volume of the solid of revolution formed when the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is rotated about the x-axis.

4. By suitably grouping terms in the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

show that the series is divergent.

By comparison with this series, or otherwise, show that:

(i) $\sum \frac{n}{1+n^2}$ (ii) $\sum \frac{1.3.5.....(2n-1)}{2.4.6.....2n}$
are also divergent.

5. The plane $2x + 3y + 6z = 7$ cuts the square

$$x^2 + y^2 + z^2 = 4$$

in the circle c. Find:

(i) the radius of c;

(ii) the co-ordinates of the centre of c;

(iii) the equation of the cylinder through c with generators parallel to the z-axis.

6. (i) If $a > 0$ and $ab > h^2$ show that the eigenvalues of the matrix

$$\begin{bmatrix} a & h \\ h & b \end{bmatrix} \quad (a, b, h \text{ being real})$$

are real and positive. Hence, assuming the relevant theorem on the reduction of a quadratic form to standard form, deduce that the equation (in plane cartesian co-ordinates)

$$ax^2 + 2hxy + by^2 = 1$$

represents an ellipse of area $\pi/(ab - h^2)^{\frac{1}{2}}$.

[Assume that the area of the ellipse $(x^2/\alpha^2) + (y^2/\beta^2) = 1$ is $\pi\alpha\beta$.]

(ii) A particle P moves in such a way that its cartesian co-ordinates (x, y) at time t are given by

$$x = p \sin(\omega t), \qquad y = q \sin\left(\omega t + \tfrac{\pi}{6}\right)$$

where p, q, ω are positive constants. Show that the path of P is an ellipse of area $\frac{1}{2}\pi pq$.

7. (i) For plane transformations show that the product of two reflections in parallel lines is a displacement and that the product of two reflections in intersecting lines is a rotation.

(ii) A plane p rests upon three fixed points which form a triangle ABC (described anticlockwise). $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ denote anticlockwise rotations of p (in its own plane about A, B, C respectively through angles $2\alpha, 2\beta, 2\gamma$ where α, β, γ denote the angles of the triangle at A, B, C respectively. If

$$\mathcal{U} = \mathcal{T}_1\mathcal{T}_2 \text{ and } \mathcal{V} = \mathcal{T}_1\mathcal{T}_2\mathcal{T}_3$$

describe the geometric transformations represented by $\mathcal{U}$ and $\mathcal{V}$. Also find all the points of p which remain invariant under the transformation $\mathcal{V}$.

8. Explain how the definite integral $\int_a^b f(x)\, dx$, for an increasing function f, is defined by dividing the interval $[a, b]$ into sub-intervals and considering areas of "inner" and "outer" rectangles with bases on these sub-intervals.

Illustrate the above when $f(x) = \log x$ with a sub-division of $[1, 2]$ at the points

$$1, r, r^2, r^3, r^4, \ldots, r^n \quad (\text{where } r^n = 2)$$

showing that the sum of the areas of the outer rectangles is

$$\log r\left(2n - \frac{1}{r-1}\right).$$

Deduce that $\int_1^2 \log x \; dx = 2\log 2 - 1$.

9. Explain how complex numbers are represented on the Argand diagram.

The three roots of the equation

$$x^3 + ax^2 + bx + c = 0$$

(where a, b, c are given complex numbers) are represented on the Argand diagram by the points A, B, C. Prove that ABC is an equilateral triangle if and only if $a^2 = 3b$.

10. A gun fires a shot from O with initial speed V at an angle α with the horizontal. If the acceleration due to gravity is constant $(= g)$ prove that the shot describes a parabola of focal length $V^2 \cos^2 \alpha/(2g)$.

If the initial speed V is fixed but the direction of firing can be varied prove that the region of vulnerability (i.e. the set of points that can be hit) consists of points within and on the paraboloid whose equation (referred to a cartesian x, y, z-frame with origin at O and z-axis vertically upwards) is

$$x^2 + y^2 + (2V^2/g)z = V^4/g^2.$$

NSW HSC Mathematics Level 1 (4 Unit) Examination 1970

1. Evaluate:

(i) $\int_0^1 \frac{du}{\sqrt{4-u^2}}$;

(ii) $\int_0^1 \frac{dx}{4-x^2}$;

(iii) $\int_0^{\pi/4} \theta \cos^2 \theta \ d\theta$;

(iv) $\int_0^1 te^{-t^2} \ dt$.

2. The minute hand OP and hour clock OQ of a clock are 4 feet and 3 feet long respectively. At the instant when the clack shows 9 o'clock find the rate (in feet per hour) at which the length PQ is increasing.

3. (i) Calculate the length of the arc of the curve $y = 2x\sqrt{x}$ between $x = 0$ and $x = 1$.

(ii) Prove that the volume common to the solid cylinders

$$x^2 + z^2 \leqslant 1 \text{ and } y^2 z^2 \leqslant 1$$

(where x, y, z are rectangular cartesian coordinates) is 16/3.

[Hint: Consider sections by planes parallel to the xy-plane].

4. If a particle moving with speed v experiences air resistance kv^2 per unit (k being a constant) prove that, in falling from rest in a vertical line through a distance s, it will acquire a speed

$$V\sqrt{1 - e^{-2ks}}$$

where $V = \sqrt{g/k}$ (the "terminal velocity") and g is the acceleration due to gravity (assumed constant).

With air resistance as above prove that a particle projected vertically upwards with initial speed U will return to the point of projection with speed W given by

$$W^{-2} = U^{-2} + V^{-2}.$$

5. Explain the terms "convergent" and "absolutely convergent" as applied to an infinite series of real numbers and prove that an absolutely convergent series is convergent.

Determine for what (real) values of x the series

$$\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots$$

is:

(i) convergent

(ii) absolutely convergent.

6. The plane $3x + 2y + 6z = 12$ cuts the x, y, z axes respectively in A, B, C. Find the equation of the sphere which passes through the four points O, A, B, C (where O is the origin) and specify its centre and radius.

Also find the radius of the circumcircle of the triangle ABC.

7. (i) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

(ii) f is a function whose domain is the set of points in the cartesian (x, y) plane excluding the origin and

$$f(P) = \frac{3x^2 - 2xy + 3y^2}{x^2 + y^2}$$

where P is the point (x, y).

Find the range of the function f.

8. Show that, in a cartesian (x, y) frame, a rotation $\mathcal{T}$ through an angle α about the origin (thereby transforming $\mathbf{r}$ into $\mathbf{r}'$) can be represented by the matrix equation $\mathbf{r}' = \mathbf{Tr}$ where

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \mathbf{T} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

If $\mathcal{R}$ denotes a refection in the line $y = x\tan\beta$ (thereby transforming $\mathbf{r}$ into $\mathbf{r}'$) find the matrix $\mathbf{R}$ such that $\mathbf{r}' = \mathbf{Rr}$.

Also form the matrix products $\mathbf{RT}$ and $\mathbf{TR}$ and hence describe geometrically the transformation $\mathcal{RT}$ and $\mathcal{TR}$.

9. (i) Define the modulus $|z|$ of aq complex number z and show that every non-zero complex number z can be written in the form $r(\cos\theta + i\sin\theta)$ where $r = |z|$.

(ii) Express $\frac{1}{1+i}$ in the form described in (i).

(iii) The four complex numbers z_1, z_2, z_3, z_4 are represented on the complex (Argand) plane by the points A, B, C, D respectively. If $z_1 - z_2 + z_3 - z_4 = 0$ and $z_1 - iz_2 - z_3 + iz_4 = 0$, determine the possible shape(s) for the quadrilateral $ABCD$.

10. Given that $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$\frac{x}{x-p} + \frac{x}{x-q} + \frac{x}{x-r} + (x - s) = 0,$$

where p, q, r, s are distinct non-zero constants:

(i) evaluate $\alpha+\beta+\gamma+\delta$ in terms of p, q, r, s and prove that:

(ii) $\frac{1}{\beta\gamma\delta}+\frac{1}{\alpha\gamma\delta}+\frac{1}{\alpha\beta\delta}+\frac{1}{\alpha\beta\gamma}=\frac{1}{qrs}+\frac{1}{prs}+\frac{1}{pqs}+\frac{1}{pqr}-\frac{3}{pqrs}$;

(iii) $\frac{p^2}{(p-\alpha)(p-\beta)(p-\gamma)(p-\delta)}+\frac{q^2}{(q-\alpha)(q-\beta)(q-\gamma)(q-\delta)}+\frac{r^2}{(r-\alpha)(r-\beta)(r-\gamma)(r-\delta)}=0.$

NSW HSC Mathematics Level 1 (4 Unit) Examination 1971

1. (i) Define $\sin^{-1} x$ and show that its derivative is $(1-x^2)^{-\frac{1}{2}}$.

(ii) Evaluate:

(a) $\int_1^2 \frac{dx}{x(x+3)}$;

(b) $\int_0^2 \frac{(x+1)\ dx}{x^2+4}$;

(c) $\int_0^1 \sin^{-1} t\ dt$.

2. A clock whose face is in the x, y-plane is movede bodily in the x-direction with a constant speed of 1 foot per second. If the clock has a sweep second hand of length 1 foot (making one complete revolution per minute) prove that the tip of this hand describes a curve whose tangent varies in inclination to the x-axis between $+\alpha$ and $-\alpha$ where

$$\tan\alpha = \left(\frac{900}{\pi^2 - 1}\right)^{-\frac{1}{2}}.$$

3. (i) Prove that the area of the parabolic segment enclosed between $y = x^2$ and the line $y = x$ is $1/6$.

(ii) Find the volume of the solid formed when the area in (i) is rotated (through one complete revolution) about the x-axis.

(iii) Write down the perpendicular distance from the point (t, t^2) to the line $y = x$.

(iv) Find the volume of the solid formed when the area in (i) is rotated (through one complete revolution) about the line $y = x$.

4. (i) Use Euclid's algorithm to find the greatest common divisor d of 221 and 104.

Also show that d can be expressed in the form

$$d = 221m + 104n$$

where m, n are integers and, moreover, that this can be done in an infinite number of ways.

(ii) What can you conclude about the greatest common divisor of two integers a, b if integers p, q exist such that

$$ap + bq = 6?$$

5. Which of the following statements (on infinite series of real numbers) are true and which false? Justify your answers by proving those that are true and by giving a counterexample for any one that is false.

(i) If $\sum u_n$ and $\sum v_n$ are convergent then so is $\sum(u_n + v_n)$.

(ii) If $\sum u_n$ and $\sum v_n$ are divergent then so is $\sum(u_n + v_n)$.

(iii) If $\sum u_n$ is convergent and $0 < u_n < 1$ (for all n) then

$$\sum \frac{u_n}{1 - u_n} \quad \text{is convergent.}$$

(iv) If $\sum u_n$ is convergent then so is $\sum u_n^2$.

6. The plane $6x + 2y + 3z = 42$ cuts the x, y, z axes respectively in A, B, C. Find:

(i) the equation of the sphere whose centre is O (the origin) and which touches the given plane;

(ii) the equation of the sphere which passes through the four points O, A, B, C;

(iii) the equation of the cylinder which passes through the curve of intersections of the above two spheres and whose generators are parallel to the x-axis.

7. The acceleration of a body moving along the x-axis is given by:

$$\begin{aligned} \frac{d^2x}{dt^2} &= -x \qquad \text{for } x \leqslant 2; \\ &= 4 - x \quad \text{for } x > 2. \end{aligned}$$

(i) If U is the function given by

$$U(x) = \frac{1}{2}x^2 \text{ for } x \leqslant 2 \text{ and } U(x) = \frac{1}{2}(x-4)^2 \text{ for } x > 2$$

deduce the fact that the quantity

$$E = \frac{1}{2}(dx/dt)^2 + U(x)$$

is a constant of the motion (i.e., is independent of the time).

(ii) At time $t = 0$ the body is placed at the origin and is given an initial velocity $v_0 = \frac{1}{2}$ in the positive x-direction. Determine the extreme points (largest and smallest x values) of the subsequent motion; hence show that the motion is simple harmonic. (Hint: Use the result in (i).)

(iii) With the same starting position, but a larger initial velocity $v_0 = 3$, find the extreme points of the subsequent motion. Is this motion simple harmonic? Explain your answer fully.

8. (i) Show that, in the Cartesian plane, a change in coordinates from (x, y) to (x^*, y^*) produced by rotating the axes through an angle α can be expressed by the transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x^* \\ y^* \end{bmatrix}.$$

(ii) Prove that, in the Cartesian plane, the equation

$$x^2 - 4xy - 2y^2 = 1$$

represents an hyperbola and find the equations of its principal axes. Illustrate by a sketch.

9. (i) Explain with the aid of a diagram (without proof) how the matrix

$$\begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$$

represents a reflection.

(ii) In the Cartesian plane the reflection of the point P in the line

$$3x - 4y = 10$$

is the point P' If the coordinates of P and P' are (x, y) and (x', y') respectively and if

$$\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix} \;, \; \mathbf{r}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

derive a matrix expressing $\mathbf{r}'$ in terms of $\mathbf{r}$.

10. (i) Given a complex number $z = x + iy$ (where x and y are real) define the modulus $|z|$ and the conjugate $\overline{z}$.

(ii) Specify the geometric locus in the complex (Argand) plane represented by the equation

$$|z - z_0| = c$$

where z_0 is a fixed complex number and c a real constant.

(iii) Given that, in the complex plane, the point P represents the complex number z and Q the number $1/z$ prove that, if P describes a circle of radius r with centre at w (where $r \neq |w|$), then Q will also describe a circle, whose centre is at the point

$$\frac{\overline{w}}{|w|^2 - r^2};$$

and find the radius of this circle.

NSW HSC Mathematics Level 1 (4 Unit) Examination 1972

1. (i) Find the greatest common divisor of 713 and 943.

(ii) If a, b, c are given integers prove that the equation

$$ax + by = c$$

has solutions in integers for x, y if and only if c is divisible by the greatest common divisor of a and b.

(iii) Determine whether the equation

$$713x + 943y = 115$$

has integer solutions for x, y and, if it has, find one such solution.

2. Explain how complex numbers are represented on the Argand diagram.

Given that, in the Argand diagram, the point P represents the complex number z and Q the number z^2 prove that if P moves on a straight line parallel to (but not coinciding with) the imaginary axis then Q will move on a certain parabola, and that all such parabolas have a common focus.

Also state what the locus of Q is when P describes the imaginary axis.

3. (i) Evaluate

(a) $\int_0^{\log 2} te^{-t}\, dt$;

(b) $\int_0^{2/3} \frac{dx}{4+9x^2}$.

(ii) The domain of the function f is the interval $-3 \leqslant x \leqslant 7$ and

$$f(x) = (x^3 - 6x^2)^{\frac{1}{2}}$$

Determine the range of the function f and sketch its graph.

Do NOT use squared paper for the graph and give answers for the range to two significant digits.

4. (i) Explain, with proof and with the aid of a diagram, why

$$\begin{bmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{bmatrix}$$

is called a reflection matrix.

(ii) Given that

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c, d are real numbers such that

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{1} \text{ and } \det \mathbf{A} = -\mathbf{1}$$

prove that $\mathbf{A}$ is a reflection matrix.

(NOTE.- $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of $\mathbf{A}$ and $\mathbf{1}$ denotes $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.)

5. (i) Describe in geometrical terms the foci (in three-dimensional cartesian space) represented by

(a) $x^2 + y^2 + z^2 + x + y + z = 0$;

(b) $x^2 + y^2 + x + y = 0$.

(ii) Explain the term "direction cosines".

The inclinations to the horiontal of two lines which are perpendicular to one another are α and β. If these two lines lie in a plane which is inclined at an angle θ to the horizontal prove that

$$\sin^2 \theta = \sin^2 \alpha + \sin^2 \beta.$$

6. Explain the terms "convergent sequence" and "convergent series".

Prove that, for $t \neq 1$, the sum of the geometric series

$$1 + t + t^2 + \cdots + t^{n-1}$$

is $(1 - t^n)/(1 - t)$.

By integrating this result between 0 and x prove that, for $0 \leqslant x < 1$, the series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots$$

is convergent and that its sum is $-\log(1 - x)$.

7. The co-ordinates of a point moving in the cartesian (x, y) plane at time t are given by

$$x = t + \sin t, y = 1 - \cos t, \text{ where } 0 \leqslant t \leqslant 2\pi.$$

(i) Show that the curve thus traced out is symmetric about the line parallel to the y-axis through the point where $t = \pi$.

(ii) Determine whether the curve has a tangent at the point where $t = \pi$.

(iii) Find the length of the curve (between $t = 0$ and $t = 2\pi$).

8. (i) With $\mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{r}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$

let $\mathbf{r}' = \mathbf{M}\mathbf{r}$ be the equation of a affine transformation $\mathcal{T}$ of a cartesian plane into itself in which the points P, Q are transformed into P', Q' respectively. Prove that

(a) $\mathcal{T}$ transforms points of the line segment PQ into points of the line segment $P'Q'$;

(b) area $\triangle OP'Q' =$ (area $\triangle OPQ$). $\det \mathbf{M}$.

(ii) Specify an affine transformation which will transform the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b > 0)$$

into a circle and hence, using the results in (i), prove that the area of the ellipse is πab.

9. In this question assume that the earth is a square of radius R and that, at a point distant r ($> R$) from the centre of the earth, the acceleration due to gravity is proportional to r^{-2} and is directed towards the earth's centre; also, neglect forces due to all causes other than the earth's gravity.

A body is projected vertically upwards from the surface of the earth with initial speed V.

(i) Prove that it will escape from the earth (i.e., never return) if and only if $V \geqslant \sqrt{2gR}$ where g is the magnitude of the acceleration due to gravity at the earth's surface.

(ii) If $V = \sqrt{2gR}$ prove that the time taken to rise to a height R above the earth's surface is $\frac{1}{3}(4 - \sqrt{2})\sqrt{R/g}$.

10. (i) Explain what is meant by a "root of multiplicity m of a polynomial".

(ii) If a polynomial $f(x)$ has a root of multiplicity m at $x = c$ prove that the derived polynomial $f'(x)$ has a root of multiplicity $(m - 1)$ at the same point.

(iii) Prove that the polynomial

$$x^3 + 3px^2 + 3qx + r,$$

has a double root (i.e., a root of multiplicity 2) if and only if

$$(pq - r)^2 = 4(p^2 - q)(q^2 - pr) \text{ and } p^2 \neq q.$$

NSW HSC Mathematics Level 1 (4 Unit) Examination 1973

Question 1.

(i) Sketch the graphs (showing the main features - do NOT use squared paper) of:

(a) $y = \sin^2(2x)$ $(-2\pi \leqslant x \leqslant 2\pi)$;

(b) $\sin(x + y) = 0$;

(c) $|x| + |y| = 1$.

(ii) Evaluate $\int_0^1 x \tan^{-1} x \, dx$.

Question 2.

(i) Define "the greatest common divisor of two integers a, b".

(ii) Given two odd integers a, b satisfying a relation

$$pa + qb = 8$$

where p, q are integers prove that a and b are co-prime (i.e., their greatest common divisor is 1).

(iii) Prove that two integers m, n must be co-prime if $m+n$ and $m-n$ are co-prime.

Question 3.

(i) Define the modulus $|z|$ of a complex number z.

(ii) If z_1 and z_2 are two complex numbers prove that

$$|z_1 z_2| = |z_1||z_2|.$$

(iii) In the Argand diagram P represents the complex number z and Q the complex number ζ given by:

$$\zeta = \frac{3z - 1}{z - 1}.$$

If P describes the circle of unit radius with centre at origin find the locus described by Q.

Question 4.

Prove that the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a \geqslant b > 0)$$

is πab.

Also find the volume of the solid formed when this area is rotated through one complete revolution about the line $y = b$.

Question 5.

Explain and derive Simpson's Rule (for three ordinates), namely,

$$\int_a^b f(x)\,dx = \frac{b-a}{6}[f(a) + 4f(\tfrac{1}{2}(a+b)) + f(b)] \text{ approximately.}$$

Also prove that Simpson's Rule yields an exact result when $f(x)$ is a cubic polynimial, i.e., when it has the form

$$f(x) = Ax^3 + Bx^2 + Cx + D.$$

Question 6.

(i) Show that, in the Cartesian plane, a change in coordinates from (x, y) to (x^*, y^*) produced by rotating the axes through an angle α can be expressed by the transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x^* \\ y^* \end{bmatrix}.$$

(ii) Assume that, by a suitable transformation of the above type, the equation

$$ax^2 + 2hxy + by^2 = 1$$

becomes

$$\lambda_1 {x^*}^2 + \lambda_2 {y^*}^2 = 1$$

where λ_1, λ_2 are the eihenvalues of the matrix

$$\begin{bmatrix} a & h \\ h & b \end{bmatrix}.$$

State and prove *necessary and sufficient* conditions on a, h, b in order that

$$ax^2 + 2hxy + by^2 = 1$$

should represent an ellipse, and, when these conditions are satisfied, prove that the area of the ellipse is $\pi/\sqrt{ab - h^2}$.

(NOTE: Results stated in any previous question on this paper may be assumed.)

Question 7.

(i) With rectangular coordinates in three-dimensional Cartesian space the equations of two planes λ, μ are respectively

$$\begin{aligned} 2x + 3y + 6z &= 1 \\ 3x + 2y - 2z &= 1. \end{aligned}$$

Find:

(a) the angle between the two planes;

(b) the equation of the sphere which touches the plane λ and has its centre at the point $(3, 1, 1)$.

(ii) In the following result (which is valid for $-1 < x < 1$)

$$\log_e\left(\frac{1+x}{1-x}\right) = 2\left[\frac{x}{1} + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right]:$$

(a) use the first two terms of the series on the right-hand side to obtain $0{\cdot}69$ as an approximate value of $\log_e 2$;

(b) use the remaining terms of the series to prove that the error in this approximation for $\log_e 2$ is less than 1%.

Question 8.

With the earth regarded as a uniform solid sphere of radius R it is given that the acceleration due to gravity at a point distant r from the earth's centre is directed towards the centre and has magnitude proportional to r^{-2} when $r \geqslant R$ and proportional to r when $r \leqslant R$; and the magnitude of this acceleration at the surface of the earth (i.e., when $r = R$) is denoted by the constant g. Suppose a narrow tunnel is bored along a diameter AB of the earth and a particle is projected from A with initialo velocity U towards B.

(a) Show that the subsequent motion is oscillatory if and only if

$$U^2 < 2gR.$$

(b) When the motion is oscillatory prove that it takes place between two point whose distance apart is

$$\frac{2R}{1 - U^2/(2gR)}.$$

(c) If $U = 0$ state, with brief justification, the period of the motion.

Question 9.

The matrix $\mathbf{C}$ is given by

$$\mathbf{C} = \begin{bmatrix} a & -b \\ -a & b \end{bmatrix}.$$

(i) Find $\mathbf{C}\mathbf{C}^{\mathrm{T}}, \mathbf{C}^{\mathrm{T}}\mathbf{C}$ and $\mathbf{C}^2$.

(ii) If n is a positive integer $\geqslant 2$ and $\lambda = a + b \neq 0$ prove that

(a) $\mathbf{C}^n = \lambda^{n-1}\mathbf{C}$;

(b) $(\mathbf{1} - \mathbf{C})^n = \mathbf{1} - \frac{1}{\lambda}\{1 - (1-\lambda)^n\}\mathbf{C}$.

$\left(\text{NOTE: } \mathbf{C}^{\mathrm{T}} \text{ denotes the transpose of } \mathbf{C} \text{ and } \mathbf{1} \text{ the unit matrix } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\right)$

Question 10.

A man inherits, on his 21st birthday (31 December, 1970), two houses - one in the city and one in the country. On the first day after this birthday, i.e., on 1 January, 1971, he moves into the city house. In this house there is a box containing one red and two white balls and in the country house there is a similar box containing one red and three white balls. Each day he draws at random a ball from the box in the house where he is, notes its colour, and then returns it to the box: if it is red he moves to spend the next day at the other house (otherwise he stays where he is and awaits the outcome of the next drawing).

(i) Calculate the probability that he is in residence at the city house:

(a) on 3 January, 1971;

(b) on the n-th day after his 21st birthday.

(ii) Assuming that he lives to a ripe old age estimate (approximately) the probability that he will die in the city house.

(NOTE: Results stated in any previous question on this paper may be assumed.)

NSW HSC Mathematics Level 1 (4 Unit) Examination 1974

Question 1.

(i) Sketch the graphs (showing the main features - do NOT use squared paper) of:

(a) $y = |\sin x|$ $(-2\pi \leqslant x \leqslant 2\pi)$;

(b) $y = \sin |x|$ $(-2\pi \leqslant x \leqslant 2\pi)$;

(c) $(x+2)(y+1) = 1$.

(ii) Assuming that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$
evaluate

(a) $\lim\limits_{x\to 0} \left[\frac{\sin^2(3x)}{x^2}\right]$;

(b) $\lim\limits_{x\to 0} \left[\frac{1-\cos(4x)}{x^2}\right]$.

Question 2.

State the Fundamental Theorem of Arithmetic (concerning the factorisation of integers into primes) and give a proof of this theorem by first showing that, if a prime p divides the product ab of two integers and does not divide a, then it must divide b.

Question 3.

(i) If z is the complex number $1+2i$ indicate on the Argand diagram the points

$$z, \overline{z}, z^2, \frac{1}{z}.$$

(ii) On the Argand diagram P represents the complex number z and Q the complex number $\frac{1}{z}$. If P lies on the straight line $x = 1$ prove that Q will lie on a certain circle and find its centre and radius.

Question 4.

In the Cartesian plane indicate (by shading) the region R consisting of those points whose coordinates (x, y) simultaneously satisfy the five relations

$$0 \leqslant x \leqslant \pi/2,\ y \geqslant 0,\ y \geqslant \sin x,\ y \leqslant \cos x,\ y \leqslant \tan x.$$

Also prove that the area of R is

$$\tfrac{1}{2}\left[2\sqrt{2} - 1 - \sqrt{5} - \log_e\left\{\tfrac{\sqrt{5}-1}{2}\right\}\right].$$

Question 5.

(i) Find the length of the arc of the curve

$$y = \frac{x^4 + 3}{6x}$$

that lies between $x = 1$ and $x = 2$.

(ii) If

$$f(x) = \int_0^{2x} \sqrt{1 + t^4}\, dt$$

evaluate $f'(2)$ where $f'(x)$ denotes the derivative of $f(x)$.

Question 6.

(i) State the values of x for which the series

$$1 - x^2 + x^4 - \cdots + (-1)^{n-1} x^{2n-2} + \cdots$$

is convergent.

Also write down an expression for the sum of its first n terms.

(ii) Determine all the values of x for which the series

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \cdots$$

is convergent. Also prove that, when it is convergent, its sum is $\tan^{-1} x$.

Question 7.

In three-dimensional Cartesian space, S denotes the sphere

$$x^2 + y^2 + z^2 = 1$$

and β the plane

$$2x + 2yz = 1.$$

(i) Find the equations of the two planes which are parallel to β and which touch S (i.e., are tangent planes to S).

(ii) If β meets the plane $z = 2$ in the line p find the equations of the two tangent planes to S which pass through p.

Question 8.

Given that

$$\mathbf{A} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix},$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{d} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \ \mathbf{r} = \begin{bmatrix} x \\ y \end{bmatrix}, \ \mathbf{r}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

describe in geometrical language (using terms such as reflection, rotation, etc.) the transformation $\mathbf{r} \to \mathbf{r}'$ in the Cartesian plane in the following cases:

(i) $\mathbf{r}' = \mathbf{A}\mathbf{r}$;

(ii) $\mathbf{r}' = \mathbf{B}\mathbf{r}$;

(iii) $\mathbf{r}' = \mathbf{B}\mathbf{r} + \mathbf{d}$;

(iv) $\mathbf{r}' = \mathbf{C}\mathbf{r}$;

(v) $\mathbf{r}' = (\mathbf{I} + \mathbf{C})^2\mathbf{r}$.

Question 9.

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \frac{1}{2}\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$.

(i) Show that the eigenvalues of $\mathbf{A}$ are 4,1 and find the corresponding eigenvectors.

(ii) Find a matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

(iii) Solve, for $\mathbf{X}$, the matrix equation $\mathbf{X}^2 = \mathbf{A}$.

Question 10.

A large vertical wall stand on horizontal ground. The nozzle of a water hose is positioned at a point C on the ground at a distance c from the wall and the water jet can be pointed in any direction from C. Also the water issues from the nozzle with speed V. (Air resistance may be neglected and the constant g denotes the acceleration due to gravity.)

(i) Prove that the jet can reach the wall above ground level if and only if $V > \sqrt{gc}$.

(ii) If $V = 2\sqrt{gc}$ prove that the portion of the wall that can be reached by the jet is a parabolic segment of height $15c/8$ and area $5\sqrt{15}c^2/2$.