Poisson traffic processes in pure jump Markov processes and generalized networks *

Cheng-Xuan CAO
School of Management
University of Science and Technology Beijing
Beijing 100080, China
Guang-Hui HSU
Institute of Applied Mathematics
Chinese Academy of Sciences
Beijing 100080, China

Abstract

In this paper, we present conditions under which the traffic processes in a pure jump Markov process with a general state space are Poisson processes, and give a simple proof of PASTA type theorem in Melamed (1982) and Walrand (1988). Furthermore, we consider a generalized network with phase type negative arrivals and show that the network has a product-form invariant distribution and its traffic processes which represent the customers exiting the network are Poisson processes.

Key words: DUAL PREDICTABLE PROJECTION; NEGATIVE ARRIVAL; PH-DISTRIBUTION; GENERALIZED NETWORK; TRAFFIC PROCESS.

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60K25 SECONDARY 60G55

---

*Supported by the National Natural Science Foundation of China

1. Introduction

Traffic processes in a Markov process are point processes associated the jump epochs of the Markov process and their behavior can be described in terms of the characteristics of the Markov process. The study of traffic processes is conductive to a better understanding of stochastic systems and networks in manufacturing, telecommunications, computer design and performance evaluation, etc. Since the stochastic systems and networks with Poisson traffic processes are much more convenient to be dealt with, an important problem in this regard is: When is a traffic process Poisson ? Poisson traffic processes in a pure jump Markov process have been discussed in the literature by using the Markov renewal method (see Melamed (1979, 1982) and Disney and Kiessler (1987)), the filtering method (see Bémaud (1981) and Varaiya and Walrand (1981)), and the time-reversal method (see Walrand (1988) and Serfozo (1989)).

In this paper, we present conditions under which traffic processes in a pure jump Markov process with a general state space are Poisson processes by means of the dual predictable projections of the traffic processes and the time-reversal process, and give a simple proof of PASTA type theorem in Melamed (1982) and Walrand (1988), which are the subject of Section 2. In Section 3, we consider a symmetric queue with phase type negative arrivals and show that the queue is quasi-reversible. The concept of negative arrivals, sometimes called signals, was first introduced by Gelenbe (1991). Recently many authors considered queueing networks with negative arrivals, but they always assumed that the negative customers arrive at the system according to a stationary Poisson process, see Gelenbe (1991,1993a,b), Gelenbe and Schassberger (1992), Chao and Pinedo (1993), Chao (1995), and Miyazawa and Wolff (1996). Gelenbe (1993b) called them Generalized networks, or G-networks for short. In Section 4, we consider a generalized network with phase type negative arrivals and show that the network has a product-form invariant distribution and its traffic processes which represent the customers exiting the network are Poisson processes.

2. Poisson traffic processes in pure jump Markov processes

Given a Polish space $(E, \mathcal{E})$ and a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, let $X = \{X_t\}_{t \geq 0}$ be a right-continuous regular pure jump Markov process with left-hand limits on $(\Omega, \mathcal{F}, \mathcal{P})$ with state space $(E, \mathcal{E})$. Let $\{\mathcal{F}_t\}_{t \geq 0}$ be the natural filtration of X, i.e.

$$\mathcal{F}_t = \sigma\{X_s, s \leq t\},\ t \geq 0$$

(we denote by $\sigma\{\dots\}$ the complete σ-field generated by $\{\dots\}$).

We assume that X has standard stationary transition probabilities

$$P_t(x, A) = P\{X_t \in A \mid X_0 = x\},\ t \geq 0,\ x \in E,\ A \in \mathcal{E},$$

and infinitesimal generator $\mathcal{A}$. Its transition rates are denoted by $q(x, A)\hat{=}\mathcal{A}1_A(x)$, $x \in E$, $A \in \mathcal{E}$, and $q(x)\hat{=}q(x, E - \{x\})$, where 1_A is the indicator function of A. We assume that $0 < q(x) < \infty$. A σ-finite measure π on $(E, \mathcal{E})$ is said to be invariant for X, if π solves the following balance equations:

$$\int_A \pi(dy)q(y) = \int_E \pi(dy)q(y, A - \{y\}),$$

for all $A \in \mathcal{E}$.

Let $\{S_n\}_{n\geq 1}$ be a sequence of jump epochs of X and put $S_0\hat{=}0$. Then S_n are clearly stopping times of $\{\mathcal{F}_t\}$. For each $n \geq 0$, define $Y_n\hat{=}X_{S_n}$. $Y = \{Y_n\}_{n\geq 0}$ is said to be the jump chain of X. We assume $\beta\hat{=}\int_E \pi(dx)q(x) < \infty$. Then the probability measure θ defined by

$$\theta(dx) = \beta^{-1}\pi(dx)q(x), \quad x \in E$$

is invariant for Y. Define N by $N(C) = \{$number of jumps of $\{X_t\}$ in the Borel set $C\}$, i.e.,

$$N(C) = \sum_{n=1}^{\infty} 1_C(S_n).$$

Let H be a subset of $E \times E - diag(E \times E)$ and N_H be the point process counting the H-transitions of $\{X_t\}$, i.e.,

$$N_H(C) = \int_C 1_H(X_{s-}, X_s)N(ds).$$

Then $N_H(C)$ is the number of jumps of X in the time period C from some x to some y for which $(x, y) \in H$. Clearly, $N_H(C)$ is finite when C is bounded and maybe infinite when C is unbounded.

Define a integer-valued random measure on $R_+ \times E$ by

$$\mu(ds, dy) = \sum_{n=1}^{\infty} \delta_{(S_n, Y_n)}(ds, dy), \tag{2.1}$$

where δ_x is the Dirac measure at $x \in R_+ \times E$. Then we have

$$N_H(C) = \int_C \int_E 1_H(X_{s-}, x)\mu(ds, dx). \tag{2.2}$$

Note that the integer-valued random measure μ has dual $\mathcal{F}_t$-predictable projection υ:

$$\upsilon(ds, dx) = q(X_{s-}, dx)ds. \tag{2.3}$$

(see Chapter 3 of Last and Brandt (1995) for the details). Let N_H^p be the dual $\mathcal{F}_t$-predictable projection of N_H. Then from (2.2) and (2.3), we have

$$N_H^p(C) = \int_C \int_E 1_H(X_{s-}, x)q(X_{s-}, dx)ds. \tag{2.4}$$

Let

$$\Lambda_H(t) = \int_E 1_H(X_{t-}, x) q(X_{t-}, dx) \tag{2.5}$$

for all t,

$$\eta(y) = \frac{1}{\pi(dy)} \int_E 1_H(x, y) q(x, dy) \pi(dx)$$

for all $y \in E$, and $\widetilde{\Lambda}_H(t) = \eta(X_{t-})$ for all t. Then we have the following theorem.

Theorem 2.1. i) Under the initial probability measure π, the present state X_t and the future of N_H: $\{N_H(t+u) - N_H(t),\ u \geq 0\}$ are independent for all t if and only if $\Lambda_H(t)$ is a fixed positive constant a for all t. In that case, N_H is a Poisson process with rate a.

ii) Under the initial probability measure π, the present state X_t and the past of N_H: $\{N_H(u),\ u \leq t\}$ are independent for all t if and only if $\widetilde{\Lambda}_H(t)$ is a fixed positive constant a for all t. In that case, N_H is a Poisson process with rate a.

Remark 2.1. Theorem 2.1 for Markov processes with countable state space was given by Serfozo (1989) and he pointed out that the results can extend to Markov processes with general state space. Now, we give another proof for general state space case.

Proof. i) If Λ_H is a fixed positive constant a, then by (2.4), (2.5) and T4 theorem (see Brémaud (1981), pp. 25), N_H is a Poisson process with intensity a. For all t, we have

$$\begin{aligned} E[N_H(t+s) - N_H(t) \mid \sigma(X_t)] &= E[N_H(t+s) - N_H(t) \mid \mathcal{F}_t] \\ &= E[\int_t^{t+s} \Lambda_H(u) du \mid \mathcal{F}_t] = sa. \end{aligned}$$

Thus, the present state X_t and the future of N_H: $\{N_H(t+u) - N_H(t),\ u \geq 0\}$ are independent for all t.

On the other hand, from the definition of N_H, we know that N_H is a simple point process. By the defination of N_H^p, we have $EN_H^p(\{t\}) = 0$ if and only if $P(N_H(t) \neq N_H(t-)) = 0$, where $N_H(t) \hat{=} N_H((0,t])$, for all t. From (2.4), we obtain that N_H is continuous in probability, i.e., $\lim_{t \to u} N_H(t) = N_H(u)$, where the limit is taken in probability. By Markov property of $\{X_t\}$, $N_H(t+u) - N_H(t)$ is independent of $\mathcal{F}_t$ for all t and $u \geq 0$. Since ${\mathcal{F}_t}^{N_H} \subset \mathcal{F}_t$ for all t, where ${\mathcal{F}_t}^{N_H}$ is the natural filtration of N_H, we have $N_H(t+u) - N_H(t)$ is independent of ${\mathcal{F}_t}^{N_H}$ for all t and $u \geq 0$. Thus N_H is a Poisson process with nonnegative deterministic cumulative intensity (compensator). Note that X is stationary, and therefore Λ_H is a fixed positive constant.

ii) By using the time-reversal of X with $\widetilde{H} = \{(x, y) \mid (y, x) \in H\}$ instead of H, we obtain the conclusion with the same argument above. $\square$

Define $\Psi = \{\Psi_n = (Y_n, Y_{n+1}), n \geq 0\}$. Then Ψ is a discrete-time Markov process on

$\mathbf{\Psi} = \{(x, y) \in E^2 \mid x \neq y\}$. Its one-step transition function is given by

$$M(x, dy) = F(x_2, dy_2) 1_{y_1 = x_2}$$

for all $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbf{\Psi}$. Assume that $\beta \hat{=} \int_E \pi(dx) q(x) < \infty$. Then

$$\eta(dx) \hat{=} \beta^{-1} \pi(dx_1) q(x_1, dx_2), \quad x = (x_1, x_2) \in \mathbf{\Psi}$$

is the invariant probability measure for Ψ. Let $\{T_n\}$ be a sequence of jump epoch of $\{(X_{t-}, X_t)\}$ in $H \in \mathcal{E}$ and define $\gamma_n \hat{=} (X_{T_n-}, X_{T_n})$ for all $n \geq 1$. Then $\{\gamma_n\}$ is a Markov chain on H with the invariant probability measure π_H defined by

$$\pi_H(dx) = \frac{\eta(dx)}{\eta(H)} 1_{x \in H}.$$

Assume that

$$\lambda \hat{=} \int_E \pi(dx) \int_E 1_H(x, y) q(x, dy) > 0. \tag{2.6}$$

Then $\{X_{T_n-}\}$ is a discrete-time Markov process with invariant probability measure π_{T-} given by

$$\pi_{T-}(dx) = \lambda^{-1} \pi(dx) \int_E 1_H(x, y) q(x, dy), \quad x \in E, \tag{2.7}$$

and $\{X_{T_n}\}$ is a discrete-time Markov process with invariant probability measure π_T given by

$$\pi_T(dy) = \lambda^{-1} \int_E q(x, dy) 1_H(x, y) \pi(dx), \quad y \in E. \tag{2.8}$$

Theorem 2.2. i) Under the initial probability measure π, the present state X_t and the future of N_H: $\{N_H(t+u) - N_H(t),\ u \geq 0\}$ are independent for all $t \geq 0$ if and only if $\pi_{T-} = \pi$. In this case, N_H is a Poisson process with rate λ.

ii) Under the initial probability measure π, the present state X_t and the past of N_H: $\{N_H(u),\ u \leq t\}$ are independent for all $t \geq 0$ if and only if $\pi_T = \pi$. In this case, N_H is a Poisson process with rate λ.

Remark 2.2. Theorem 2.2 is given by Melamed (1982) and Walrand (1988) (see, p.75). Now, we give a simple proof of it by using Theorem 2.1.

Proof. i) If $\pi_{T-} = \pi$, then

$$\int_E 1_H(x, y) q(x, dy) = \lambda.$$

It follows from (2.5) that $\Lambda_H(t) = \lambda$. If the present state X_t and the future of N_H: $\{N_H(t+u) - N_H(t),\ u \geq 0\}$ are independent for all $t \geq 0$, then by Theorem 2.1, $\Lambda_H(t)$ is a fixed positive constant a, for all t, i.e.,

$$\Lambda_H(t) = E[\Lambda_H(t)] = a$$

for all t. Thus, by (2.5), (2.6) and (2.7), we have $\pi_{T-} = \pi$. The rest of the conclusion comes from Theorem 2.1 .

ii) By using the time-reversal of X with $\widetilde{H} = \{(x, y) \mid (y, x) \in H\}$ instead of H, we obtain the conclusion with the same argument above. $\square$

Now, we consider the quasi-reversibility of a queue. The evolution of a queue is described by a Markov process $\{X_t\}$. Its arrival process A^c and departure process D^c correspond to certain transitions of $\{X_t\}$ in the state space E, for each $c \in \mathbf{C}$, where $\mathbf{C}$ is the set of classes of customers. That is,

$$A_t^c = \sum_{0<s\leq t} 1_{\mathbf{A^c}}(X_{s-}, X_s), \quad D_t^c = \sum_{0<s\leq t} 1_{\mathbf{D^c}}(X_{s-}, X_s),$$

where $\mathbf{A^c}$ and $\mathbf{D^c}$ are disjoint subsets of $\{(x, y) \in E \mid x \neq y\}$, for each $c \in \mathbf{C}$.

Corollary 2.1. Suppose the queue is stationary with the invariant probability measure π. Then it is quasi-reversible if and only if for all $c \in \mathbf{C}$,

$$\int_E 1_{\mathbf{A^c}}(x, y) q(x, dy) = \lambda_c, \text{ for all } x \in E,$$

and

$$\frac{1}{\pi(dy)} \int_E 1_{\mathbf{D^c}}(x, y) q(x, dy) \pi(dx) = \mu_c, \text{ for all } y \in E,$$

where λ_c is independent of $x \in E$, and μ_c is independent of $y \in E$. In this case, arrival processes are independent Poisson processes with rates λ_c, $c \in \mathbf{C}$ and so are the departure processes with rates μ_c, $c \in \mathbf{C}$.

Remark 2.3. Corollary 2.1 is a generalization of (3.1) and (3.2) in Walrand (1988) (see pp. 90-91). It immediately follows from Theorem 2.1.

3. A symmetric queue with phase type negative arrivals

In this section, we consider a symmetric queue within which customers are located in positions 1, 2, $\cdots$, n, where n is the total number of customers present. Assume that:

(i) Regular customers arrive at the queue as independent Poisson processes with rate λ_c^+, for class $c \in \mathbf{C}$ (a given countable set). When a regular customer arrives, he moves into position l, $l = 1, 2, \cdots, n+1$, with probability $\gamma(l, n+1)$; customers previously in positions l, $l+1$, $\cdots$, n move to positions $l+1$, $l+2$, $\cdots$, $n+1$ respectively.

(ii) The distribution of the required service time of a class c customer is $PH(\alpha_c, S_c)$ of order m for $c \in \mathbf{C}$.

(iii) The total service effort of the queue is dependent on the state of the queue and is supplied at rate $\psi(n)$.

(iv) A proportion $\gamma(l, n)$ of the total service effort is distributed to the customer in position l, $l = 1, 2, \cdots, n$; with his departure, customers in positions $l+1$, $l+2$, $\cdots$, n move into positions l, $l+1$, $\cdots$, $n-1$ respectively.

(v) In addition, as soon as a class c regular customer arrives at the queue and joins some position, a negative arrival process in this position starts. The required time until the negative arrival occurs has a $PH(\beta_c,\ U_c)$ distribution of order m', for $c \in \mathbf{C}$.

(vi) For negative arrival processes, the actual total rate of progress in all positions is dependent on the state of the queue and is equal to $\psi(n)$. A proportion $\gamma(l,n)$ of the actual total rate of progress is distributed to the negative arrival process in position l, $1 \le l \le n$. A negative arrival in some position l results in a service completion of the regular customer and the end of the corresponding negative arrival process in this position. With the departure of the regular customer, customers in positions $l+1$, $l+2$, $\cdots$, n move into positions l, $l+1$, $\cdots$, $n-1$ respectively.

(vii) All the regular customer arrival processes, the required times for negative arrivals and the required service times are mutually independent.

These $\gamma(l,n)$'s satisfy

$$\sum_{l=1}^{n} \gamma(l,n) = 1.$$

Let S_c^- be a random variable with distribution $PH(\beta_c,\ U_c)$ of order m', and S_c be a random variable with distribution $PH(\alpha_c,\ S_c)$ of order m for $c \in \mathbf{C}$. Then $S_c^- \wedge S_c$ is a random variable with distribution $PH(\beta_c \otimes \alpha_c, U_c \otimes I + I \otimes S_c)$ of order mm', for $c \in \mathbf{C}$. Let $L_c = U_c \otimes I + I \otimes S_c$, $\varphi_c = \beta_c \otimes \alpha_c$. Then it is easy to know that the queue can be described by a Markov process $\{X_t\}$ with state space $E = \{x\}$, where

$$x = (c_1, \phi_1; c_2, \phi_2; \cdots; c_n, \phi_n), \text{ and } \phi_i = (\phi_i', \phi_i''),\ 1 \le i \le n, \tag{3.1}$$

when there are n customers in the queue, the customer in position i is of class c_i with service phase ϕ_i'', and the phase of negative arrival is ϕ_i'. We define that $x = 0$ when $n = 0$.

For each $x = (c_1, \phi_1; c_2, \phi_2; \cdots; c_n, \phi_n) \in E$, we define

$$\begin{cases} A_{c\phi}^{l} x = (c_1, \phi_1; \cdots; c_{l-1}, \phi_{l-1}; c, \phi; c_l, \phi_l; \cdots; c_n, \phi_n),\ 1 \le l \le n,\ n \ge 1, \\ A_{c\phi}^{l} x = (c_1, \phi_1; \cdots; c_n, \phi_n; c, \phi),\ l = n+1,\ n \ge 0, \\ D_{c_l\phi_l}^{l} x = (c_1, \phi_1; \cdots; c_{l-1}, \phi_{l-1}; c_{l+1}, \phi_{l+1}; \cdots; c_n, \phi_n),\ 1 \le l \le n,\ n \ge 1, \\ T_{\phi}^{l} x = (c_1, \phi_1; \cdots; c_l, \phi; \cdots; c_n, \phi_n),\ 1 \le l \le n,\ n \ge 1. \end{cases}$$

Then, the transition rate matrix of the queue $Q = (q(x,y))_{x,y\in E}$ is given by

$$\begin{cases} q(x, A^l_{c\phi}x) = \lambda^+_c \gamma(l, n+1)\varphi_c(\phi),\ 1 \le l \le n+1,\ n \ge 0, \\ \\ q(x, D^l_{c_l\phi_l}x) = \gamma(l,n)L_{0c_l}(\phi_l)\psi(n), 1 \le l \le n,\ n \ge 1, \\ \\ q(x, T^l_\phi x) = \gamma(l,n)L_{c_l}(\phi_l, \phi)\psi(n)1_{\phi\ne\phi_l},\ 1 \le l \le n,\ n \ge 1, \end{cases} \tag{3.2}$$

where $L_{0c_l} = -L_{c_l}e$ (e is the unit column vector), and $q(x,y) = 0$ for all other state $y \in E - \{x\}$.

Let $v_c = (v_c(\phi_1), v_c(\phi_2), \cdots, v_c(\phi_{mm'}))$ be the probability distribution that solves

$$v_c[L_c + L_{0c}\varphi_c] = 0,$$

and $\mu^*_c \hat{=} v_c L_{0c}$, for each $c \in \mathbf{C}$. Assume that the Markov process is irreducible. Then we have the following theorem.

Theorem 3.1. If

$$\lim_{l\to\infty} \frac{1}{\psi(l)} \sum_{c\in\mathbf{C}} \frac{\lambda^+_c}{\mu^*_c} < 1$$

then the invariant distribution for the queue described above is given by

$$\pi(x) = b \prod_{l=1}^{n} \frac{\lambda^+_{c_l} v_{c_l}(\phi_l)}{\mu^*_{c_l}\psi(l)}, \tag{3.3}$$

for all $x \ne 0$, and $\pi(0) = b$, where

$$b = (1 + \sum_{n=1}^{\infty} \frac{1}{\prod_{l=1}^{n}\psi(l)} (\sum_{c\in\mathbf{C}} \frac{\lambda^+_c}{\mu^*_c})^n)^{-1}$$

is the normalization constant.

Furthermore, the queue is quasi-reversible when the system is in equilibrium. In this case, the departure processes of class c, $c \in \mathbf{C}$, regular customers are independent Poisson processes with rates λ^+_c, $c \in \mathbf{C}$ respectively.

Proof. Let $\widetilde{Q} = (\widetilde{q}(x,y))_{x,y\in E}$ be the transition rate matrix of the time-reversal process. It is given by

$$\begin{cases} \widetilde{q}(D^l_{c_l\phi_l}x, x) = \lambda^+_{c_l}\gamma(l,n)\widetilde{\varphi}_{c_l}(\phi_l),\ 1 \le l \le n,\ n \ge 1, \\ \\ \widetilde{q}(A^l_{c\phi}x, x) = \gamma(l, n+1)\widetilde{L}_{0c}(\phi)\psi(n+1),\ 1 \le l \le n+1,\ n \ge 0, \\ \\ \widetilde{q}(T^l_\phi x, x) = \gamma(l,n)\widetilde{L}_{c_l}(\phi, \phi_l)\psi(n)1_{\phi\ne\phi_l},\ 1 \le l \le n,\ n \ge 1, \end{cases} \tag{3.4}$$

where

$$\widetilde{L}_c(\phi, \overline{\phi}) = L_c(\overline{\phi}, \phi)\frac{v_c(\overline{\phi})}{v_c(\phi)},\ \widetilde{L}_{0c}(\phi) = \frac{\mu^*_c\varphi_c(\phi)}{v_c(\phi)},\ \widetilde{\varphi}_c(\phi) = \frac{L_{0c}(\phi)v_c(\phi)}{\mu^*_c},$$

for $c \in \mathbf{C}$ and any $\phi = (\phi', \phi'')$, $\overline{\phi} = (\overline{\phi}', \overline{\phi}'')$. To show that there is a constant b that normalizes π given in (3.3), by Kelly's Lemma (see Walrand (1988) p.63), one must verify that

$$\pi(x)q(x,y) = \pi(y)\widetilde{q}(y,x), \tag{3.5}$$

for all $x, y \in E$. But this follows immediately from (3.2), (3.3) and (3.4).

The quasi-reversibility of the queue comes from Kelly's Lemma. But we may also verify it by using Corollary 2.1. In fact, for all $x \in E$ given by (3.1),

$$\sum_{y \in E} q(x,y) 1_{\mathbf{A}^c}(x,y) = \sum_{l=1}^{n+1} \sum_{\phi=1}^{mm'} q(x, A^l_{c\phi}x) = \lambda^+_c,$$

and for all $y = (c_1, \phi_1; \cdots; c_n, \phi_n) \in E$,

$$\frac{1}{\pi(y)} \sum_{x \in E} \pi(x) q(x,y) 1_{\mathbf{D}^c}(x,y) = \frac{1}{\pi(y)} \sum_{l=1}^{n+1} \sum_{\phi=1}^{mm'} \pi(A^l_{c\phi}y) q(A^l_{c\phi}y, y) = \lambda^+_c. \quad \square$$

4. A generalized network with phase type negative arrivals

In this section, we consider a generalized network with J symmetric queues defined above. Let $\mathbf{C}$ be the countable set of regular customer classes. We assume that regular customers of class, $c \in \mathbf{C}$, arrive at queue j from the outside of the network according to independent Poisson processes with respective rate Λ^+_{jc}. When a regular class c customer completes his service at queue j, he goes to queue k as a regular class c' customer with probability $p_{jc;kc'}$, and leaves the network as a regular class c customer with probability $p_{jc;0}$. These transition probabilities satisfy

$$\sum_{k,c'} p_{jc;kc'} + p_{jc;0} = 1, \ 1 \le j \le J, \ c \in \mathbf{C}.$$

For each $j = 1, 2, \cdots, J$, let n_j be the total number of customers in the queue j. We assume that queue j operates in the following manner:

i) When a regular customer arrives, he moves into position l, $l = 1, 2, \cdots, n_j + 1$, with probability $\gamma_j(l, n_j + 1)$; customers previously in positions l, $l+1$, $\cdots$, n_j move to positions $l+1$, $l+2$, $\cdots$, $n_j + 1$ respectively.

(ii) The distribution of the required service time of a class c customer is $PH(\alpha_{jc}, S_{jc})$ of order m_j, for $c \in \mathbf{C}$.

(iii) The total service effort at queue j is dependent on the state of the queue and is supplied at rate $\psi_j(n_j)$.

(iv) A proportion $\gamma_j(l, n_j)$ of the total service effort is distributed to the customer in position l, $l = 1, 2, \cdots, n_j$; with his departure, customers in positions $l+1$, $l+2$, $\cdots$, n_j move into positions l, $l+1$, $\cdots$, $n_j - 1$ respectively.

(v) In addition, as soon as a class c regular customer arrives at queue j and joins some position, a negative arrival process in this position starts. The required time until the negative arrival occurs has a $PH(\beta_{jc},\ U_{jc})$ distribution of order m'_j, for $c \in \mathbf{C}$.

(vi) For negative arrival processes, the actual total rate of progress in all positions is dependent on the state of the queue and is equal to $\psi_j(n_j)$. A proportion $\gamma_j(l, n_j)$ of the actual total rate of progress is distributed to the negative arrival process in position l, $1 \le l \le n_j$. A negative arrival in some position l results in a service completion of the regular customer and the end of the corresponding negative arrival process in this position. This regular customer will either leave the network, or go to another queue with probabilities defined above. With the departure of the regular customer in position l, customers in positions $l+1,\ l+2,\ \cdots,\ n_j$ move into positions $l,\ l+1,\ \cdots,\ n_j - 1$ respectively.

(vii) All the regular customer arrival processes, the required times for negative arrivals and the required service times are mutually independent.

These $\gamma_j(l, n_j)$'s satisfy

$$\sum_{l=1}^{n_j} \gamma_j(l, n_j) = 1.$$

For each $j = 1, 2, \cdots, J$ and $c \in \mathbf{C}$, let S^-_{jc} be a random variable with distribution $PH(\beta_{jc}, U_{jc})$ of order m'_j, and S_{jc} be a random variable with distribution $PH(\alpha_{jc},\ S_{jc})$ of order m_j. Then $S^-_{jc} \wedge S_{jc}$ is a random variable with distribution $PH(\beta_{jc} \otimes \alpha_{jc}, U_{jc} \otimes I + I \otimes S_{jc})$ of order $m_j m'_j$. We define $L_{jc} \hat{=} U_{jc} \otimes I + I \otimes S_{jc}$, $\varphi_{jc} = \beta_{jc} \otimes \alpha_{jc}$. Let $v_{jc} = (v_{jc}(\phi_1), v_{jc}(\phi_2), \cdots, v_{jc}(\phi_{m_j m'_j}))$ be the probability distribution that solves

$$v_{jc}[L_{jc} + L_{j0c}\varphi_{jc}] = 0,$$

and $\mu^*_{jc} \hat{=} v_{jc} L_{j0c}$, where $L_{j0c} = -L_{jc} e$ (e is a unit column vector of appropriate dimension). It is easy to know that the network can be described by a Markov process $\{X_t\}$, whose state is given by

$$x = (x_1, x_2, \cdots, x_J), \tag{4.1}$$

where x_j denotes the state of queue j defined as (3.1), i.e.,

$$x_j = (c_{j1}, \phi_{j1}; c_{j2}, \phi_{j2}; \cdots; c_{jn_j}, \phi_{jn_j}),\ j = 1, 2, \cdots, J.$$

We assume that the state process of the network is irreducible, and denote by D_{jc} the traffic process which represents the class c customers exiting the network from queue j, for $j = 1, 2, \cdots, J$ and $c \in \mathbf{C}$. Then we have the following theorem.

Theorem 4.1. If

$$\lim_{l \to \infty} \frac{1}{\psi_j(l)} \sum_{c \in \mathbf{C}} \frac{\lambda^+_{jc}}{\mu^*_{jc}} < 1,\ j = 1, 2, \cdots, J, \tag{4.2}$$

where λ^+_{jc} is a solution of the traffic equation:

$$\lambda^+_{jc} = \Lambda^+_{jc} + \sum_{i=1}^{J} \sum_{c' \in \mathbf{C}} \lambda^+_{ic'} p_{ic';jc}, \tag{4.3}$$

for $c \in \mathbf{C}$ and $j = 1, 2, \cdots, J$. Then the invariant distribution for the state process of the network described above is given by

$$\pi(x) = \prod_{j=1}^{J} \pi_j(x_j), \tag{4.4}$$

where $\pi_j(x_j)$ is the invariant distribution for queue j:

$$\pi_j(x_j) = b_j \prod_{l=1}^{n_j} \frac{\lambda^+_{jc_{jl}} v_{jc_{jl}}(\phi_{jl})}{\mu^*_{jc_{jl}} \psi_j(l)}, \tag{4.5}$$

for all $x_j \neq 0$, and $\pi_j(0) = b_j$, where

$$b_j = (1 + \sum_{n_j=1}^{\infty} \frac{1}{\prod_{l=1}^{n_j} \psi_j(l)} (\sum_{c \in \mathbf{C}} \frac{\lambda^+_{jc}}{\mu^*_{jc}})^n)^{-1}$$

is the normalization constant.

Furthermore, when the network is in equilibrium, for all $c_j \in \mathbf{C}$ and $j = 1, 2, \cdots, J$, traffic processes D_{1c_1}, D_{2c_2},$\cdots$, D_{Jc_J} are independent Poisson processes with rate $\lambda^+_{1c_1} p_{1c_1;0}$, $\lambda^+_{2c_2} p_{2c_2;0}$,$\cdots$, $\lambda^+_{Jc_J} p_{Jc_J;0}$ respectively, and their past and the present state of the network are independent at any time t.

Proof. From (4.2) and (4.3), it is easy to verify that (4.4) is the invariant distribution for the state process of the network with the similar argument used in Theorem 3.1.

For all $y = (y_1, y_2, \cdots, y_J)$, where $y_j = (c_{j1}, \phi_{j1}; c_{j2}, \phi_{j2}; \cdots; c_{jn_j}, \phi_{jn_j})$, for each $j = 1, 2, \cdots, J$, we have

$$\frac{1}{\pi(y)} \sum_{l=1}^{n_j+1} \sum_{\phi_j=1}^{mm'} \pi(A^l_{jc_j\phi_j} y) \gamma(l, n_j+1) L_{j0c_j}(\phi_j) p_{jc_j;0}$$

$$= \lambda^+_{jc_j} p_{jc_j;0},$$

where

$$A^l_{jc\phi_j} y = (y_1, \cdots, y_j, c_{j1}, \phi_{j1}; \cdots; c_{jl-1}, \phi_{jl-1}; c, \phi_j; c_{jl}, \phi_{jl}; \cdots; c_{jn_j}, \phi_{jn_j}, y_{j+1}, \cdots, y_J),$$

for all $c_j \in \mathbf{C}$ and $j = 1, 2, \cdots, J$. Thus, the conclusion comes from Corollary 2.1. □

References

[1] Brémaud, P. (1980) *Point Processes and Queue: Martingale Dynamics.* Springer–Verlag, New York.

[2] Chao, X. (1995) Networks of queues with customers, signals and arbitrary service time distributions. *Opns. Res.* **43,** 537-544.

[3] Chao, X. and Pinedo, M. (1993) On generalized networks of queues with positive and negative arrivals. *Probability in the Engineering and Informational Sciences* **7,** 301-334.

[4] Disney, R.L. and Kiessler, P.C. (1987) *Traffic Processes in Queueing Networks: A Markov Renewal Approach.* Johns Hopkins University Press, Baltimore.

[5] Gelenbe, E. (1991) Product form queueing networks with positive and negative customers. *J. Appl. Prob.* **28,** 656-663.

[6] Gelenbe, E. (1993a) G-networks with signals and batch removal. *Probability in the Engineering and Informational Sciences* **7,** 335-342.

[7] Gelenbe, E. (1993b) G-networks with triggered customer movement. *J. Appl. Prob.* **30,** 742-748.

[8] Gelenbe, E. and Schassberger, R. (1992) Stability of product-form G-networks, *Probability in the Engineering and Informational Sciences* **6,** 271-276.

[9] Last, G. and Brand, A.(1995) *Marked Point Processes on the Real Line: The Dynamic Approach.* Springer, New York.

[10] Melamed, B. (1979) On Poisson traffic processes in discrete-state Markovian systems with applications to queueing theory. *Adv. Appl. Prob.* **11,** 218-239.

[11] Melamed, B. (1982) On Markov jump processes imbedded at jump epochs and their queueing-theoretic applications. *Math. Operat. Res.* **7,** 111-128.

[12] Miyazawa, M. and Wolff, R.W. (1996) Symmetric queues with batch departures and their networks. To appear in *Adv. Appl. Prob.*.

[13] Serfozo, R.F. (1989) Poisson functionals of Markov processes and queueing networks. *Adv. Appl. Prob.* **21,** 595–611.

[14] Varaiya, P. and Walrand, J. (1981) Flows in queueing networks: a martingale approach. *Math. Oper. Res.* **6** 387–404.

[15] Walrand, J.(1988) *An Introduction to Queueing Networks.* Prentice–Hall, Englewood Cliffs N.J.