Asymptotic Poisson Traffic Processes in Queueing Systems

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Abstract

In this paper, we present conditions under which the finite-dimension distributions of a simple point process formed by some jump times of a Markov process converge weakly to those of a Poisson process and we discuss the some traffic processes of stationary Markovian queues which the finite-dimension distributions of them converge weakly to those of Poisson processes. Subsequently, we prove that the finite-dimensional distributions of the departure process of $GI/M/1$ queue converge weakly to those of Poisson process in heavy traffic.

Key Words: Queueing system; Stochastic intensity; Time reversal; Limit theorem; Traffic process.

1 Introduction

The study of the traffic processes in a queueing system was initiated by Burke (1956) and Reich (1957). They showed that in stationary the departure process of an $M/M/1$ queue is Poisson. Beutler and Melamed (1978) extended this result by showing that the departure processes from the nodes in a Jackson network are independent Poisson processes. Subsequently, Walrand and Varaiya (1981) showed that the traffic process on a link of Jackson network in stationary is Poisson if and only if that link is not part of a loop. Most of papers which studied the traffic processes in queueing systems discussed when the traffic process is Poisson. For example, Brémaud (1980), Disney and Kiessler (1987), Kelly (1979), Walrand (1988) and Whittle (1986). However, under the relaxed condition, that traffic processes tend to be quite complicated; see Disney and König (1985). Thus, it is of interest to know when the traffic process is asymptotic Poisson; see Whitt (1984), Franken et al. (1981) and references there.

In this paper, we give some conditions under which a sequence of the finite-dimensional distributions of simple point processes converge weakly to those of a Poisson process, and discuss some asymptotic Poisson traffic processes in stationary Markovian queues At last, we prove that the finite-dimensional distributions of the departure process of $GI/M/1$ queue converge weakly to those of Poisson process in heavy traffic. In Section 2, we give the definition of natural filtration and stochastic intensity (also see Brémaud (1980), p. 28). Lemma 2.1 is a particular case of Theorem 1 in Kabanov et al. (1981). By Lemma 2.1, we obtain Lemma 2.2 and theorem 2.3. In Section 3, we discuss some asymptotic Poisson traffic processes in stationary Markov queueing systems. In Section 4, we prove that the finite-dimensional distributions of the departure process of $GI/M/1$ queue converge weakly to those of Poisson process in heavy traffic.

2 Preliminaries

Given a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a family $\{\mathcal{F}_t\}_{t\in R}$ of the sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s < t$ which is called a filtration. If the stochastic process $\{X_t\}_{t\in R}$ is such that for all t, X_t is $\mathcal{F}_t$-measurable, then $\{\mathcal{F}_t\}$ is called a filtration of $\{X_t\}$ and $\{X_t\}$ is said to be $\mathcal{F}_t$-adapted. Let

$$\mathcal{F}_t{}^X = \sigma\{X_s, s \le t\}, \quad t \in R$$

(we denote by $\sigma\{\dots\}$ the complete σ-field generated by $\{\dots\}$). Then $\{\mathcal{F}_t{}^X\}$ is a filtration of stochastic process $\{X_t\}$ and $\{\mathcal{F}_t{}^X\}$ is said to be the natural filtration of $\{X_t\}$.

Let N be a simple point process with $EN(C) < \infty$, for all bounded $C \in \mathcal{B}(R)$, let $\{\mathcal{F}_t\}$ be a filtration of N, and let $\{\lambda_t\}$ be a non-negative $\mathcal{F}_t$-predictable process. Then the process $\{\lambda_t\}$ is called an $\mathcal{F}_t$-intensity of N if it is locally integrable (i.e.,

$$\int_C \lambda_s \, ds < \infty,$$

for all bounded $C \in \mathcal{B}(R)$) and if

$$E[N((a,b]) \mid \mathcal{F}_a] = E[\int_a^b \lambda_s ds \mid \mathcal{F}_a], \tag{2.1}$$

for all $(a,b] \in \mathcal{B}(R)$.

Lemma 2.1. *Let $\{N^n\}_{n\geq 1}$ be a sequence of simple point processes and $\{\Lambda^n\}_{n\geq 1}$ be their stochastic intensities. If there is a non-negative deterministic function Λ such that*

$$\int_C \Lambda_s^n \, ds \xrightarrow{d} \int_C \Lambda_s \, ds,$$

for all bounded $C \in \mathcal{B}(R)$, then

$$N^n \xrightarrow{\mathcal{L}} N,$$

where N is a Poisson process with intensity Λ.

Remark. Notation "$\xi^n \xrightarrow{d} \xi$ " means the convergence of the random variables ξ^n to ξ in distribution; "$X^n \xrightarrow{\mathcal{L}} X$" means weak convergence of the finite-dimension distributions of the stochastic processes X^n to those of X.

Proof. See Kabanov et al. (1980). □

We have the following lemma by using lemma 2.1.

Lemma 2.2. *Let $\{N^n\}_{n\geq 1}$ be a sequence of simple point processes and $\{\Lambda^n\}_{n\geq 1}$ be their stochastic intensities. If there is a non-negative deterministic function Λ such that Λ^n a.s. uniformly converge to Λ on all bounded $C \in \mathcal{B}(R)$, i.e., there exist a set $M \in \mathcal{F}$ such that $P(M) = 0$, and $\Lambda^n(\omega)$ uniformly converges to Λ on all bounded $C \in \mathcal{B}(R)$, for any $\omega \in M^c$, then $N^n \xrightarrow{\mathcal{L}} N$, where N is a Poisson process with intensity Λ.*

Proof. By the condition, for any fixed $\omega \in M^c$, and each $\epsilon > 0$, there exists $n'(\omega) \in \mathbf{N}$, if $n > n'(\omega)$, then

$$\mid \Lambda_s^n(\omega) - \Lambda \mid < \epsilon,$$

for all $s \in C$. Therefore

$$\begin{aligned} & \mid \int_C \Lambda_s^n(\omega) dx - \int_C \Lambda_s \, ds \mid \\ \leq & \int_C \mid \Lambda_s^n(\omega) - \Lambda_s \mid \, ds \\ < & \mid C \mid \epsilon. \end{aligned}$$

(The notation $\mid C \mid$ means the Lebesgue measure of C). Thus

$$\int_C \Lambda_s^n \, ds \to \int_C \Lambda_s \, ds \quad a.s..$$

Using Lemma 2.1, we get $N^n \xrightarrow{\mathcal{L}} N$. $\square$

Let $X^n = \{X_t^n\}_{t\in R}$, $n \geq 1$ be a sequence of right-continuous pure jump Markov processes on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a countable state space $(E, \mathcal{E})$. We assume that each $X^n (n \geq 1)$ has finite number of jumps in any finite time period and it is irreducible. The transition rates of $X^n (n \geq 1)$ are

$$q^n(x,y), \text{ for all } x \neq y,\ q^n(x) = \sum_{y\neq x} q^n(x,y) < \infty,\ q^n(x,x) = -q^n(x),\ x \in E.$$

We define the point processes $N^n, n \geq 1$ associated with the jump times of Markov processes $X^n, n \geq 1$ by

$$N^n(C) = \sum_{t\in C} f(X_{t-}^n, X_t^n),$$

where f is a non-negative function from $E \times E$ into $\{0,\ 1\}$. Then, for any $s \leq t$, the Lévy formula holds:

$$E[\sum_{u\in(s,\ t]} f(X_{u-}^n, X_u^n) \mid {\mathcal{F}_s}^{X^n}] = E[\int_{(s,\ t]} \sum_{y\neq X_u^n} q^n(X_u^n, y) f(X_u^n, y) du \mid {\mathcal{F}_s}^{X^n}].$$

Therefore, if $f : E \times E \longrightarrow \{0,\ 1\}$ is such that, for all bounded $C \in \mathcal{B}(R)$,

$$\int_C \sum_{y\neq X_t^n} q^n(X_t, y) f(X_t^n, y) dt < \infty,$$

and let

$$\Lambda_t^n = \sum_{y\neq X_t^n} q^n(X_t^n, y) f(X_t^n, y), n \geq 1.$$

Then Λ_t^n is the ${\mathcal{F}_t}^{X^n}$-intensity of N^n $(n \geq 1)$.

Theorem 2.3. *$\{N^n\}_{n\geq 1}$ and $\{X^n\}_{n\geq 1}$ are defined as above. We assume that each $X^n (n \geq 1)$ is stationary with equilibrium distribution $\pi^n (n \geq 1)$. Let*

$$\overline{\Lambda}_t^n = \pi^n(X_t^n)^{-1} \sum_{y\in E} \pi^n(y) q^n(y, X_t^n) f(y, X_t^n). \tag{2.2}$$

If for any bounded $A \in \mathcal{B}(R)$, $\overline{\Lambda}_t^n$ a.s. uniformly converge to a constant $\lambda > 0$ on A, then

$$N^n \xrightarrow{\mathcal{L}} \mathcal{P}(\lambda),$$

where $\mathcal{P}(\lambda)$ denotes the time-homogeneous Poisson process with parameter λ.

Proof. Let $\overline{X}_t^n = \widetilde{X}_{-t}^n$, $t \in R$, where $\widetilde{X}_t^n = X_{t-}^n$, $n \geq 1$, then $\overline{X}^n$ is the right-continuous time-reversal of X^n for each $n \geq 1$.

Since each X^n $(n \geq 1)$ is stationary, it follows that each $\overline{X}^n$ is an irreducible stationary pure jump regular Markov process and its transition rates are

$$\overline{q}^n(x,y) = \pi^n(x)^{-1} \pi^n(y) q^n(y,x), \quad x, y \in E.$$

Define a point process $\overline{N}^n$ on R by

$$\overline{N}^n(C) = \sum_{t \in C} \bar{f}(\overline{X}^n_{t-}, \overline{X}^n_t), \quad C \in \mathcal{B}(R),$$

where $\bar{f}(x,y) = f(y,x)$, for all $x, y \in E$. Clearly, for each $C \in \mathcal{B}(R)$,

$$\overline{N}^n(C) = N^n(-C).$$

So $\overline{N}^n$ is the time-reversal of N^n (see Serfoso (1989)). By Lemma 2.2, we have

$$\overline{N}^n \xrightarrow{\mathcal{L}} \mathcal{P}(\lambda).$$

Thus $N^n \xrightarrow{\mathcal{L}} \mathcal{P}(\lambda)$. $\square$

3 Asymptotic Poisson Traffic Processes in Stationary Markov Queueing Systems

First, we consider a sequence of birth and death queueing systems with feedback. Let $\{X^n\}_{n \geq 1}$ be a sequence of Markov processes and we suppose that each X^n $(n \geq 1)$ has the state space $Z_+ = \{0, 1, 2, \ldots\}$ and transition rates

$$q^n(x, x+1) = \lambda^n_x, \quad x \in Z_+,$$

$$q^n(x, x-1) = \mu_x(1 - p^n)1_{(x \geq 1)}, \quad x \in Z_+,$$

where λ^n_x and μ_x are positive, for all $x \in Z_+$, $0 < p^n < 1$, and $q^n(x,y) = 0$, for all other states $y \in Z_+ - \{x\}$. Then this process represents the number of customers in a queueing system in which customers arrive at the rate λ^n_x, when $x \in Z_+$ customers are present and the customers complete service at the rate μ_x when there are $x \in N$ customers in the system. If a customer completes service, then he leaves the system with probability $1 - p^n$ or feeds back to the queue with probability p^n. We assume that $\mu_x \leq M$, for all $x \in N$ and a fixed constant $M > 0$.

Let D^n, F^n and E^n be respectively the departure process, the feedback stream and the service facility input stream (see P.38 of Brémaud (1980)). Let

$$\mathcal{H}^n_t = \mathcal{F}_t^{X^n} \bigvee \mathcal{F}_t^{F^n}, \quad t \in R.$$

Then E^n has the $\mathcal{H}^n_t$-intensity

$$\lambda^n_{X^n_t} + \mu_{X^n_t} p^n 1_{(X^n_t > 0)}.$$

If

$$\mu^n_0 \triangleq 1 + \sum_{x \in N} \frac{\lambda^n_0 \lambda^n_1 \ldots \lambda^n_{x-1}}{\mu_1 \mu_2 \ldots \mu_x (1 - p^n)^x} < \infty,$$

then each $X^n(n \geq 1)$ has unique equilibrium distribution

$$\pi^n(0) = \frac{1}{\mu_0^n},$$
$$\pi^n(x) = \pi^n(0)[\mu_1\mu_2 \ldots \mu_x(1-p^n)^x]^{-1}\lambda_0^n\lambda_1^n \ldots \lambda_{(x-1)}^n, \quad x \in N.$$

Theorem 3.1. i) *If $p^n \to 0$ and λ_x^n uniformly converge to a constant $\lambda > 0$ on $x \in Z_+$, as $n \to \infty$, then*

$$E^n \xrightarrow{\mathcal{L}} \mathcal{P}(\lambda).$$

ii) *We assume that each $X^n(n \geq 1)$ is stationary with equilibrium distribution π^n. If λ_x^n uniformly converge to a constant $\lambda > 0$ on $x \in Z_+$, as $n \to \infty$, then*

$$D^n \xrightarrow{\mathcal{L}} \mathcal{P}(\lambda).$$

Proof. i) By the condition of the theorem, for each $\epsilon_1 > 0$, there exists $n' \in N$, for all $n > n'$,

$$\mid \lambda_x^n - \lambda \mid < \epsilon_1 \quad \text{on } x \in Z_+.$$

For each $\epsilon_2 > 0$, there exists $n'' \in N$, for all $n > n''$, such that

$$p^n < \epsilon_2.$$

Let

$$\Lambda_t^n = \lambda_{X_t^n}^n + \mu_{X_t^n} p^n 1_{(X_t^n > 0)}, \; \bar{n} = max\{n', n''\}.$$

Then for all $n > \bar{n}$, we have

$$\begin{aligned}
\mid \Lambda_t^n - \lambda \mid &= \mid \lambda_{X_t^n}^n + \mu_{X_t^n} p^n 1_{(X_t^n > 0)} - \lambda \mid \\
&= \mid \sum_{x \in Z_+} [\lambda_x^n + \mu_x p^n 1_{(x>0)} - \lambda] 1_{(X_t^n = x)} \mid \\
&\leq \sum_{x \in Z_+} [\mid \lambda_x^n - \lambda \mid + \mu_x p^n] 1_{(X_t^n = x)} \\
&< \sum_{x \in Z_+} (\epsilon_1 + M\epsilon_2) 1_{(X_t^n = x)} \\
&= \epsilon_1 + M\epsilon_2.
\end{aligned}$$

Therefore, λ_t^n uniformly converge to $\lambda > 0$ on $t \in R_+$. By Lemma 2.2, we obtain

$$E^n \xrightarrow{\mathcal{L}} \mathcal{P}(\lambda).$$

ii) By the condition of the theorem, for each $\epsilon > 0$, there exists $n' \in N$, for $n > n'$ and $x \in Z_+$, such that

$$\mid \lambda_x^n - \lambda \mid < \epsilon.$$

Let

$$\overline{\Lambda}_t^n = \pi^n(X_t^n)^{-1} \sum_{y\in Z} \pi^n(y) q^n(y, X_t^n) f(y, X_t^n),$$

$$f(x,y) = 1_{(x=y+1)}, \text{ for all } x, y \in Z_+.$$

Then

$$\begin{aligned}\overline{\Lambda}_t^n &= \pi^n(X_t^n)^{-1}\pi^n(X_t^n+1)q^n(X_t^n+1, X_t^n) \\ &= \sum_{x\in Z_+} \pi^n(x)^{-1}\pi^n(x+1)q^n(x+1,x)1_{(X_t^n=x)} \\ &= \sum_{x\in Z_+} \lambda_x^n 1_{(X_t^n=x)} \\ &= \lambda_{X_t^n}^n.\end{aligned}$$

Therefore

$$\begin{aligned}\mid \overline{\Lambda}_t^n - \lambda \mid &= \mid \lambda_{X_t^n}^n - \lambda \mid \\ &= \mid \sum_{x\in Z_+} (\lambda_x^n - \lambda) 1_{(X_t^n=x)} \mid \\ &\le \sum_{x\in Z_+} \mid \lambda_x^n - \lambda \mid 1_{(X_t^n=x)} \\ &< \epsilon \sum_{x\in Z_+} 1_{(X_t^n=x)} \\ &= \epsilon.\end{aligned}$$

By Lemma 2.2, we obtain $D^n \xrightarrow{\mathcal{L}} \mathcal{P}(\lambda)$. $\square$

Second, we consider a sequence of Markov queueing systems with compound Poisson arrivals. Suppose each X^n $(n \ge 1)$ has the state space Z_+ and transition rates

$$q^n(x, x+m) = \lambda_x^n p^{m-1}(1-p)1_{(m\ge 1)}, \quad x \in Z_+,$$

$$q^n(x, x-1) = \mu_x^n 1_{(x\ge 1)}, \quad x \in Z_+,$$

where λ_x^n, μ_x^n are positive, $0 < p < 1$ and $q^n(x,y) = 0$, for all other states $y \in Z_+ - \{x\}$. this process represent the number of customers in queueing system in which batches of customers arrive at the rate λ_x^n when x customers are present and the number of customers in a batch has a geometric distribution with parameter p. The customers depart at the rate μ_x^n when x are in the system.

$X^n(n \ge 1)$ has unique equilibrium distribution

$$\pi^n(x) = \pi^n(0)\lambda_0^n(\mu_1^n\mu_2^n \ldots \mu_x^{n-1})^{-1} \prod_{k=1}^{x-1}(\lambda_k^n + p\mu_k^n), \quad x \in N, \tag{2.3}$$

if $\sum_{x\in Z_+}\pi^n(x)<\infty$ (see Serfozo (1989)).

Theorem 3.2. *Suppose each* $X^n(n\geq 1)$ *is stationary and* $\mu_0^n=0$. *If* $\lambda_x^n+p\mu_x^n$ *uniformly converges to constant* $\lambda>0$ *on* $x\in Z_+$, *as* $n\to\infty$. *Then*

$$D^n\xrightarrow{\mathcal{L}}\mathcal{P}(\lambda),$$

as $n\to\infty$, *where* D^n *is the departure process of* X^n.

Proof. Let

$$\overline{\Lambda}_t^n=\pi^n(X_t^n)^{-1}\sum_{y\in Z_+}\pi^n(y)q^n(y,X_t^n)f(y,X_t^n),$$

where

$$f(x,y)=1_{(x>y)},\ \text{for all } x,y\in Z_+.$$

Then

$$\begin{aligned}\overline{\Lambda}_t^n&=\pi^n(0)^{-1}\pi^n(1)q^n(1,0)1_{(X_t^n=0)}+\pi^n(X_t^n)^{-1}\pi^n(X_t^n+1)q^n(X_t^n+1,X_t^n)1_{(X_t^n>0)}\\&=\sum_{x\in Z_+}[\pi^n(0)^{-1}\pi^n(1)q^n(1,0)1_{(x=0)}+\pi^n(x)^{-1}\pi^n(x+1)q^n(x+1,x)1_{(x>0)}]1_{(X_t^n=x)}\\&=\sum_{x\in Z_+}(\lambda_x^n+p\mu_x)1_{(X_t^n=x)}\\&=\lambda_{X_t^n}^n+p\mu_{X_t^n}.\end{aligned}$$

By the condition, for each $\epsilon>0$, there exists $n'\in N$, if $n>n'$, then for all $x\in Z_+$

$$|\ \lambda_x^n+p\mu_x^n-\lambda\ |<\epsilon.$$

Therefore

$$\begin{aligned}|\ \overline{\Lambda}_t^n-\lambda\ |&=|\ \lambda_{X_t^n}^n+p\mu_{X_t^n}^n-\lambda\ |\\&=|\ \sum_{x\in Z_+}(\lambda_x^n+p\mu_x^n)1_{(X_t^n=x)}-\lambda\ |\\&\leq\sum_{x\in Z_+}|\ \lambda_x^n+p\mu_x^n-\lambda\ |\ 1_{(X_t^n=x)}\\&<\epsilon.\end{aligned}$$

Thus, $\overline{\Lambda}_t^n$ uniformly converges to λ on $t\in R$. By Theorem 2.3, we obtain

$$D^n\xrightarrow{\mathcal{L}}\mathcal{P}(\lambda).$$

This completes the proof. □

3 Asymptotic Poisson Departure From $GI/M/1$ Queue

Let $A = \{A(t)\}_{t\geq 0}$ be a renewal process, $T_m (m \geq 1)$ be the epochs of successive jumps of A, $T_0 = 0$, let $G(x) = P[T_m - T_{m-1} \leq x]$, for all $x \geq 0$ and $m \geq 0$,

$$\lambda^{-1} = \int_0^\infty x dG(x), \quad \lambda > 0.$$

Let $\overline{D} = \{\overline{D}(t)\}_{t\geq 0} = \mathcal{P}(\mu),\ \mu > 0$. We suppose that A and $\overline{D}$ are two independent processes having no common jumps and they are the right-continuous with left hand-limits processes. If a stochastic process Q satisfies the stochastic integral equation:

$$Q(t) = Q(0) + A(t) - \overline{D}(t) + \int_0^t 1_{Q(\tau-)=0} d\overline{D}(\tau), \tag{4.1}$$

where $Q(0)$ is a random variable on Z_+. Then, Q is the state process of classical $GI/M/1$ queue with arrival process A and service process $\overline{D}$.

Let $\{t_n\}_{n\geq 1}$ be a sequence of real numbers with $\lim_{n\to\infty} t_n = \infty$, and let

$$D^n(t) = \overline{D}(t + t_n) - \overline{D}(t_n) - \int_{t_n}^{t+t_n} 1_{Q(\tau-)=0} d\overline{D}(\tau), \quad t \geq 0.$$

Then each $D^n = \{D^n(t)\}_{t\geq 0}$ $(n \geq 1)$ is a simple point process, where $D^n(t)$ is the number of customers who departure from the system during a time interval $(t_n,\ t+t_n]$. We denote the successive jump times of $\overline{D}$ on $(t_n,\ \infty)$ by $\tau_m (m \geq 1)$ and let $\tau'_m = \tau_m - t_n$, $m \geq 1$. Then $\{\tau'_m\}_{m\geq 1}$ is the sequence of successive jump times of $\{\overline{D}(t+t_n) - \overline{D}(t_n)\}_{t\geq 0}$. Thus,

$$\begin{aligned} D^n(t) &= \overline{D}(t + t_n) - \overline{D}(t_n) - \sum_{t_n<\tau_m\leq t+t_n} 1_{Q(\tau_m-)=0} \\ &= \overline{D}(t + t_n) - \overline{D}(t_n) - \sum_{0<\tau'_m\leq t} 1_{Q((\tau'_m+t_n)-)=0} \\ &= \overline{D}(t + t_n) - \overline{D}(t) - \int_0^t 1_{Q((s+t_n)-)=0} d\{\overline{D}(s + t_n) - \overline{D}(t_n)\}. \end{aligned}$$

Let $\overline{Q}^n(t) = Q(t + t_n)$, $t \geq 0$, and $\{\widetilde{T}_i\}_{i\geq 1}$ be a sequence of jump times of Q on $(t_n,\ \infty)$. Then

$$D^n(t) = \sum_{i=1}^\infty 1_{Q(\widetilde{T}_{i-1})>Q(\widetilde{T}_i)} 1_{t_n<\widetilde{T}_i\leq t+t_n},$$

where $\widetilde{T}_0$ is the last jump time of Q before $\widetilde{T}_1$. Let $\overline{T}^n_i = \widetilde{T}_i - t_n$, $i \geq 1$, $\overline{T}^n_0 = 0$. Then $\{\overline{T}^n_i\}_{i\geq 1}$ is a sequence of jump times of $\overline{Q}^n$, and

$$\begin{aligned} D^n(t) &= \sum_{i=1}^\infty 1_{Q(\overline{T}^n_{i-1}+t_n)>Q(\overline{T}^n_i+t_n)} 1_{\overline{T}^n_i\leq t} \\ &= \sum_{i=1}^\infty 1_{\overline{Q}^n(\overline{T}^n_{i-1})>\overline{Q}^n(\overline{T}^n_i)} 1_{\overline{T}^n_i\leq t}. \end{aligned}$$

Thus, D^n is a $\mathcal{F}_t^{\overline{Q}^n}$-adapted process, and $\mathcal{F}_t^{D^n} \subset \mathcal{F}_t^{\overline{Q}^n}$, $t \geq 0$.

Theorem 4.1. *Assume that* $\rho = \dfrac{\lambda}{\mu} \geq 1$. *Then* $D^n \xrightarrow{\mathcal{L}} \mathcal{P}(\mu)$.

Proof. Since $\overline{D}$ is $\mathcal{F}_t^{\overline{D}}$-Poisson process with $\mathcal{F}_t^{\overline{D}}$-intensity $\mu > 0$, let

$$\overline{D}^n(t) = \overline{D}(t + t_n) - \overline{D}(t_n), \quad t \geq 0.$$

Then $\overline{D}^n(t)$ is $\mathcal{F}_{t+t_n}^{\overline{D}}$-Poisson process with $\mathcal{F}_{t+t_n}^{\overline{D}}$-intensity μ. Using the fact $\mathcal{F}_t^{\overline{D}^n} \subset \mathcal{F}_{t+t_n}^{\overline{D}}$, $t \geq 0$, we obtain that $\overline{D}^n$ is $\mathcal{F}_t^{\overline{D}^n}$-Poisson process with $\mathcal{F}_t^{\overline{D}^n}$-intensity μ, and the compensator of D^n with respect to $\mathcal{F}_t^{\overline{Q}^n}$ is

$$\begin{aligned} \widetilde{D}^n(t) &= \mu t - \mu \int_0^t 1_{\overline{Q}^n(s-)=0} ds \\ &= \mu t - \mu \int_0^t 1_{\overline{Q}^n(s)=0} ds. \end{aligned} \tag{4.2}$$

From a classical result of $GI/M/1$ queue: if $\rho \geq 1$, then

$$\lim_{n\to\infty} P[\overline{Q}^n(t) = 0] = 0,$$

by using the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{n\to\infty} E \int_0^t 1_{\overline{Q}^n(s)=0} ds &= \lim_{n\to\infty} \int_0^t P[\overline{Q}^n(s) = 0] ds \\ &= 0. \end{aligned}$$

Thus,

$$\int_0^t 1_{\overline{Q}^n(s)=0} ds \xrightarrow{L_1} 0. \tag{4.3}$$

From (4.2) and (4.3), we have $\widetilde{D}^n(t) \xrightarrow{d} \mu t$, $t \geq 0$. By Lemma 2.1, we obtain $D^n \xrightarrow{\mathcal{L}} \mathcal{P}(\mu)$. This completes the proof. □

Remark. For $M/M/1$ queue, let $\rho \geq 1$. Then from theorem 4.1, we have $D^n \xrightarrow{\mathcal{L}} \mathcal{P}(\mu)$. This is a generalization of theorem 2.2 in Cohen [4, p. 201] in heavy traffic.

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