QUEUEING MODELS WITH BATCH SERVICES AND CATASTROPHES

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Abstract

In this paper, we first consider a single queue model with batch services and catastrophes and give its stationary distribution, show that the departure process of the queue is Poisson. Then we consider a queueing network model with batch services and catastrophes and show that the network has a product-form stationary distribution and its traffic processes which represent the customers exiting the network are independent Poisson processes.

Key words: Queueing model; Poisson process; traffic process; batch service; catastrophe.

1. Introduction

Modeling and analysis of migration processes with catastrophes is important in population genetics (see Brockwell et al. ([3]), Karlin and Tavare ([8]), Gripenberg ([6]), Brockwell ([2]) and their references). Chao ([4]) first considered a queueing network model with catastrophes which can be used to study the migration processes with catastrophes and computer networks with virus infections. In this paper, we consider a queueing network with batch services and catastrophes which is a generalization of the model described in Chao ([4]). Recently, many authors have considered queueing networks with batch services, see for example Henderson and Taylor ([7]), Boucherie and van Dijk ([1]), Chao et al. ([5]), Miyazawa and Wolff ([9]). But the queueing network with batch services and catastrophes has not been addressed before.

In this paper, we first consider a single queue model with batch services and catastrophes and give its stationary distribution, show that the departure process of the queue is Poisson, which are the subject of Section 2. In Section 3, we consider a queueing network model with batch services and catastrophes and show that the network has a product-form stationary distribution and its traffic processes which represent the customers exiting the network are independent Poisson processes.

2. A Single Queue Model

Consider a single queue model with batch services and catastrophes. Customers arrive at the queue according to a Poisson process with rate λ. Let K be a positive integer constant. On a service completion, if the number of customers in the system is not less than K, the K customers leave the queue as a full batch; otherwise all the customers in the queue are scraped from the system as a partial batch (see Chao et al. ([5]) or Miyazawa and Wolff ([9])). The server serves customers with an exponential rate μ regardless of the number of customers present in the queue. Assume that upon a service completion a full batch leaves the queue as a single customer. In addition, catastrophes arrive at the queue according to a Poisson process with rate δ. Whenever a catastrophe arrives the queue, all the customers in the queue are destroyed immediately and the server is ready to serve new customers.

It is easy to know that the queue can be described by a Markov process $\{X_t\}$ with state space $E = \{0, 1, 2, \cdots\}$. Its transition rates are given by

$$\begin{cases} q(n, n+1) = \lambda, \\ q(n, n-K) = \mu 1_{n \geq K} + \delta 1_{n=K}, \\ q(n, 0) = (\mu 1_{n<K} + \delta 1_{n \neq K}) 1_{n \geq 1}, \end{cases} \tag{2.1}$$

for all $n \in E$, and $q(n, n') = 0$, for all other states $n' \in E - \{n\}$, where $1_{(\cdot)}$ is the indicator function of $(\cdot)$.

Theorem 2.1. If the following equation:

$$\mu \rho^{K+1} - (\lambda + \mu + \delta)\rho + \lambda = 0 \tag{2.2}$$

have a solution ρ with $\rho < 1$, then the stationary distribution of the queue described above is given by

$$\pi(n) = (1-\rho)\rho^n, \ n = 1, 2, \cdots. \tag{2.3}$$

Furthermore, when the queue is stationary, the departure process of the queue is a Poisson process with rate $\rho^K \mu$, its past and the present state of the queue are independent at any time t.

Proof. From (2.1), we have

$$\begin{cases} q(n-1, n) = \lambda 1_{n \geq 1}, \\ q(n+K, n) = \mu + \delta 1_{n=0}, \\ q(n, 0) = (\mu 1_{n<K} + \delta 1_{n \neq K}) 1_{n \geq 1}, \end{cases} \tag{2.4}$$

for all $n \in E$, and $q(n', n) = 0$, for all other states $n' \in E - \{n\}$.

Note that π can be obtain through the following global balance equations:

$$\lambda \pi(0) = \sum_{n=1}^{K} \mu \pi(n) + \delta \sum_{n=1}^{\infty} \pi(n), \tag{2.6}$$

$$(\lambda + \mu + \delta)\pi(n) = \lambda \pi(n-1) + \mu \pi(n+K), \ n \geq 1. \tag{2.7}$$

If $\rho < 1$, then we easy verify that (2.3) is a unique solution of (2.6) and (2.7) with $\sum_{n=0}^{\infty} \pi(n) = 1$.

Since

$$\frac{1}{\pi(n)}\mu\pi(n+K,n) = \mu\rho^K,$$

for all $n \in E$, we obtain the rest of the conclusions by Theorem 2.3 in Serfozo ([10]). $\square$

Remark 2.1. We easily verify that there is a unique solution of equation (2.2) in $(0,1)$.

3. The Queueing Network Model

Consider a queueing network with J queues and single class of customers. Customers arrive at queue j from the outside of the network according to a Poisson process with rate λ_j, $j = 1, 2, \cdots, J$. Let K_j be a positive integer constant, for each $j = 1, 2, \cdots, J$. On a service completion at queue j ($j = 1, 2, \cdots, J$), if the number of customers in the queue is not less than K_j, the K_j customers leave the queue as a full batch; otherwise all the customers in the queue are scraped from the network. The server at queue j ($j = 1, 2, \cdots, J$) serves customers with an exponential rate μ_j regardless of the number of customers present in the queue. Assume that upon a service completion at queue i a full batch goes to queue j as a single customer with probability p_{ij} or as a catastrophe with probability q_{ij}, and it leaves the network as a single customer with probability p_{i0}, $i, j = 1, 2, \cdots, J$.

These routing probabilities satisfy

$$p_{i0} + \sum_{j=1}^{J}(p_{ij} + q_{ij}) = 1,\ i = 1, 2, \cdots, J. \tag{3.1}$$

In addition, catastrophes arrive at queue j from the outside of the network according to a Poisson process with rate δ_j, $j = 1, 2, \cdots, J$. Whenever a catastrophe arrives at a queue, either from the outside or from another queue, all the customers in the queue are destroyed immediately and the server is ready to serve new customers.

It is easy to know that the network can be described by a Markov process by a Markov process $\{X_t\}$, whose state is given by

$$n = (n_1, n_2, \cdots, n_J), \tag{3.2}$$

where n_j denote the number of the customers in the queue j, $j = 1, 2, \cdots, J$. We denote the state space of the Markov process by E. Its transition rates are given by

$$\begin{cases} q(n, n + e_i) = \lambda_i, \\ q(n, n - K_i e_i + e_j) = \mu_i p_{ij} 1_{n_i \geq K_i}, \\ q(n, n - K_i e_i) = \mu_i p_{i0} 1_{n_i \geq K_i} + \mu_i (\sum\limits_j q_{ij} 1_{n_j = 0}) 1_{n_i \geq K_i} + \delta_i 1_{n_i = K_i}, \\ q(n, n - K_i e_i - n_j e_j) = \mu_i q_{ij} 1_{n_i \geq K_i} 1_{n_i \geq 1}, \\ q(n, n - n_i e_i) = \delta_i 1_{n_i \neq K_i} 1_{n_i \geq 1} + \mu_i 1_{1 \leq n_i < K_i}, \end{cases} \tag{3.3}$$

where e_i is a J-dimensional vector with 1 in i-th position and all other elements being 0.

Let α_i, β_i and ρ_i are the solutions of the following traffic equations:

$$\alpha_i = \lambda_i + \sum_j \rho_j^{K_j} \mu_j p_{ji}, \tag{3.4}$$

$$\beta_i = \delta_i + \sum_j \rho_j^{K_j} \mu_j q_{ji}, \tag{3.5}$$

$$\mu_i \rho_i^{K_i + 1} - (\alpha_i + \mu_i + \beta_i) \rho_i + \alpha_i = 0, \tag{3.6}$$

for $i = 1, 2, \cdots, J$. Assume that the state process of the network is irreducible, and denote D_j by the traffic process which represents customers exiting the network from queue j, for $j = 1, 2, \cdots, J$. Then we have following theorem:

Theorem 3.1. If $\rho_j < 1$, for all $j = 1, 2, \cdots, J$, then the stationary distribution for the state process of the network described above is given by

$$\pi(n) = \prod_{j=1}^{J} \pi_j(n_j), \tag{3.7}$$

where $\pi_j(n_j)$ is the stationary distribution for queue j:

$$\pi_j(n_j) = \rho_j^{n_j}(1-\rho_j),\ \ j = 1, 2, \cdots, J. \tag{3.8}$$

Furthermore, when the system is stationary, traffic processes D_1, $D_2,\cdots,$ D_J are independent Poisson processes with rates $\mu_1\rho_1^{K_1}p_{10}$, $\mu_2\rho_2^{K_2}p_{20},\cdots,$ $\mu_J\rho_J^{K_J}p_{J0}$ respectively, and their past and the present state of the network are independent at any time t.

Proof. From (3.3), we have

$$\begin{cases} q(n-e_i,n) = \lambda_i 1_{n_i\geq 1}, \\ q(n+K_ie_i-e_j,n) = \mu_i p_{ij} 1_{n_j\geq 1}, \\ q(n+K_ie_i,n) = \mu_i p_{i0} + \mu_i \sum\limits_j q_{ij} 1_{n_j=0} + +\delta_i 1_{n_i=0}, \\ q(n+K_ie_i+me_j,n) = \mu_i q_{ij} 1_{n_j=0} 1_{m\geq 1}, \\ q(n+me_i,n) = \delta_i 1_{n_i=0} 1_{1\leq m\neq K_i} + \mu_i 1_{1\leq m<K_i} 1_{n_i=0}. \end{cases} \tag{3.9}$$

From (3.3) and (3.8), we obtain the global balance equation:

$$\begin{aligned} &\pi(n)\sum_{i=1}^J(\lambda_i+\mu_i 1_{n_i\geq 1}+\delta_i 1_{n_i\geq 1}) = \\ &\sum_{i=1}^J \lambda_i\pi(n-e_i)1_{n_i\geq 1} + \sum_{i=1}^J\sum_{j=1}^J \mu_i p_{ij} 1_{n_j\geq 1}\pi(n+K_ie_i-e_j) \\ &+\sum_{i=1}^J(\mu_i p_{i0} + \sum_{j=1}^J \mu_i q_{ij} 1_{n_j=0} + \delta_i 1_{n_i=0})\pi(n+K_ie_i) \\ &+\sum_{i=1}^J\sum_{j=1}^J\sum_{m=1}^\infty \mu_i q_{ij} 1_{n_j=0}\pi(n+K_ie_i+me_j) \\ &+\sum_{i=1}^J\sum_{m=1}^\infty(\delta_i 1_{n_i=0} 1_{1\leq m\neq K_i} + \mu_i 1_{1\leq m<K_i} 1_{n_i=0})\pi(n+me_i). \end{aligned} \tag{3.9}$$

Since

$$1_{n_i=0} = 1 - 1_{n_i\geq 1},\ \ \alpha_i = \sum_{m=1}^{K_i}\rho_i^m\mu_i + \frac{\rho_i\beta_i}{1-\rho_i},$$

$$\alpha_i - \rho_i\mu_i - \rho_i\beta_i - \frac{\rho_i^2\beta_i}{1-\rho_i} - \sum_{m=1}^{K_i-1}\rho_i^{m+1}\mu_i = 0,$$

for each $i = 1, 2, \cdots, J$. If $\rho_j < 1$, for all $j = 1, 2, \cdots, J$, then from (3.4), (3.5) and (3.6), we easily verify that (3.7) is the stationary distribution of the network.

For all $n = (n_1, n_2, \cdots, n_J) \in E$, we have

$$\begin{aligned} &\frac{1}{\pi(n)}\mu_i p_{i0}\pi(n + K_i e_i) \\ &= \mu_i \rho_i^{K_i} p_{i0}, \end{aligned}$$

where $i = 1, 2, \cdots, J$. Thus, the rest of the conclusions comes from Theorem 2.3 in Serfozo ([10]). $\square$

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