Asymptotic Poisson departure from G/G/1 queue

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Abstract

In this paper, we give conditions under which the finite-dimensional distributions of the departure process of $G/G/1$ queue converge weakly to those of Poisson process. In particular, we give some simple conditions under which the departure processes of the birth-death queueing system and $G/M/1$ queue are asymptotic Poisson processes.

Key Words: Point process; Compensator; Weak convergence; departure process

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1 Introduction

The study of the departure processes in a queueing system was initiated by Burke [3] and Reich [13]. They showed that in stationary the departure process of an $M/M/1$ queue is Poisson. Beutler and Melamed [1] extended this result by showing that the departure processes from the nodes in a Jackson network are independent Poisson processes. Subsequently, Walrand and Varaiya [16] showed that the traffic process on a link of Jackson network in stationary is Poisson if and only if that link is not part of a loop. Most of papers which studied the traffic processes in queueing systems discussed when the traffic process is Poisson. For example, Brémaud [2], Disney and Kiessler [6], Kelly [10], Walrand [15] and Whittle [18]. However, under the relaxed condition, that departure processes tend to be quite complicated; see Disney and König [7]. Thus, it is of interest to know when the departure process is asymptotic Poisson; see Whitt [17], Franken et al. [8] and references there.

In this paper, we consider the departure process of $G/G/1$ queue and show that its finite-dimensional distributions converge weakly to those of Poisson process under some conditions. As the special cases, we consider the birth-death queueing system and $G/M/1$ queue and prove that their departure processes are asymptotic Poisson under some simple conditions. In Section 2, we give definition of natural filtration and the compensator of a stochastic process. lemma 1 is a particular case of theorem 1 in Kabanov et al. [9]. In section 3, we show that the finite-dimensional distributions of the departure process of $G/G/1$ queue converge weakly to those of Poisson processes under some conditions. In particular, we consider the departure processes of the birth-death queueing system and $G/M/1$ queue and show that their finite-dimensional distributions converge weakly to those of Poisson processes under some simple conditions.

2 Preliminaries

Given a complete probability space $(\Omega, \mathcal{F}, P)$ and a family $\{\mathcal{F}_t\}_{t\in R_+}$ of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s < t$ is called a filtration. If the stochastic process $\{X(t)\}_{t\in R_+}$ is such that for all t, $X(t)$ is $\mathcal{F}_t$-measurable, then $\{\mathcal{F}_t\}$ is called a filtration of $\{X(t)\}$ and $\{X(t)\}$ is said to be $\mathcal{F}_t$-adapted. Let ${\mathcal{F}_t}^X = \sigma\{X(s), s \le t\}$, $t \in R_+$ (we denote by $\sigma\{\dots\}$ the complete σ-field generated by $\{\dots\}$). Then $\{{\mathcal{F}_t}^X\}$ is a filtration of stochastic process $\{X(t)\}$ and $\{{\mathcal{F}_t}^X\}$ is said to be the natural filtration of $\{X(t)\}$.

Let N be a simple point process with $EN(t) < \infty$, for all $t \in R_+$, let $\{\mathcal{F}_t\}$ be a filtration of N and it satisfies "the usual conditions" (see Liptser et al. [11], p. 1). A non-negative $\mathcal{F}_t$-predictable increasing process A on R_+ is called the compensator of N with respect to $\mathcal{F}_t$, if for every non-negative $\mathcal{F}_t$–predictable process C,

$$E[\int_o^\infty CdN] = E[\int_o^\infty CdA].$$

Lemma 1 Let $\{N^n\}_{n\geq 1}$ be a sequence of simple point processes on R_+ having compensators $\{A^n\}_{n\geq 1}$ for their respective filtrations $\{\mathcal{F}_t^n\}_{n\geq 1}$. If there is a non-negative deterministic continuous function A such that $A^n(t) \xrightarrow{d} A(t)$, for all $t \in R_+$, then

$$N^n \xrightarrow{\mathcal{L}} N,$$

where N is a Poisson process with mean measure A.

Remark Notation "$\xi^n \xrightarrow{d} \xi$" means the convergence of the random variables ξ^n to ξ in distribution; "$X^n \xrightarrow{\mathcal{L}} X$" means weak convergence of the finite-dimensional distributions of the stochastic processes X^n to those of X.

Remark If $A(t) = \lambda t$, where λ is a positive constant, then N is a time-homogeneous Poisson process with rate λ, we shall denote it by $\mathcal{P}(\lambda)$.

Proof See Kabanov et al. [9] $\square$

3 Asymptotic Poisson Departure from $G/G/1$ Queue

Let $A = \{A(t)\}_{t\geq 0}$ be a renewal process, $T_m (m \geq 1)$ be the epochs of successive jumps of A, $T_0 = 0$, let $G(x) = P[T_m - T_{m-1} \leq x]$, for all $x \geq 0$ and $m \geq 0$,

$$\lambda^{-1} = \int_0^\infty xdG(x), \quad \lambda > 0.$$

Let $\overline{D}^n = \{\overline{D}^n(t)\}_{t\geq 0}$, $n \geq 1$ is a sequence of simple point processes. We suppose that A and $\overline{D}^n$ are two independent processes having no common jumps and they are the right-continuous with left-hand limits processes. If stochastic process Q^n satisfies the stochastic integral equation:

$$Q^n(t) = Q^n(0) + A(t) - \overline{D}^n(t) + \int_0^t 1_{Q^n(\tau-)=0} d\overline{D}^n(\tau), \tag{1}$$

where $Q^n(0)$ is a random variable on Z_+. Then, Q^n is the state process of the $G/G/1$ queue which arrival process is A and service process is $\overline{D}^n$. Being different from the classical $G/G/1$ queue, the service process of this $G/G/1$ queue is not renewal process.

Let $\{t_n\}_{n\geq 1}$ be a sequence of real numbers with $\lim\limits_{n\to\infty} t_n = \infty$, and

$$D^n(t) = \overline{D}^n(t+t_n) - \overline{D}^n(t_n) - \int_{t_n}^{t+t_n} 1_{Q^n(\tau-)=0} d\overline{D}^n(\tau), \quad t \geq 0.$$

Then $D^n = \{D^n(t)\}_{t\geq 0}$ is a simple point process, where $D^n(t)$ is the number of customers who departure from the system during the time interval $(t_n,\ t+t_n]$. We denote the successive jump times of $\overline{D}^n$ on $(t_n,\ \infty)$ by $\tau_m (m \geq 1)$ and let $\tau'_m = \tau_m - t_n$, $m \geq 1$. Then $\{\tau'_m\}_{m\geq 1}$ is the sequence of successive jump times of $\{\overline{D}^n(t+t_n) - \overline{D}^n(t_n)\}_{t\geq 0}$,

$$\begin{aligned} D^n(t) &= \overline{D}^n(t+t_n) - \overline{D}^n(t_n) - \sum_{t_n<\tau_m\leq t+t_n} 1_{Q^n(\tau_m-)=0} \\ &= \overline{D}^n(t+t_n) - \overline{D}^n(t_n) - \sum_{0<\tau'_m\leq t} 1_{Q^n((\tau'_m+t_n)-)=0} \\ &= \overline{D}^n(t+t_n) - \overline{D}^n(t_n) - \int_0^t 1_{Q^n((s+t_n)-)=0} d\{\overline{D}^n(s+t_n) - \overline{D}^n(t_n)\}. \end{aligned}$$

Let $\overline{Q}^n(t) = Q^n(t+t_n)$, $t \geq 0$, and $\{\widetilde{T}_i\}_{i\geq 1}$ be a sequence of jump times of Q^n on $(t_n,\ \infty)$. Then

$$D^n(t) = \sum_{i=1}^{\infty} 1_{Q^n(\widetilde{T}_{i-1})>Q^n(\widetilde{T}_i)} 1_{t_n<\widetilde{T}_i\leq t+t_n},$$

where $\widetilde{T}_0$ is the last jump time of Q^n before $\widetilde{T}_1$. Let $\overline{T}_i^n = \widetilde{T}_i - t_n$, $i \geq 1$, $\overline{T}_0^n = 0$. Then $\{\overline{T}_i^n\}_{i\geq 1}$ is a sequence of jump times of $\overline{Q}^n$, and

$$\begin{aligned} D^n(t) &= \sum_{i=1}^{\infty} 1_{Q^n(\overline{T}_{i-1}^n+t_n)>Q^n(\overline{T}_i^n+t_n)} 1_{\overline{T}_i^n\leq t} \\ &= \sum_{i=1}^{\infty} 1_{\overline{Q}^n(\overline{T}_{i-1}^n)>\overline{Q}^n(\overline{T}_i^n)} 1_{\overline{T}_i^n\leq t}\ . \end{aligned}$$

Thus, D^n is the $\mathcal{F}_t^{\overline{Q}^n}$-adapted process, $\mathcal{F}_t^{D^n} \subset \mathcal{F}_t^{\overline{Q}^n}$, $t \geq 0$.

Let Λ^n be the stochastic intensity of simple point process $\{\overline{D}^n(t+t_n) - \overline{D}^n(t_n)\}_{t\geq 0}$ with respect to its nature filtration. Then the compensator of D^n with respect to $\mathcal{F}_t^{\overline{Q}^n}$ is

$$\widetilde{D}^n(t) = \int_0^t \Lambda^n(s)ds - \int_0^t 1_{\overline{Q}^n(s-)=0} \Lambda^n(s)ds. \tag{2}$$

Theorem 2 Assume that there exists a constant $M > 0$, such that $\Lambda^n(t) \leq M$ and $\Lambda^n(t) \xrightarrow{d} \mu$, for all $t \geq 0$, where $\mu > 0$ is a fixed constant. If $\lim_{n\to\infty} P[\overline{Q}^n(t) = 0] = 0$, then $D^n \xrightarrow{\mathcal{L}} \mathcal{P}(\mu)$.

Remark The first condition in thorem 2 means that the finite-dimensional distributions of the service process of this system converge weakly to those of Poisson process with rate μ and the second condition in it means the traffic of this system is heavy.

Proof By the condition, we have $\Lambda^n(t) \xrightarrow{L_1} \mu$, for all $t \geq 0$, and

$$E \mid \int_0^t \Lambda^n(s)ds - \mu t \mid \leq \int_0^t E \mid \Lambda^n(s) - \mu \mid ds.$$

Since $E \mid \Lambda^n(s) - \mu \mid \leq E\Lambda^n(s) + \mu \leq M + \mu$, by Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_0^t E \mid \Lambda^n(s) - \mu \mid ds = 0.$$

Thus

$$\int_0^t \Lambda^n(s)ds \xrightarrow{L_1} \mu t. \tag{3}$$

On the other hand,

$$\begin{aligned} E \int_0^t 1_{\overline{Q}^n(s-)=0} \Lambda^n(s)ds &\leq M\, E \int_0^t 1_{\overline{Q}^n(s-)=0} ds \\ &= M \int_0^t P[\overline{Q}^n(s) = 0]ds. \end{aligned}$$

By Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} E \int_0^t 1_{\overline{Q}^n(s-)=0} \Lambda^n(s)ds = 0.$$

Thus

$$\int_0^t 1_{\overline{Q}^n(s-)=0} \Lambda^n(s)ds \xrightarrow{L_1} 0, \text{ as } n \to \infty. \tag{4}$$

From (2), (3) and (4), we obtain $\widetilde{D}^n(t) \xrightarrow{d} \mu t$, for all $t \geq 0$, as $n \to \infty$. Therefore,

$$D^n \xrightarrow{\mathcal{L}} \mathcal{P}(\mu).$$

This completes the proof. □

Now, we consider some special cases of thorem 2. First, consider a sequence of birth-death queueing system. Let $\{Q^n\}_{n\geq 1}$ be a sequence of Markov processes, each Q^n $(n \geq 1)$ has the state space $Z_+ = \{0, 1, 2, \cdots\}$ and transition rates

$$\begin{aligned} q^n(x, x+1) &= \lambda, \quad x \in Z_+, \\ q^n(x, x-1) &= \mu_x^n 1_{(x\geq 1)}, \quad x \in Z_+, \end{aligned}$$

where λ and μ_x^n are positive, for all $x \in Z_+$, $n \geq 1$, and $q^n(x,y) = 0$, for all other states $y \in Z_+ - \{x\}$. Each Q^n $(n \geq 1)$ represents the state process of a queueing system which arrival process is $\mathcal{P}(\lambda)$ and service process is a simple point process with $\mathcal{F}_t^{Q^n}$-intensity $\mu_{Q^n}^n$.

Corollary 3 If μ_x^n uniformly converge to a constant $\mu > 0$ on $x \in Z_+$, as $n \to \infty$, and

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{\prod_{i=1}^{j} \mu_i^n} = \infty, \tag{5}$$

$$\sum_{j=1}^{\infty} \left(\frac{1}{\lambda} + \frac{\mu_1^n}{\lambda^2} + \cdots + \frac{\mu_j^n \cdots \mu_2^n}{\lambda^n}\right) = \infty, \tag{6}$$

for each $n \geq 1$. Then

$$D^n \xrightarrow{\mathcal{L}} \mathcal{P}(\mu).$$

Proof From the first condition, for each $\epsilon > 0$, there exists a number $n' \in N$, for all $n > n'$,

$$\mid \mu_x^n - \mu \mid < \epsilon, \quad \text{on } x \in Z_+,$$

and

$$\begin{aligned} \mid \mu_{Q^n(t)}^n - \mu \mid &= \mid \sum_{x \in Z_+} [\mu_x^n - \mu] 1_{Q^n(t)=x} \mid \\ &< \epsilon \sum_{x \in Z_+} 1_{Q^n(t)=x} \\ &= \epsilon. \end{aligned}$$

Thus, $\mu_{Q^n(t)}^n$ uniformly converges to $\mu > 0$ on $t \in R_+$. Therefore, there exists a constant $M > 0$ such that

$$\mu_{Q^n(t)}^n = \sum_{x \in Z_+} \mu_x^n 1_{Q^n(t)=x} \leq M,$$

and $\mu_{Q^n(t)}^n \xrightarrow{d} \mu$, as $n \to \infty$. Note that $\lim_{n \to \infty} P[\overline{Q}^n(t) = 0] = 0$ comes from (5) and (6), by thorem 2, we obtain $D^n \xrightarrow{\mathcal{L}} \mathcal{P}(\mu)$. This completes the proof. □

In (1), let $\overline{D} \hat{=} \overline{D}^n = \mathcal{P}(\mu)$, $\mu > 0$, for all $n \geq 1$. Then the stochastic process Q which satisfies the stochastic integral equation (1) is the state process of classical $G/M/1$ queue with arrival process A and service process $\overline{D}$.

Corollary 4 For the G/M/1 queue, if $\rho = \dfrac{\lambda}{\mu} \geq 1$, then $D^n \xrightarrow{\mathcal{L}} \mathcal{P}(\mu)$.

Proof Since $\overline{D}$ is $\mathcal{F}_t^{\overline{D}}$-Poisson process with $\mathcal{F}_t^{\overline{D}}$-intensity $\mu > 0$, let

$$\widehat{D}^n = \overline{D}(t+t_n) - \overline{D}(t_n), \quad t \geq 0.$$

Then $\widehat{D}^n$ is $\mathcal{F}_{t+t_n}^{\overline{D}}$-Poisson process with $\mathcal{F}_{t+t_n}^{\overline{D}}$-intensity μ. Using the fact $\mathcal{F}_t^{\widehat{D}^n} \subset \mathcal{F}_{t+t_n}^{\overline{D}}$, $t \geq 0$, we obtain that $\widehat{D}^n$ is $\mathcal{F}_t^{\widehat{D}^n}$-Poisson process with $\mathcal{F}_t^{\widehat{D}^n}$-intensity μ, and the compensator of D^n with respect to $\mathcal{F}_t^{\overline{Q}^n}$ is

$$\begin{aligned}\widetilde{D}^n(t) &= \mu t - \mu \int_0^t 1_{\overline{Q}^n(s-)=0} ds \\ &= \mu t - \mu \int_0^t 1_{\overline{Q}^n(s)=0} ds.\end{aligned}$$

Using a classical result of $G/M/1$ queue: if $\rho \geq 1$, then

$$\lim_{n\to\infty} P[\overline{Q}^n(t) = 0] = 0.$$

By thorem 2, we obtain $D^n \xrightarrow{\mathcal{L}} \mathcal{P}(\mu)$. $\square$

Remark For $M/M/1$ queue, if $\rho \geq 1$, then from corollary 4, we have $D^n \xrightarrow{\mathcal{L}} \mathcal{P}(\mu)$. This is a generalization of theorem 2.2 in Cohen ([4], p. 201) in heavy traffic.

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