Poisson traffic processes in RGSMP and generalized networks *

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Abstract

In this paper, we present conditions under which the traffic processes in a generalized semi-Markov process with reallocation (RGSMP) are Poison processes, and give an easy-to-use criterion for the quasi-reversibility of queues which can be described by RGSMP's. By applying these results to a multiple class generalized network with general service time distributions, we prove that its stationary state distribution is of a product-form and the traffic processes which represent the customers exiting the network are Poisson processes.

Key words: Dual predictable projection; Negative arrival; RGSMP; Generalized network; Poisson process; Traffic process.

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1. Introduction

Traffic processes in queueing networks are an important operating facet of such models, as well as valuable in the study of valid decompositions of networks. If we find some traffic processes in a network Poisson, then it often renders the mathematical analysis tractable. Generalized semi-Markov processes (GSMP's) have been introduced to study insensitivity of queueing models in a unified way, see Schassberger (1977a,b, 1978, 1986), Burman (1981), Franken et al (1982), Baccelli and Brémaud (1987), Henderson (1989), Henderson and Taylor (1991). A generalized semi-Markov process with reallocation (RGSMP), which was first introduced by Miyazawa (1993), is a generalization of GSMP and is useful for studying a wide range of stochastic models which can not be described by ordinary GSMP's, see Miyazawa (1991, 1993, 1994), Miyazawa et al. (1995) and Miyazawa and Wolff (1996). Poisson traffic processes in GSMP's have been discussed in the literature by using the embedded Markov chain method, see Franken et al (1982) and Baccelli and Brémaud (1987). We will now discuss traffic processes in RGSMP's and their networks with different method.

In this paper, we first recall the notion of an RGSMP in Section 2. Then we present conditions under which the traffic processes in an RGSMP are Poisson processes by using the dual predictable projections of traffic processes and their time-reversal processes, and give an easy-to-use criterion for the quasi-reversibility of queues which can be described by RGSMP's, that are the subject of Section 3. In Section 4, we consider a symmetric queue with general service time distributions and negative arrivals, and show that the queue is quasi-reversible. Here the concept of negative arrival, sometimes called signal, was first introduced by Gelenbe (1991). Recently many authors considered queueing networks with negative arrivals, see Gelenbe (1991, 1992, 1993a,b), Chao and Pinedo (1993), Chao (1995), and Miyazawa and Wolff (1996). Gelenbe (1993b) called them Generalized networks, or G-networks for short. In Section 5, we consider a multiple class generalized network with general service time distributions and negative arrivals, and show that the network has a product-form stationary distribution and the traffic processes which represent the customers exiting the network are Poisson processes.

2. Generalized semi-Markov processes with reallocation

In this section, we recall the notion and results of a generalized semi-Markov process with reallocation (RGSMP) which are cited from Miyazawa et al. (1995). Let G, S and D be countable sets. An element $g \in G$ is called a macrostate. With each macrostate $g \in G$ is associated a finite subset of S, $A(g)$, called the set of active sites for g. Each site $s \in A(g)$ has a clock which counts the attained lifetime of the site at a given epoch. The type of the clock at each site $s \in A(g)$ is given by a mapping γ_g: $A(g) \to D$, where D is called the set of clock types. Denote $s(g) \hat{=} \gamma_g(s)$, for simplicity. Let H_d be the lifetime distribution for each $d \in D$. Assume that H_d is differentiable with bounded and continuous density h_d and $\mu_d^{-1} \hat{=} \int_0^\infty (1 - H_d(u))du < \infty$, for each $d \in D$. We define $d(g) \hat{=} \mid A(g) \mid$, i.e., the number of elements in $A(g)$. Under the macrostate $g \in G$, a clock at site $s \in A(g)$ is increasing with speed r_{sg}. These speeds r_{sg} verify the conditions

$$\begin{cases} r_{sg} \geq 0, \quad \text{if } s \in A(g); \\ \sum\limits_{s \in A(g)} r_{sg} > 0. \end{cases} \tag{2.1}$$

It means that, for each $g \in G$, all clocks have non-negative speeds and at least one clock is increased with positive speed. When the clock at a site $s \in A(g)$ completes its lifetime, which is called expiry of the site, a transition of the macrostate g to a macrostate g' occurs and new sites may be activated, while other clocks at sites in $A(g) - \{s\}$ keep their attained lifetimes but may be reallocated to sites in $A(g') - U'$, where $U' \subset A(g')$ denotes the set of the new active sites activated at such a transition epoch. This reallocation is described by a one-to-one mapping

$$\Gamma_{sgg'}: \ A(g) - \{s\} \to A(g') - U',$$

such that

$$\gamma_{g'}(s') = \gamma_g(\Gamma_{sgg'}^{-1}(s')), \text{ for all } s' \in A(g') - U'.$$

Let $\{X(t)\}$ be a process taking values in the macrostate space G. Given that $X(t) = g$ and a site $s \in A(g)$ expires, the process goes to state $g' \in G$ and activates the set of new active sites $U' \subset A(g')$ with probability $p(g, s, g', U')$. Assume that the transition probability $p(g, s, g', U')$ does not depend on the earlier history before the considered jump epoch. For each time t, let $R_s(t) \geq 0$ be the attained lifetime of the clock at time t on site $s \in A(g)$. Define a vector-valued variable $Y(t)$ by $(X(t), \ R_s(t); \ s \in A(g))$ for each time t. It is easy to know that the joint process $\{Y(t)\}$ is a Markov process. We call $\{Y(t)\}$ a generalized semi-Markov process with reallocation, or an RGSMP for short. Let E be the state space of $\{Y(t)\}$, and we assume that the sample paths of $\{Y(t)\}$ are right-continuous with left-limits. Given that the site $s \in A(g)$ expires, let Q_s be the transition measure of the RGSMP $\{Y(t)\}$. Then it is given by

$$Q_s(y, dy') = \sum_{U' \subset A(g')} p(g, s, g', U') \prod_{s' \in A(g) - \{s\}} 1_{y'_{\Gamma_{sgg'}(s')} = y_{s'}} \times \prod_{s'' \in U'} 1_{y'_{s''} = 0} dy'_1 \cdots dy'_{d(g')},$$

for $y = (g, y_s; s \in A(g)) \in E$, $y' = (g', y'_s; s \in A(g')) \in E$ and $s \in A(g)$. If H_d is exponential for all $d \in D$, then $\{X(t)\}$ is a Markov process, which is often referred to the pure Markov case.

In the pure Markov case, if $\{X(t)\}$ is irreducible positive recurrent, then a set $\{\eta(g); \ g \in G\}$ of positive numbers $\eta(g)$ with

$$\sum_{g \in G} \eta(g) = 1 \tag{2.2}$$

gives a stationary distribution for the Markov process $\{X(t)\}$ if and only if $\{\eta(g);\ g \in G\}$ is a solution of the following system of global balance equations:

$$\eta(g) \sum_{s\in A(g)} r_{sg}\mu_{s(g)} = \sum_{g'\in G} \eta(g') \sum_{s'\in A(g')} \sum_{U\subset A(g)} r_{s'g'}\mu_{s'(g')}p(g', s', g, U), \tag{2.3}$$

for every $g \in G$. Moreover, the embedded Markov chain of $\{X(t)\}$, considered at jump epochs, is positive recurrent if and only if

$$\sum_{g\in G} \sum_{s\in A(g)} r_{sg}\mu_{s(g)}\eta(g) < \infty. \tag{2.4}$$

Throughout this paper we assume that (2.3) has a unique solution $\{\eta(g);\ g \in G\}$ with $\eta(g) > 0$ for every $g \in G$ and $\sum_{g\in G} \eta(g) = 1$.

We now consider the general case of the RGSMP. Let D' be the subset of D such that H_d is not exponential for all $d \in D'$. Let $A'(g) = \{s : s \in A(g), s(g) \in D'\}$. Consider the following system of so-called local balance equations:

$$\eta(g)r_{sg}\mu_{s(g)} = \sum_{g'\in G} \sum_{s'\in A(g')} \sum_{\{U:s\in U\}} r_{s'g'}\mu_{s'(g')}\eta(g')p(g', s', g, U), \tag{2.5}$$

for every $g \in G$ and every $s \in A'(g)$.

If $\{\eta(g)\}$ satisfies the conditions (2.3), (2.4) and (2.5), then the stationary distribution of the RGSMP $\{Y(t)\}$, π, is given by

$$P(X(t) = g, R_s(t) \le y_s; s \in A(g)) = \eta(g) \prod_{s\in A(g)} \mu_{s(g)} \int_0^{y_s} \overline{H}_{s(g)}(u)du, \tag{2.6}$$

for all $g \in G$ and $y_s \ge 0$, where $\overline{H}_d = 1 - H_d$, $d \in D$

3. Poisson traffic processes in RGSMP

Given a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, a family $\{\mathcal{F}_t\}_{t\ge0}$ of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_u \subset \mathcal{F}_t$ whenever $u < t$ is called a filtration. If a stochastic process $\{Y(t)\}$ is such that for all t, $Y(t)$ is $\mathcal{F}_t$-measurable, then $\{\mathcal{F}_t\}$ is called a filtration of $\{Y(t)\}$ and $\{Y(t)\}$ is said to be $\mathcal{F}_t$-adapted. Let ${\mathcal{F}_t}^Y = \sigma\{Y(u), u \le t\}$ (we denote by $\sigma\{\dots\}$ the complete σ-field generated by $\{\dots\}$). Then $\{{\mathcal{F}_t}^Y\}$ is a filtration of the stochastic process $\{Y(t)\}$ and $\{{\mathcal{F}_t}^Y\}$ is said to be the natural filtration of $\{Y(t)\}$.

Let $\{T_n\}_{n\ge1}$ be a sequence of jump epochs of an RGSMP $\{Y(t)\}$. Then T_n are clearly stopping times of $\{\mathcal{F}_t^Y\}$. Define an integer-valued random measure on $\mathbf{R}_+ \times E$ by

$$\mu(dt, dy) = \sum_{n=1}^{\infty} \delta_{(T_n,Y(T_n))}(dt, dy), \tag{3.1}$$

where δ_x is the Dirac measure at $x \in \mathbf{R}_+ \times E$. Note that the integer-valued random measure μ has the dual $\mathcal{F}_t$-predictable projection υ given by

$$\upsilon(dt, dy) = \sum_{s \in A(X(t-))} r_{sX(t-)} Q_s(Y(t-), dy) \lambda_{s(X(t-))}(R_s(t-))dt, \tag{3.2}$$

where

$$\lambda_d(x) = \frac{h_d(x)}{\overline{H}_d(x)}, \text{ for all } x \in \mathbf{R}_+ \text{ and } d \in D$$

(see Davis (1984) and Chapter 3 of Last and Brandt (1995) for the details). Let $N(t) \hat{=} \sum_{n=1}^{\infty} 1_{T_n \le t}$ be the number of jumps in $(0, t]$ and assume that $EN(t) < \infty$, for all t. Let H be a subset of $E \times E - diag(E \times E)$ and N_H be the point process counting the H-transitions of $\{Y(t)\}$, i.e.,

$$\begin{aligned} N_H(t) &= \int_0^t 1_H(Y(u-), Y(u))N(du) \\ &= \int_0^t \int_E 1_H(Y(u-), y)\mu(du, dy). \end{aligned} \tag{3.3}$$

N_H represents the number of jumps of Y in the time period $(0, t]$ from some x to some y such that $(x, y) \in H$. Let N_H^p be the dual ${\mathcal{F}_t}^Y$-predictable projection of $N_H(t)$. Then from (3.2) and (3.3), we have

$$\begin{aligned} N_H^p(t) &= \int_0^t \int_E 1_H(Y(u-), y)\upsilon(du, dy) \\ &= \int_0^t \int_E 1_H(Y(u-), y) \sum_{s \in A(X(u-))} r_{sX(u-)} Q_s(Y(u-), dy) \lambda_{s(X(u-))}(R_s(u-))du. \end{aligned}$$

Let

$$\Lambda_H(t) = \int_E 1_H(Y(t-), y) \sum_{s \in A(X(t-))} r_{sX(t-)} \lambda_{s(X(t-))}(R_s(t-)) Q_s(Y(t-), dy).$$

Then

$$N_H^p(t) = \int_0^t \Lambda_H(u)du. \tag{3.4}$$

When Y is stationary with the stationary distribution π, we define

$$\zeta(x) = \frac{1}{\pi(dx)} \int_E 1_H(y, x) \sum_{s \in A(g)} r_{sg} Q_s(y, dx) \lambda_{s(g)}(y_s)\pi(dy),$$

for all $x \in E$, and $\overline{\Lambda}_H(t) \hat{=} \zeta(Y(t-))$, for all t. Then, we have the following theorem.

Theorem 3.1. i) Under the initial probability measure π, the present state $Y(t)$ and the future of N_H: $\{N_H(t+u) - N_H(t),\ u \ge 0\}$ are independent for all t if and only if $\Lambda_H(t)$ is a fixed positive constant λ for all t. In that case, N_H is a Poisson process with rate λ.

ii) Under the initial probability measure π, the present state $Y(t)$ and the past of N_H: $\{N_H(u),\ u \le t\}$ are independent for all t if and only if $\overline{\Lambda}_H(t)$ is a fixed positive constant λ for all t. In that case, N_H is a Poisson process with rate λ.

Proof. i) If Λ_H is a fixed positive constant λ, then by (3.4) and T4 theorem (see Brémaud (1981), p. 25), N_H is a Poisson process with intensity λ. By Markov property of Y, we have

$$\begin{aligned} E[N_H(t+u) - N_H(t) \mid \sigma(Y(t))] &= E[N_H(t+u) - N_H(t) \mid \mathcal{F}_t^Y] \\ &= E[\int_t^{t+u} \Lambda_H(s)ds \mid \mathcal{F}_t^Y] = u\lambda. \end{aligned}$$

Thus, the present state $Y(t)$ and the future of N_H: $\{N_H(t+u) - N_H(t),\ u \ge 0\}$ are independent for all t.

On the other hand, from the definition of N_H, we know that N_H is a simple point process. By (3.4) and the definition of N_H^p, we obtain that N_H is continuous in probability, i.e., $\lim_{t\to u} N_H(t) = N_H(u)$, where the limit is taken in probability. By Markov property of Y, $N_H(t+u) - N_H(t)$ is independent of $\mathcal{F}_t^Y$ for all t and $u \ge 0$. Since ${\mathcal{F}_t}^{N_H} \subset {\mathcal{F}_t}^Y$ for all t, we have $N_H(t+u) - N_H(t)$ is independent of ${\mathcal{F}_t}^{N_H}$ for all t and $u \ge 0$. Thus N_H is a Poisson process with nonnegative deterministic cumulative intensity (compensator). Note that X is stationary, and therefore Λ_H is a fixed positive constant.

ii) Since Y is stationary, we may consider Y on the real line $\mathbf{R}$. Let $\overline{Y}$ is the right-continuous time-reversal of Y. Then $\overline{Y}$ is a stationary RGSMP. Define a point process $\overline{N}_H$ on $\mathbf{R}$ by

$$\overline{N}_H(C) = \int_C 1_H(\overline{Y}(u), \overline{Y}(u-))N(du),\ C \in \mathcal{B}(\mathbf{R}),$$

where $\mathcal{B}(\mathbf{R})$ is the Borel σ-field on $\mathbf{R}$. Then $\overline{N}_H$ is the time-reversal of N_H and its dual ${\mathcal{F}_t}^{\overline{Y}}$-predictable projection $\overline{N}_H^p$ is given by

$$\overline{N}_H^p(C) = \int_C \overline{\Lambda}_H(u)du,\ C \in \mathcal{B}(\mathbf{R}).$$

Thus, we obtain the conclusion by applying the results in i) to $\overline{Y}$. This completes the proof. □

Now, we consider the quasi-reversibility of a queue. The evolution of a queue is described by an RGSMP $\{Y(t)\}$. Its arrival process A^c and departure process D^c correspond to certain transitions of $\{Y(t)\}$ in the state space E, for each $c \in \mathbf{C}$, where $\mathbf{C}$ is the set of classes of customers. That is,

$$A^c(t) = \sum_{0<s\le t} 1_{\mathbf{A^c}}(Y(s-), Y(s)), \quad D^c(t) = \sum_{0<s\le t} 1_{\mathbf{D^c}}(Y(s-), Y(s)),$$

where $\mathbf{A^c}$ and $\mathbf{D^c}$ are disjoint subsets of $\{(y', y) \in E \times E \mid y' \neq y\}$, for each $c \in \mathbf{C}$.

Corollary 3.2. Suppose that the queue is stationary with the stationary distribution π. Then the queue is quasi-reversible if and only if for all $c \in \mathbf{C}$,

$$\int_E 1_{\mathbf{A^c}}(y', y) \sum_{s \in A(g')} r_{sg'} \lambda_{s(g')}(y'_s) Q_s(y', dy) = \lambda_c,$$

for all $y' = (g', y'_s; s \in A(g')) \in E$, and

$$\frac{1}{\pi(dy)} \int_E 1_{\mathbf{D^c}}(y', y) \sum_{s \in A(g')} r_{sg'} Q_s(y', dy) \lambda_{s(g')}(y'_s) \pi(dy') = \mu_c,$$

for all $y = (g, y_s; s \in A(g)) \in E$, where $\lambda_c > 0$ is independent of $y' \in E$, and $\mu_c > 0$ is independent of $y \in E$. In this case, the arrival processes are independent Poisson processes with rates λ_c, $c \in \mathbf{C}$ and so are the departure processes with rates μ_c, $c \in \mathbf{C}$.

Remark 3.3. Corollary 3.2 is a generalization of (3.1) and (3.2) in Walrand (1988) (see pp. 90-91). It immediately follows from Theorem 3.1.

4. A symmetric queue with negative arrivals

In this section, we consider a queue within which customers are located in positions 1, 2, $\cdots$, n, where n is the total number of customers in the queue. Assume that:

(i) Regular customers arrive at the queue as independent Poisson processes with rates λ_c^+, for class $c \in \mathbf{C}$ (a given countable set). When a regular customer arrives, he moves into position l, $l = 1, 2, \cdots, n+1$, with probability $\delta(l, n+1)$; customers previously in positions l, $l+1$, $\cdots$, n move to positions $l+1$, $l+2$, $\cdots$, $n+1$ respectively.

(ii) The required service time of a class c customer has a general distribution $H_c(\cdot)$.

(iii) The total service effort of the queue is independent of the state of the queue and is supplied at rate 1.

(iv) A proportion $\delta(l, n)$ of the total service effort is distributed to the customer in position l, $l = 1, 2, \cdots, n$; with his departure, customers in positions $l+1$, $l+2$, $\cdots$, n move into positions l, $l+1$, $\cdots$, $n-1$ respectively.

(v) In addition, as soon as a class c regular customer arrives at the queue and joins some position, a negative arrival process in this position starts. The required time until the negative arrival occurs has a general distribution $G_c(\cdot)$.

(vi) For negative arrival processes, the actual total rate of progress in all positions is equal to 1 and independent of the state of the queue. A proportion $\delta(l, n)$ of the actual total rate of progress is distributed to the negative arrival process in position l, $1 \leq l \leq n$. A negative arrival in position l results in a service completion of the regular customer and the end of the corresponding negative arrival process in this position. With the departure of the regular customer, customers in positions $l+1$, $l+2$, $\cdots$, n move into positions l, $l+1$, $\cdots$, $n-1$ respectively.

(vii) All the regular customer arrival processes, the required times for negative arrivals and the required service times are mutually independent.

These $\delta(l,n)$'s satisfy

$$\sum_{l=1}^{n} \delta(l,n) = 1.$$

It is easy to know that the queue has the same stochastic behaviour as a modified system without negative arrivals, where the required service time of a class c customer is replaced by the minimum of the original required service time and a random variable with general distribution $G_c(\cdot)$. Thus it can be described by an RGSMP $\{Y(t)\}$, whose state is

$$y = (g, y_s; s \in A(g)), \tag{4.1}$$

or by a Markov process $\{\overline{Y}(t)\}$, whose state is

$$y = (g, y_s; s \in A(g) - \{0\}), \tag{4.2}$$

where $g = (c_1, c_2, \cdots, c_n)$ and $A(g) = \{0, 1, \cdots, n\}$, when there are n customers in the queue, the customer in position i is of class c_i and the amount of service for the modified system already performed in position i is y_i, for $1 \le i \le n$. The site 0 corresponds to a regular customer arrival and y_0 represents the time since the last regular customer arrival. We denote the state spaces of the two processes by E and $\overline{E}$ respectively. Let $F_c(x) = 1 - \overline{G}_c(x)\overline{H}_c(x)$ for all $x \in \mathbf{R}_+$ and $\mu_c^* \hat{=} (\int_0^\infty \overline{F}_c(x)dx)^{-1}$, $c \in \mathbf{C}$.

Theorem 4.1. If $\sum_{c\in\mathbf{C}} \lambda_c^+ (\mu_c^*)^{-1} < 1$, then the stationary distribution for the queue described by the Markov process $\{\overline{Y}(t)\}$ is given by

$$\overline{\pi}(y) = \eta(g) \prod_{i=1}^{n} \mu_{c_i}^* \int_0^{y_i} \overline{F}_{c_i}(u)du. \tag{4.3}$$

where $\eta(g)$ is given by

$$\eta(g) = (1 - \sum_{c\in\mathbf{C}} \frac{\lambda_c^+}{\mu_c^*}) \prod_{i=1}^{n} \frac{\lambda_{c_i}^+}{\mu_{c_i}^*}. \tag{4.4}$$

The probability that there are n customers in the system is

$$(1 - \sum_{c\in\mathbf{C}} \frac{\lambda_c^+}{\mu_c^*})(\sum_{c\in\mathbf{C}} \frac{\lambda_c^+}{\mu_c^*})^n. \tag{4.5}$$

Furthermore, the queue is quasi-reversible when the system is in equilibrium. In this case, the departure processes of regular customers are independent Poisson processes with respective rates λ_c^+, for $c \in \mathbf{C}$.

Proof. From (2.3), (2.4), (2.5) and (2.6), we easily verify that the stationary distribution for the RGSMP $\{Y(t)\}$ is

$$\pi(y) = \eta(g)(1 - e^{-\lambda_{c_0}^+ y_0}) \prod_{i=1}^{n} \mu_{c_i}^* \int_0^{y_i} \overline{F}_{c_i}(u)du. \tag{4.2}$$

where $\lambda_{c_0}^+ = \sum_{c\in\mathbf{C}} \lambda_c^+$ and $\eta(g)$ is given by (4.4). Letting $y_0 \to \infty$, we obtain(4.3). (4.5) comes from (4.4).

Now, we verify the quasi-reversibility of queue when the system is in equilibrium. For all $y' = (g', y'_s; s \in A(g')) \in E$, where $g' = (c'_1, c'_2, \cdots, c'_n)$, $A(g') = \{0, 1, \cdots, n\}$,

$$\int_E 1_{\mathbf{A^c}}(y', y) \sum_{s\in A(g')} r_{sg'} \lambda_{s(g')}(y'_s) Q_s(y', dy) = \sum_{l=1}^{n+1} \delta(l, n+1)\lambda_c^+ = \lambda_c^+,$$

and for all $y \in E$ given by (4.1),

$$\frac{1}{\pi(dy)} \int_E 1_{\mathbf{D^c}}(y', y) \sum_{s\in A(g')} r_{sg'} Q_s(y', dy) \lambda_{s(g')}(y'_s) \pi(dy')$$

$$= \sum_{l=1}^{n+1} \int_0^\infty \lambda_c^+ \delta(l, n+1) F_c(dy'_l) = \lambda_c^+,$$

Thus, the conclusion comes from Corollary 3.2. □

5. A multiple class generalized network

In this section, we consider a generalized network with J queues defined above. Let $\mathbf{C}$ be the countable set of regular customer classes. We assume that regular customers of different classes arrive at queue j from the outside of the network according to independent Poisson processes with rates Λ_{jc}^+, $c \in \mathbf{C}$ respectively. When a regular class c customer completes his service at queue j, he goes to queue k as a regular class c' customer with probability $p_{jc;kc'}$ and leaves the network as a regular class c customer with probability $p_{jc;0}$. These transition probabilities satisfy

$$\sum_{k,c'} p_{jc;kc'} + p_{jc;0} = 1, \ 1 \le j \le J, \ c \in \mathbf{C}.$$

For each $j = 1, 2, \cdots, J$, let n_j be the total number of customers in the queue j. Assume that queue j operates in the following manner:

i) When a regular customer arrives, he moves into position l, $l = 1, 2, \cdots, n_j + 1$, with probability $\delta_j(l, n_j + 1)$; customers previously in positions l, $l + 1$, $\cdots$, n_j move to positions $l + 1$, $l + 2$, $\cdots$, $n_j + 1$ respectively.

(ii) The required service time of a class c customer at queue j has a general distribution $H_{jc}(\cdot)$.

(iii) The total service effort at queue j is independent of the state of the queue and is supplied at rate 1.

(iv) A proportion $\delta_j(l, n_j)$ of the total service effort is distributed to the customer in position l, $l = 1, 2, \cdots, n_j$; with his departure, customers in positions $l + 1$, $l + 2$, $\cdots$,

n_j move into positions l, $l+1$, $\cdots$, n_j-1 respectively.

(v) In addition, as soon as a class c regular customer arrives at queue j and joins some position, a negative arrival process in this position starts. The required time until the negative arrival occurs has a general distribution $G_{jc}(\cdot)$.

(vi) For negative arrival processes, the actual total rate of progress in all positions is equal to 1 and independent of the state of the queue. A proportion $\delta_j(l, n_j)$ of the actual total rate of progress is distributed to the negative arrival process in position l, $1 \le l \le n_j$. A negative arrival in position l results in a service completion of the regular customer and the end of corresponding negative arrival process in this position. This regular customer will either leave the network, or go to another queue with probabilities defined above. With the departure of the regular customer, customers in positions $l+1$, $l+2$, $\cdots$, n move into positions l, $l+1$, $\cdots$, $n-1$ respectively.

(vii) All the regular customer arrival processes, the required times for negative arrivals and the required service times are mutually independent.

These $\delta_j(l, n_j)$'s satisfy

$$\sum_{l=1}^{n_j} \delta_j(l, n_j) = 1, \; j = 1, 2, \cdots, J$$

It is easy to know that the network can be described by a Markov process $\{Y(t)\}$, whose state is given by

$$y = (y_1, y_2, \cdots, y_J), \tag{5.1}$$

where y_j denotes the state of queue j defined as (4.1):

$$y_j = (g_j, y_{js}; s \in A(g_j)), \tag{5.2}$$

or by a Markov process $\{\overline{Y}(t)\}$, whose state is given by

$$\overline{y} = (\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_J), \tag{5.3}$$

where $\overline{y}_j$ denotes the state of queue j defined as (4.2):

$$\overline{y}_j = (g_j, y_{js}; s \in A(g_j) - \{0\}), \tag{5.4}$$

where $g_j = (c_{j1}, c_{j2}, \cdots, c_{jn_j})$ and $A(g_j) = \{0, 1, \cdots, n_j\}$, when there are n_j customers in the queue j, for $j = 1, 2, \cdots, J$. We denote the state space of the two Markov processes by E and $\overline{E}$ respectively. The site 0 corresponds to a regular customer arrival and y_{j0} corresponds to the time since the last regular customer arrival at queue j.

Let λ^+_{jc} be solution of the traffic equations:

$$\lambda^+_{jc} = \Lambda^+_{jc} + \sum_{i=1}^{J} \sum_{c' \in \mathbf{C}} \lambda^+_{ic'} p_{ic';jc}, \tag{5.5),}$$

for all $c \in \mathbf{C}$ and $j = 1, 2, \cdots, J$. Let $F_{jc}(x) = 1 - \overline{G}_{jc}(x)\overline{H}_{jc}(x)$ for all $x \in \mathbf{R}_+$ and $\mu^*_{jc} = (\int_0^\infty \overline{F}_{jc}(x)dx)^{-1}$, $c \in \mathbf{C}$, $j = 1, 2, \cdots, J$. Let D_{jc} be the traffic process which represents the customers of class c exiting the network from queue j, for each $j = 1, 2, \cdots, J$, and $c \in \mathbf{C}$. Then, we have following theorem:

Theorem 5.1. If

$$\sum_{c\in\mathbf{C}} \frac{\lambda^+_{jc}}{\mu^*_{jc}} < 1, \ j = 1, 2, \cdots, J, \tag{5.6}$$

then the stationary distribution of the state process of the network described by the Markov process $\{\overline{Y}(t)\}$ is given by

$$\overline{\pi}(y) = \prod_{j=1}^{J} \pi_j(\overline{y}_j), \tag{5.7}$$

where $\overline{\pi}_j(y_j)$ is the stationary distribution for queue j:

$$\pi_j(\overline{y}_j) = \eta_j(g_j) \prod_{i=1}^{n_j} \mu^*_{jc_{ji}} \int_0^{y_{ji}} \overline{F}_{jc_{ji}}(u)du, \tag{5.8}$$

and $\eta_j(g_j)$ is given by

$$\eta_j(g_j) = (1 - \sum_{c\in\mathbf{C}} \frac{\lambda^+_{jc}}{\mu^*_{jc}}) \prod_{i=1}^{n_j} \frac{\lambda^+_{jc_{ji}}}{\mu^*_{jc_{ji}}}, \tag{5.9}$$

for $j = 1, 2, \cdots, J$.

Furthermore, the network is quasi-reversible when the system is in equilibrium. In this case, for each $c \in \mathbf{C}$, traffic processes D_{1c}, $D_{2c}, \cdots$, D_{Jc} are independent Poisson processes with rates $\lambda^+_{1c}p_{1c;0}$, $\lambda^+_{2c}p_{2c;0}, \cdots$, $\lambda^+_{Jc}p_{Jc;0}$ respectively, the past of them and the present state of the network are independent for all time t.

Proof. By the definition of $\{Y(t)\}$, we know that $\{Y(t)\}$ is an RGSMP. From (5.5), (5.6), (2.3), (2.4), (2.5), and (2.6) we easily verify that the stationary distribution for the state process of the network described by $\{Y(t)\}$ is

$$\pi(y) = \prod_{j=1}^{J} \pi_j(y_j), \tag{5.10}$$

where $\pi_j(y_j)$ is the stationary distribution for queue j:

$$\pi_j(y_j) = \eta_j(g_j)(1 - e^{-\lambda^+_{jc_{j0}} y_{j0}}) \prod_{i=1}^{n_j} \mu^*_{jc_{ji}} \int_0^{y_{ji}} \overline{F}_{jc_{ji}}(u)du, \tag{5.11}$$

where $\lambda^+_{jc_{j0}} = \sum_{c\in\mathbf{C}} \lambda^+_{jc}$ and $\eta_j(g_j)$ is given by (5.9), for $j = 1, 2, \cdot, J$. Thus, we obtain (5.7) and (5.8) with the similar argument used in Theorem 4.1.

Let $y' = (y'_1, y'_2, \cdots, y'_J)$, where $y'_j = (g'_j, y'_{js}; s \in A(g'_j))$, $g'_j = (c'_{j1}, c'_{j2}, \cdots, c'_{jn_j})$, and $A(g'_j) = \{0, 1, \cdots, n'_j\}$, for $j = 1, 2, \cdots, J$. We put $g' \hat{=} (g'_1, g'_2, \cdots, g'_J)$. Then for all $y \in E$, we have

$$\frac{1}{\pi(dy)}\int_E 1_{\mathbf{D_{jc}}}(y',y)\sum_{s\in A(g')} r_{sg'}Q_s(y',dy)\lambda_{s(g')}(y'_s)p_{jc;0}\pi(dy')$$

$$=\sum_{l=1}^{n_j+1}\int_0^\infty \lambda^+_{jc}\delta_j(l,n_j+1)p_{jc;0}F_{jc}(dy'_{jl})=\lambda^+_{jc}p_{jc;0},$$

where $\mathbf{D}_{jc}$ denotes the set of traffic transitions which represent the class c customers exiting the network from queue j, $j=1,2,\cdots,J$. Thus, the conclusion comes from Corollary 3.2. $\square$

References

[1] F. Baccelli and P. Brémaud, Palm Probabilities and Stationary Queues, Lecture Notes in Statistics 41 (Springer-Verlag, Berlin, 1987).

[2] P. Brémaud, Point Processes and Queue: Martingale Dynamics (Springer–Verlag, New York, 1980).

[3] D. Y. Burman, Insensitivity in queueing systems, Adv. Appl. Prob. 13 (1981) 846-859.

[4] X. Chao, Networks of queues with customers, signals and arbitrary service time distributions, Opns. Res. 43 (1995) 537-544.

[5] X. Chao and M. Pinedo, On generalized networks of queues with positive and negative arrivals, Probability in the Engineering and Informational Sciences, 7 (1993) 301-334.

[6] M. H. A. Davis, Piecewise-deterministic Markov processes: a general class of non-diffusion stochastic models, J. R. Statist. Sco. B 46 (1984) 353-388.

[7] P. Franken, D. König, U. Arndt and V. Schmidt, Queues and Point Processes (Wiley, Chichester, 1982).

[8] E. Gelenbe, Product form queueing networks with positive and negative customers, J. Appl. Prob. 28 (1991) 656-663.

[9] E. Gelenbe and R. Schassberger, Stability of product-form G-networks, Probability in the Engineering and Informational Sciences 6 (1992) 271-276.

[10] E. Gelenbe, G-networks with signals and batch removal, Probability in the Engineering and Informational Sciences 7 (1993a) 335-342.

[11] E. Gelenbe, G-networks with triggered customer movement, J. Appl. Prob. 30 (1993b) 742-748.

[12] W. Henderson, Insensitivity and reversed Markov processes, Adv. Appl. Prob. 26 (1989) 242-258.

[13] W. Henderson and P. Tayor, Insensitivity of processes with interruptions, J. Appl. Prob. 28 (1991) 656-663.

[14] G. Last and A. Brand (1995) *Marked Point Processes on the Real Line: The Dynamic Approach.* Springer, New York.

[15] B. Melamed, On Poisson traffic processes in discrete-state Markovian systems with applications to queueing theory, Adv. Appl. Prob. 11 (1979) 218-239.

[16] B. Melamed, On Markov jump processes imbedded at jump epochs and their queueing-theoretic applications, Math. Operat. Res. 7 (1982) 111-128.

[17] M. Miyazawa, The characterization of the stationary distributions of the supplemented self-clocking jump process, Math. Operat. Res. 16 (1991) 547-565.

[18] M. Miyazawa, Insensitivity and product-form decomposability of reallocatable GSMP, Adv. Appl. Prob. 25 (1993) 415-437.

[19] M. Miyazawa, Rate conservation law: a survey, Queueing Systems 15 (1994) 1-58.

[20] M. Miyazawa, R. Schassberger and V. Schmidt, On the structure of an insensitive generalized semi-Markov process with reallocation and point-process input, Adv. Appl. Prob. 27 (1995) 203-225.

[21] M. Miyazawa and R. Wolf, Symmetric queues with batch departures and their networks, to appear in Adv. Appl. Prob. (1996).

[22] R. Schassberger, Insensitivity of steady state distribution of generalized semi-Markov processes, Part I, Ann. Prob. 5 (1977a) 87-99.

[23] R. Schassberger, Insensitivity of steady state distribution of generalized semi-Markov processes, Part II, Ann. Prob. 6 (1977b) 85-93.

[24] R. Schassberger, Insensitivity of steady state distribution of generalized semi-Markov processes with speed, Adv. Appl. Prob. 10 (1978), 836-851.

[25] R. Schassberger, Two remarks on insensitive stochastic models, Adv. Appl. Prob. 18 (1986) 791-814.

[26] J. Walrand, An Introduction to Queueing Networks (Prentice–Hall, Englewood Cliffs, N.J., 1988).