BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS TO THE HOMOGENIZATION OF PARTIAL DIFFERENTIAL EQUATIONS

By
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Abstract / Resumé

Abstract. In this thesis, we investigate a class of Backward Stochastic Differential Equations (BSDE's) and give some applications to the homogenization of semi-linear Partial Differential Equations (PDE's).
We first establish existence, uniqueness and stability results for reflected BSDE's when the coefficient f is locally Lipschitz and the terminal condition is only square integrable. Our proofs are based on approximation techniques.
With the same spirit but different techniques, we extend our results on existence, uniqueness and stability in many directions. First, the coefficient is "almost" quadratic in its two variables y and z, i.e. $| f(t,\omega,y,z) | \leq \overline{\eta} + M(| y |^{\alpha} + | z |^{\alpha})$ for some $\alpha < 2$. Second the coefficient satisfies a locally monotonicity condition. Third, The coefficient is neither locally Lipschitz in the variable y nor in the variable z. Moreover, the terminal data is assumed to be square integrable only.
We finally prove some homogenization results for semi-linear PDE's by using an approach based upon the nonlinear Feynman-Kac formula developed in [74] and [68]. This gives a probabilistic formulation for the solutions of systems of semi-linear PDE's via the BSDE's. The problem then reduces to study the stability properties of BSDE's.

Resumé. Dans cette thèse, nous étudions une classe des Equations Différentielles Stochastiques Rétrogrades (EDSRs) et nous donnons quelques applications à l'homogénéisation des Equations aux Dérivées Partielles (EDPs).
Dans un premier temps, nous établissons des résultats d'existence, d'unicité et de stabilité quand le coefficient f est localement Lipschitzien et la condition terminale ξ est seulement de carré intégrable. Nos démonstrations sont basées sur des techniques d'approximation.
Dans le même esprit mais avec des techniques différentes, nous généralisons nos résultats d'existence, d'unicité et de stabilité dans plusieurs directions. d'une part, le coefficient est à croissance presque quadratique par rapport à ses deux arguments y et z, i.e. $| f(t,\omega,y,z) | \leq \overline{\eta} + M(| y |^{\alpha} + | z |^{\alpha})$ pour $\alpha < 2$, et d'autre part, il vérifie une condition de type monotonie locale en la variable y. En outre, la condition vérifiée par rapport à la variable z est plus faible que la condition de Lipschitz locale.
Finalement, nous prouvons quelques résultats d'homogénéisation aux EDPs en utilisant une approche basée sur la formule de Feynman-Kac généralisée et développée dans [74] et [68]. Ceci nous donne une représentation probabiliste pour les systèmes d'EDPs via les EDSRs. Le problème est alors réduit à étudier la stabilité des EDSRs.

Introduction

0.1 Backward stochastic differential equations

It was mainly during the last decade that the theory of backward stochastic differential equations took shape as a distinct mathematical discipline. This theory has found a wide field of applications as in stochastic optimal control and stochastic games (see Hamadène and Lepeltier [40]) and at the same time, in mathematical finance, the theory of hedging and non-linear pricing theory for imperfect markets (see El Karoui and Peng and Quenez [27]). Backward stochastic differential equations also appear to be a powerful tool for constructing Γ–martingale on manifolds (see Darling [22]) and they provide probabilistic formulae for solutions to partial differential equations (see Pardoux and Peng [67]).

Consider the following linear backward stochastic differential equation:

$$\begin{cases} dY_s = \left[Y_s\beta_s + Z_s^*\gamma_s + \varphi_s\right]ds - Z_s^* dW_s \\ Y_T = \xi. \end{cases} \tag{0.1}$$

As well known the equation was first introduced by Bismut [14, 16] when he was studying the adjoint equations associated with the stochastic maximum principle in optimal stochastic control. It is used in the context of mathematical finance as the model behind Black and Scholes formula for the pricing and hedging options. The equation (0.1) tells how to price the marginal value of the resource represented by the state variable in a random environment. Here, we solve for Y and Z, Y stands for the price while Z stands for the uncertainty between the present and terminal times.

The starting point of the development of general BSDE

$$\begin{cases} -dY_s = f(s, Y_s, Z_s)ds - Z_s^* dW_s \\ Y_T = \xi, \end{cases} \tag{0.2}$$

is the paper of Pardoux Peng [67]. Since then, BSDEs have been extensively studied. Note that, since the boundary condition is given at the terminal time T, it is not really natural for the solution Y_t to be adapted at each time t to the past of the Brownian motion W_s before time t. The presence of Z_t seems superfluous. However, we point out that it is the presence of this process that makes it possible to find adapted process Y_t to satisfy (0.2). Hence, a solution of BSDE (0.2) on the probability space of Brownian motion, as mentioned above, is a pair (Y, Z) of adapted processes that satisfies (0.2) almost surely. There is a vast literature on the subject (see, for example, Antonelli [2], Duffie and Epstein [26], Nualart and Schoutens [57], Pardoux [66], Peng [74]).

In [67], Pardoux and Peng have established the existence and uniqueness of the solution of equation (0.2) under the uniform Lipschitz condition, i.e. there exists a constant $K > 0$ such that

$$| \ f(\omega, t, y, z) - f(\omega, t, y', z') \ | \leq K \left(| \ y - y' \ | + | \ z - z' \ | \right), \tag{0.3}$$

for all $y, y' \in \mathbb{R}^d, z, z' \in \mathbb{R}^{d \times n}$, and $(\omega, t) \in \Omega \times [0, 1]$.

The existence and uniqueness of reflected backward stochastic differential equation (RBSDE) in a convex domain, via penalization method, have been proved by Gegout-Petit and Pardoux [36] under hypothesis (0.3) (see also Ouknine [61]). In the case where the solution is forced to remain above an obstacle, El Karoui et *al.* [28] have derived an existence result for reflected BSDE with Lipschitz conditions by Picard iteration method as well as a penalization argument (see also [41]). In this case, the solution is a triple (Y, Z, K), where K is an increasing process, satisfying

$$\begin{cases} -dY_s = f(s, Y_s, Z_s)ds - Z_s^* dW_s + dK_t \\ Y_T = \xi. \end{cases} \tag{0.4}$$

The existence and uniqueness of reflected backward stochastic differential equation (RBSDE) with jumps

$$\begin{cases} -dY_s = f(s, Y_s, Z_s)ds - Z_s^* dW_s + dK_t + \displaystyle\int U_s(e) \, \mu(de, ds) \\ Y_T = \xi, \end{cases} \tag{0.5}$$

both in one-dimensional and multidimensional cases, have been proved by Hamadène and Ouknine [41] and by Ouknine [61] under Lipschitz conditions on the coefficient via penalization argument. Moreover, Tang and Li [78] have applied the idea of Pardoux and Peng [67] for BSDE to get the first result on the existence and uniqueness of an adapted solution to a BSDE with Poisson jumps for a fixed terminal time and with Lipschitzian coefficients.

The assumption (0.3) (Lipschitz) is usually not satisfied in many problems, for example in finance (see Remark 2.3 of Chapter 2). So it is important to find weaker conditions, than the Lipschitz one, under which the BSDE has a unique solution. Now the question is: *Are there any weaker conditions than the Lipschitz continuity under which the BSDE has a unique solution?*

0.2 Some answers

Since the result of Pardoux and Peng [67], several works have attempted to relax the Lipschitz condition and the growth of the generator function, see Pardoux and Peng [69], Lepeltier and San Martin [49], Hamadène [39], Dermoune et *al* [24], Barles and Kobylanski [44], N'zi [58] and N'zi-Ouknine [59]. Most of these works deal only with real-valued BSDEs and the terminal condition ξ is bounded because of their dependence on the use of the comparison theorem for BSDEs (see Theorem 1.8), the uniqueness does not hold in general. Furthermore, the multidimensional case is also studied even though the comparison theorem does not hold. However, in general, the existence and uniqueness results are obtained only under weaker condition with respect to Y and Lipschitz with respect to Z (see Bahlali et *al* [6], Briand and Carmona [17], Darling and Pardoux [23], Hamadène [38], Mao [53] and Pardoux [64]). Let us mention nevertheless an exception: in [3], Bahlali has established an existence and uniqueness result for the solution of BSDEs (without reflection) under locally Lipschitz condition with respect to Y and Z.

0.3 Homogenization of PDE's via BSDE's

In [12], Bensoussan et *al.* studied the homogenization of linear second order partial differential operators using a probabilistic approach, based upon the linear Feynman-Kac formula. They left the question of studying the nonlinear case by the probabilistic method as an area open to investigation.

Recently, Pardoux and Peng [67, 68] have generalized the Feynman-Kac formula to take into account semi-linear PDE's. This generalization is based upon the theory of backward stochastic differential equations. More precisely, let u be the solution of the following system of semi-linear parabolic PDEs:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \frac{1}{2}Tr(\sigma\sigma^*\Delta u)(t,x) + b\nabla u(t,x) + f(t,x,u(t,x),\nabla u\sigma(t,x)) = 0 \\ u(T,x) = g(x). \end{cases} \tag{0.6}$$

Introducing $\{Y^{s,x}, Z^{s,x};\ s \leq t \leq T\}$ the adapted solution of the backward stochastic differential equation

$$\begin{cases} -dY_t = f(t, X_t^{s,x}, Y_t, Z_t)ds - Z_t^* dW_t \\ Y_T = g(X_T^{t,x}), \end{cases} \tag{0.7}$$

where $(X^{s,x})$ denotes the solution of the following stochastic differential equation

$$\begin{cases} dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t \\ X_s = x, \end{cases} \tag{0.8}$$

then we have, for each $(t,x) \in [0,T] \times \mathbb{R}^n$,

$$u(t,x) = Y_t^{t,x}, \tag{0.9}$$

both in the sense that any classical solution of the PDE (0.6) is equal to Y_t, and Y_t is –in the case where all coefficients are continuous– a viscosity solution of the PDE (0.6). This formula is the generalization of the well known Feynman-Kac formula.

It is then by now well known that systems of parabolic semi-linear are closely related to BSDE's. From the knowledge of BSDE's, we can derive some results on systems of semi-linear PDE's (see Pardoux and Peng [67], [68]). This correspondence reduces Bensoussan et *al* question to a question of stability of BSDEs. This last idea has been used in Pardoux, Veretennikov [71] to give averaging results for semi-linear PDEs where the nonlinear term is a function of the solution and not depend on the gradient, in Pardoux [65] and Ouknine, Pardoux [73] to prove homogenization property for a system of semi-linear PDEs of parabolic type, with rapidly oscillating periodic coefficients, a singular drift and a singular coefficient of the zero-th order term. Furthermore, let us recall that other homogenization results have been proved by Buckdahn and *al.* [18], Gaudron, Pardoux [37], and Lejay [48] where a divergence operators has been involved. On the other hand, from the knowledge of systems of semi-linear PDE's, we can derive some results on BSDE's (see Ma et *al.* [52] for more details). Now the question is: *How to obtain homogenization results for semi-linear variational inequalities and for semi-linear PDE's with singular coefficient?*

0.4 Results

In this thesis, we present some new results in the theory of BSDE's and give some applications to the homogenization of semi-linear PDE's. In particular, we provide answers to the questions we have raised above.

First, we establish existence and uniqueness results for the following type of multidimensional reflected backward stochastic differential equations, $E(\xi, f)$, with jumps

$$\begin{cases} (1)\, Z \text{ and } U \text{ are predictable processes and} \\ \quad E\left(\int_0^1 \mid Z_t \mid^2 dt + \int_0^1 \int_U \mid U_s(e) \mid^2 \lambda\,(de)\, ds\right) < +\infty \\ (2)\ Y_t = \xi + \int_t^1 f(s, Y_s, Z_s, U_s) ds - \int_t^1 Z_s dW_s - \int_t^1 \int U_s\,(e)\, \mu\,(de, ds) + K_1 - K_t \\ (3) \text{ the process Y is right continuous having left-hand limits (càdlàg)} \\ (4)\, \text{K is absolutely continuous, } K_0 = 0, \text{ and } \int_0^{\cdot} (Y_t - \alpha_t)\, dK_t \leq 0 \\ \text{for every } \alpha_t, \text{ progressively measurable process, which is right continuous having} \\ \text{left-hand limits and takes values into } \overline{\Theta} \\ (5)\ Y_t \in \overline{\Theta},\ 0 \leq t \leq 1\ .a.s., \end{cases}$$

for the case where the generator f is locally Lipschitz with respect to (y, z, u), that is: for each $N > 0$, there exists L_N such that:

$$\mid f(t, y, z, u) - f(t, y, z', u') \mid \leq L_N \left(\mid y - y' \mid + \mid z - z' \mid + \|u - u'\|\right)\ \mathbb{P} - a.s., a.e.\ t \in [0, 1]$$
$$\text{and } \forall y, y', z, z', u, u' \text{ such that } \mid y \mid \leq N, \mid y' \mid \leq N, \mid z \mid \leq N, \mid z' \mid \leq N, \|u\| \leq N, \|u'\| \leq N.$$

We don't impose any boundedness condition on the terminal data. It will be assumed square integrable only and this is important for applications, while the Lipschitz constant L_N behaves as $\sqrt{\log(N)}$ or satisfies the following condition

$$\lim_{N \to +\infty} \frac{\exp(L_N^2 + 2L_N)}{(L_N^2 + 2L_N) N^{2(1-\alpha)}} = 0,$$

for $0 \leq \alpha < 1$. Usually the method used to prove existence of solution to BSDE's consists in constructing a solution via successive approximations. Although this method is a powerful tool under globally Lipschitz hypothesis on the coefficient, it fails when the assumptions are only local. Thus, new techniques must be used. We adopt the following method: we approximate f by a sequence of Lipschitz functions f_n, then we consider the sequence of solutions (Y^n, Z^n, K^n, U^n) of equation $E(\xi, f_n)$ and finally we prove that (Y^n, Z^n, K^n, U^n) converges, for a suitable family of semi-norms, to the process (Y, Z, K, U) which is a solution to equation $E(\xi, f)$. Using the same idea we also extend our result to the case when f is locally monotone with respect to the state variable y and locally Lipschitz with respect to z.

More generally, we extend our results essentially in two directions. First, the coefficient grow "almost" in quadratic fashion in the two variables y and z, i.e. $\mid f(t, \omega, y, z) \mid \leq \overline{\eta} + M(\mid y \mid^\alpha + \mid z \mid^\alpha)$ for some $\alpha < 2$. Second the coefficient may be no locally Lipschitz. For example, our coefficient can take the form: $|z|\sqrt{|\log|z||}$ or $|y||\log|y||$. The method we use here develops the ones used previously: we approximate f by a sequence $(f_n)_{n>1}$ of Lipchitz functions via a suitable family of semi-norms. Then we use an appropriate localization to identify the limit as a solution of the equation (E^f), where (E^f) denotes the BSDE $E(f,\xi)$ without reflection and no jumps part. The main difference idea here stays in the fact that we apply Itô formula to $(|Y^{f_n}-Y^{f_m}|^2+\varepsilon)^\beta$ for some $0<\beta<1$ and $\varepsilon>0$, instead of $|Y^{f_n}-Y^{f_m}|^2$ as usually done. This allows us to treat multidimensional BSDE with super-linear growth coefficient in the both variables y and z. We prove the existence and uniqueness of solution for a small time duration, then we use the continuation procedure to extend the result to an arbitrarily prescribed time duration. The stability of the solution is established by similar arguments.

To illustrate our result, let us consider the following example: Let $\varepsilon>0$ and
$f_1(t,\omega,y,z)=g(t,\omega,y)\big[|z|\sqrt{|\log|z||}1_{|z|<\varepsilon}+h(z)1_{\varepsilon\leq|z|\leq 1+\varepsilon}+|z|\sqrt{\log|z|}1_{|z|>1+\varepsilon}\big]$
where g is a bounded function which is continuous in y such that $g(t,\omega,0)=0$ and $\langle y-y',g(t,y)-g(t,y')\rangle\leq 0$. h is a lipchitz and positive function which is choosing such that f_1 is continuous.
Let $f_2(t,\omega,y,z)$ be a continuous function in (y,z) such that:
i) There exist $M>0$, and $\eta\in\mathbb{L}^1([0,T]\times\Omega)$: $\langle y,f_2(t,\omega,y,z)\rangle\leq\eta+M|y|^2+M\mid z\mid^2$
ii) There exist $M>0$, $1<\alpha<2$ and $\overline{\eta}\in\mathbb{L}^{\frac{2}{\alpha}}([0,T]\times\Omega)$: $\mid f_2(t,\omega,y,0)\mid\leq\overline{\eta}+M\mid y\mid^\alpha$.
iii) There exists a constant $C>0$:

$$\begin{aligned}&\langle y-y',f_2(t,y,z)-f_2(t,y',z')\rangle\\&\leq C\mid y-y'\mid^2\big[1+|\log\mid y-y'\mid|\big]+C\mid y-y'\mid\mid z-z'\mid\big[1+\sqrt{|\log\mid z-z'\mid|}\big].\end{aligned}$$

Our work shows that equation $(E^{f_1+f_2})$ has a unique solution. It should be noted that this example is not covered by the previous papers.

Second, we prove some homogenization results for semi-linear PDE's by using an approach based upon the nonlinear Feynman-Kac formula (0.9) developed in [74] and [68]. This gives a probabilistic formulation for the solutions of systems of semi-linear PDE's via the BSDE's. The problem then reduces to study the stability properties of BSDE's. To be more precise, let $u^\varepsilon:[0,t]\times\mathbb{R}\longrightarrow\mathbb{R}$ be a solution of the system of multivalued partial differential equations associated to a lower semi-continuous, proper and convex function $\phi:\mathbb{R}^k\longrightarrow(-\infty,+\infty]$

$$\begin{cases}\dfrac{\partial u^\varepsilon}{\partial s}(s,x)-L_\varepsilon u^\varepsilon(s,x)-f(x,u^\varepsilon(s,x))\in\partial\phi(u^\varepsilon(s,x)),for\quad s\in[0,t]\\ u^\varepsilon(0,x)=g(x),u^\varepsilon(t,x)\in\overline{Dom(\phi)},x\in\mathbb{R}^d,\end{cases}$$

where L_ε is a second order operator and f,g are given functions, then one has

$$u^\varepsilon(t,x)\longrightarrow u(t,x),\quad\text{as}\quad\varepsilon\quad\text{goes to}\quad 0,$$

where u is the viscosity solution of the system of multivalued PDEs

$$\begin{cases}\dfrac{\partial u}{\partial s}(s,x)-Lu(s,x)-f(x,u(s,x))\in\partial\phi(u(s,x)),for\quad s\in[0,t]\\ u(0,x)=g(x),u(t,x)\in\overline{Dom(\phi)},x\in\mathbb{R}^d.\end{cases}$$

In order to prove this result, we use the probabilistic interpretation (0.9). Then we establish the stability for the following reflected BSDE

$$\begin{cases} Y_s^{t,x,\varepsilon} = g(X_t^{t,x,\varepsilon}) + \int_s^t f(X_r^{t,x,\varepsilon}, Y_r^{t,x,\varepsilon})dr - \int_s^t Z_r^{t,x,\varepsilon} dB_r + K_t^{t,x,\varepsilon} - K_s^{t,x,\varepsilon} \\ K_t^{t,x,\varepsilon} = -\int_0^t U_s^{t,x,\varepsilon} ds, \quad (Y^\varepsilon, U^\varepsilon) \in \mathbf{Gr}(\partial\phi), \end{cases}$$

in the Meyer and Zheng topology [56] via double approximation schemes: Yosida approximation on the reflection term and the usual homogenization approximation.

0.5 Outline of the thesis

The thesis is organized as follows.

In **Chapter 1**, we present, under classical assumptions and by means of a Picard approximation scheme, an existence and uniqueness theorem for solutions of BSDE's. In particular, we obtain a result for linear BSDE's which are classical in finance. Then, we state various properties concerning BSDE's. A probabilistic interpretations for PDE's is also presented.

In **Chapter 2**, we prove existence and uniqueness results of solution of reflected multi-dimensional backward stochastic differential equation with jumps in d-dimensional convex region. Our contribution in this topic is to weaken the Lipschitz assumption on the data (ξ, f). This is done with locally Lipschitz coefficient f and an only square integrable terminal condition ξ. We give, under the same assumptions, a stability result: more precisely, let (f_n) be a sequence of processes which converges to f locally uniformly and (ξ^n) a sequence of random variable which converge to ξ in $\mathbb{L}^2(\Omega)$, then the solutions Y^n of reflected BSDE $E(\xi^n, f_n)$ converges to Y the solution of $E(\xi, f)$ (see Theorem 2.14 of this chapter). We also study the case when the generator f has a super-linear growth of the following type: $C(1+ \mid y \mid \sqrt{\mid \log \mid y \mid\mid})$, $C(1+ \mid y \mid \sqrt{\mid \log \mid \log \mid y \mid\mid\mid})$...
We would like to mention here that the main device of our proof is an approximation technique. Such an idea was recently given in Bahlali [3].

Chapter 3 is devoted to the study of existence and uniqueness results for reflected backward stochastic differential equation with monotone and locally monotone coefficient and squared integrable terminal data. Precisely, let the generator f satisfying the following assumptions:
(i) f is continuous in (y, z) for almost all (t, ω),
(ii) There exist $M > 0$ and $0 \leq \alpha \leq 1$ such that $\mid f(t, \omega, y, z) \mid \leq M(1+ \mid y \mid^\alpha + \mid z \mid^\alpha)$.
(iii) There exists μ_N such that:

$$\langle y - y', f(t, y, z) - f(t, y', z)\rangle \leq \mu_N \mid y - y' \mid^2; \ \mathbb{P} - a.s., a.e.t \in [0, 1] \text{ and}$$
$$\forall y, y', z \text{ such that } \mid y \mid \leq N, \mid y' \mid \leq N, \mid z \mid \leq N.$$

(iv) For each $N > 0$, there exists L_N such that:

$$\mid f(t, y, z) - f(t, y, z') \mid \leq L_N \mid z - z' \mid; \ \mid z \mid, \mid z' \mid \leq N.$$

Then, if L_N satisfies

$$\lim_{N \to +\infty} \frac{\exp(L_N^2 + 2\mu_N^+)}{(L_N^2 + 2\mu_N^+) N^{2(1-\alpha)}} = 0,$$

the reflected BSDE $E(\xi, f)$ has a unique solution. In particular, if there exists a constant $L \geq 0$ such that: $L_N^2 + 2\mu_N^+ \leq L + 2(1-\alpha)\log N$ the reflected BSDE $E(\xi, f)$ has also a unique solution (see Theorem 3.6).

More generally, we extend our results essentially in two directions. First, the coefficient grow "almost" in quadratic fashion in the two variables Y and z, i.e. $\mid f(t,\omega,y,z) \mid \leq \overline{\eta} + M(\mid y \mid^\alpha + \mid z \mid^\alpha)$ for some $\alpha < 2$. Second the coefficient may be no locally Lipschitz. For example, our coefficient can take the form: $|z|\sqrt{|\log|z||}$ or $|y||\log|y||$.

In **chapter 4**, we study the limit of solutions of multivalued semi-linear partial differential equations involving a second order differential operator of parabolic type where the nonlinear term is a function of the solution and does not depend on the gradient. Our basic tool is the approach given by Pardoux [66] and Ouknine [61]. The weak convergence of the associated reflected backward stochastic differential equation involving the subdifferential operator of a lower semi-continuous, proper and convex function is proved in the sense of Meyer and Zheng topology [56]. An homogenization result for solutions of semi-linear PDE's in Sobolev spaces is also established.

In **Chapter 5**, we combine BSDE with the theory of diffusion approximation, as in Papanicolaou, Stroock, Varadhan [63], Pardoux, Veretennikov [71] and Ethier, Kurtz [32]. Firstly, in order to prove averaging result for a system of semi-linear PDE's of second order of parabolic type, with rapidly oscillating periodic coefficients, a singular drift and singular coefficients of the zero and second-th order term. Secondly, to prove averaging result of a singular Cauchy problem by introducing BSDE with local time (see Dermoune et *al* [24]).

In **Chapter 6**, we prove the convergence of the viscosity solution of a semi-linear variational inequality (SVI for short) involving a second order differential operator of parabolic type with periodic coefficients and highly oscillating term, using again the Meyer and Zheng topology and the weak convergence of an associated reflected backward stochastic differential equation. Roughly speaking, let u^ε be the viscosity solution of the following semi-linear variational inequality

$$\begin{cases} \forall s \in [0,t],\ x \in \mathbb{R}^d \\ \dfrac{\partial u^\varepsilon}{\partial s}(s,x) - \frac{1}{2}\sum_{i,j=1}^d a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j}(s,x) - \sum_{i=1}^d(\frac{1}{\varepsilon}b_i(\frac{x}{\varepsilon}) + c_i(\frac{x}{\varepsilon}))\dfrac{\partial u^\varepsilon}{\partial x_i}(s,x) \\ \qquad\qquad -(\frac{1}{\varepsilon}e(\frac{x}{\varepsilon}, u^\varepsilon(s,x)) - f(\frac{x}{\varepsilon}, u^\varepsilon(s,x))) \in \partial\phi(u^\varepsilon(s,x)) \\ \\ u^\varepsilon(0,x) = g(x), \quad u^\varepsilon(s,x) \in Dom(\phi) = \mathbf{cl}(\Theta), \end{cases}$$

where ϕ is a lower semi-continuous, proper and convex function. Then, we have

$$u^\varepsilon(t,x) \longrightarrow u(t,x), \quad \text{as} \quad \varepsilon \quad \text{goes to} \quad 0,$$

where u is the viscosity solution of the system of semi-linear variational inequality with some constant coefficients:

$$\begin{cases} \forall s \in [0,t],\ x \in \mathbb{R}^d \\ \big[\dfrac{\partial u}{\partial s}(s,x) - \frac{1}{2}\sum_{i,j=1}^d A_{ij}\dfrac{\partial^2 u}{\partial x_i \partial x_j}(s,x) - \sum_{i=1}^d C_i(u(s,x))\dfrac{\partial u}{\partial x_i}(s,x) \\ \qquad\qquad -D(u(s,x))\big] \in \partial I_{\overline{\Theta}}(u(s,x)) \\ u(0,x) = g(x). \end{cases}$$

Chapter 1

Introductory Material on Backward Stochastic Differential Equations

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This introductory chapter is intended to give a thorough description of BSDE's and then we present existence and uniqueness results under classical Lipshitz conditions. A probabilistic interpretations for PDE's is given. Some basic facts, which are widely used throughout the thesis, are also presented.

1.1 A background on Backward stochastic differential equations

Let consider a filtered space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, W_t, t \in [0,1])$ be a complete Wiener space in $\mathbb{R}^n$, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $(\mathcal{F}_t, t \in [0,1]\,)$ is a right continuous increasing family of complete sub $\sigma-$algebras of $\mathcal{F}$, $(W_t, t \in [0,1])$ is a standard Wiener process in $\mathbb{R}^n$ with respect to $(\mathcal{F}_t, t \in [0,1]\,)$. We assume that

$$\mathcal{F}_t = \sigma\left[W_s, s \leq t\right] \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of $\mathbb{P}$-null sets. Now, we define the following two objects:
(A.1) A terminal value $\xi \in L^2\left(\Omega, \mathcal{F}_1, \mathbb{P}\right)$.
(A.2) A function process f defined on $\Omega \times [0,1] \times \mathbb{R}^k \times \mathbb{R}^{k\times n}$ with values in $\mathbb{R}^k$ and satisfies the following assumptions:

(i) for all $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times n}$: $\;(\omega, t) \longrightarrow f\left(\omega, t, y, z\right)$ is $\mathcal{F}_t-$ progressively measurable

(ii) $\mathbb{E}\displaystyle\int_0^1 \mid f(t,0,0) \mid^2 dt < +\infty$

(iii) for some $K > 0$ and all $y, y' \in \mathbb{R}^k, z, z' \in \mathbb{R}^{k\times n}$, and $(\omega, t) \in \Omega \times [0,1]$
$\mid\; f\left(\omega, t, y, z\right) - f(\omega, t, y', z') \mid\leq K\left(\mid y - y' \mid + \mid z - z' \mid\right).$

We denote by $\mathbb{L}$ the set of $\mathbb{R}^k \times \mathbb{R}^{k\times n}$–valued processes (Y, Z) defined on $\mathbb{R}_+ \times \Omega$ which are $\mathcal{F}_t$–adapted and such that:

$$\|(Y,Z)\|^2 = \mathbb{E}\left(\sup_{0\leq t\leq 1} |Y_t|^2 + \int_0^1 |Z_s|^2 ds\right) < +\infty.$$

The couple $(\mathbb{L}, \|.\|)$ is then a Banach space.
Let us now introduce our BSDE: Given a data (f, ξ) we want to solve the following backward stochastic differential equation:

$$Y_t = \xi + \int_t^1 f(s, Y_s, Z_s)ds - \int_t^1 Z_s dW_s, \quad 0 \le t \le 1. \tag{1.1}$$

Definition 1.1. *A solution of equation (1.1) is a pair of processes (Y, Z) which belongs to the space $(\mathbb{L}, \|.\|)$ and satisfies equation (1.1).*

We now make more precise the dependence of the norm of the solution (Y, Z) upon the data (ξ, f).

Proposition 1.2. *Let assumptions **(A.1)**, **(A.2)**$(i)-(iii)$ hold. Then there exists a constant C, which depends only on K, such that*

$$\mathbb{E} \sup_{0\le t\le 1} |Y_t|^2 + \mathbb{E}\left(\int_0^1 | Z_t |^2 dt\right) \le C\mathbb{E}\left(| \xi |^2 + \int_0^1 | f(t,0,0) |^2 dt\right)$$

$$| Y_t |^2 \le \mathbb{E}\left[e^{a(1-t)} | \xi |^2 + \int_0^1 e^{a(s-t)} | f(s,0,0) |^2 ds/\mathcal{F}_t\right],$$

where $a = 1 + 2K + 2K^2$.

Before proving Proposition 1.2, let us first prove the inequality

$$\mathbb{E} \sup_{0\le s\le 1} | Y_s |^2 + \mathbb{E}\left(\int_0^1 |Z_s|^2 ds\right) < \infty. \tag{1.2}$$

Define for each $n \in I\!N$, the stopping time

$$\tau_n = \inf\{0 \le t \le 1;\ |Y_t| \ge n\},$$

and the processes

$$Y_t^n = Y_{t\wedge\tau_n}.$$

By noting

$$Z_t^n = 1_{[0,\tau_n]}(t)\, Z_t,$$

we have

$$Y_t^n = \xi + \int_t^1 1_{[0,\tau_n]}(s)\, f(s, Y_s^n, Z_s^n)ds - \int_t^1 Z_s^n dW_s, \quad 0 \le t \le 1.$$

If we apply Itô's formula to the process $|Y_t^n|^2$, then

$$|Y_t^n|^2 + \int_t^1 |Z_s^n|^2 ds = |\xi|^2 + 2\int_t^1 1_{[0,\tau_n]}(s)\,(Y_s^n)^* f(s, Y_s^n, Z_s^n)ds - \int_t^1 \langle Y_s^n, Z_s^n dW_s\rangle,$$

which implies

$$\mathbb{E}\left(|Y_t^n|^2 + \int_t^1 |Z_s^n|^2 ds\right) \le \mathbb{E}\,|\xi|^2 + \mathbb{E}\int_t^1 \left(|f(s,0,0)|^2 + \left(1 + 2K + 2\alpha^2\right)|Y_s^n|^2\right) ds$$
$$+ \frac{K}{2\alpha^2}\mathbb{E}\int_t^1 |Z_s^n|^2 ds.$$

If we take $\frac{K}{2\alpha^2} \leq \frac{1}{2}$, we get

$$\mathbb{E}\,|Y_t^n|^2 + \frac{1}{2}\mathbb{E}\int_t^1 |Z_s^n|^2\,ds \leq C\left(1 + \mathbb{E}\int_t^1 |Y_s^n|^2\,ds\right).$$

Now it follows from Gronwall's lemma that

$$\sup_{n\in\mathbb{N}^*}\sup_{0\leq t\leq 1}\mathbb{E}|Y_t^n|^2 \leq C.$$

On the other hand,

$$\sup_{n\in\mathbb{N}^*}\mathbb{E}(\int_0^1 |Z_s^n|^2 ds) < +\infty.$$

From Fatou's lemma, we can see that

$$\sup_{0\leq t\leq 1}\mathbb{E}|Y_t|^2 < +\infty.$$

Burkholder-Davis-Gundy inequality implies that

$$\mathbb{E}\sup_{0\leq t\leq 1}|Y_t|^2 < +\infty.$$

It follows that $\tau_n \uparrow 1$ *a.s.* Using again Fatou's lemma, we obtain

$$\mathbb{E}\left(\int_0^1 |Z_s|^2\,ds\right) < +\infty.$$

Proof of Proposition 1.2. Since (Y,Z) satisfies (1.1) and (1.2), $\mathbb{E}\int_t^1 \langle Y_s, Z_s dW_s\rangle = 0$, because the local martingale, $\{\mathbb{E}\int_0^t \langle Y_s, Z_s dW_s\rangle, 0\leq t\leq 1\}$ is uniformly integrable martingale from the Burkholder-Davis's inequality for stochastic integrals (see M. T. Barlow and P. Protter [11], proposition 3) and the fact that

$$\mathbb{E}\sup_{0\leq t\leq 1}|\int_t^1 (Y_s)^*\,Z_s dW_s| \leq C\left(\mathbb{E}\sup_{0\leq t\leq 1}|Y_t|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^1 |Z_s|^2\,ds\right)^{\frac{1}{2}} < \infty.$$

From Itô's formula, **(A.2)**(iii) and Schwarz's inequality,

$$\begin{aligned}|Y_t|^2 + \int_t^1 |Z_s|^2\,ds &= |\xi|^2 + 2\int_t^1 (Y_s)^*\,f(s,Y_s,Z_s)ds - 2\int_t^1 \langle Y_s, Z_s dW_s\rangle. \\ &\leq |\xi|^2 + \int_t^1\left(\mid f(s,0,0)\mid^2 + (1+2K+2K^2)\,|Y_s|^2 + \frac{1}{2}\,|Z_s|^2\right)ds - 2\int_t^1 \langle Y_s, Z_s dW_s\rangle.\end{aligned}$$

Taking expectation and using Gronwall's lemma we get

$$\sup_{0\leq t\leq 1}\mathbb{E}\,|Y_t|^2 + \mathbb{E}\left(\int_0^1 \mid Z_t\mid^2\,dt\right) \leq C\mathbb{E}\left(\mid\xi\mid^2 + \int_0^1 \mid f\,(t,0,0)\mid^2\,dt\right) < +\infty.$$

Then the result follows from the Burkholder-Davis-Gundy inequality. The second result follows by taking the conditional expectation in the following inequality

$$e^{at}\,|Y_t|^2 + \frac{1}{2}\int_t^1 e^{as}\,|Z_s|^2\,ds \leq e^{a}\,|\xi|^2 + \int_t^1 e^{as}\mid f(s,0,0)\mid^2\,ds - 2\int_t^1 e^{as}\langle Y_s, Z_s dW_s\rangle.$$

■

We shall now prove existence and uniqueness for BSDE (1.1) under conditions **(A.1)** and **(A.2)**.

Theorem 1.3. *Under conditions **(A.1)**, **(A.2)**$(i)-(iii)$, there exists a unique solution for equation (1.1).*

Proof Theorem 1.3.
Existence. First, let us prove that the BSDE

$$Y_t = \xi + \int_t^1 f(s)ds - \int_t^1 Z_s dW_s,$$

has one solution.
Let

$$Y_t \;=\; \mathbb{E}\left(\xi + \int_0^1 f(s)ds/\mathcal{F}_t\right),$$

and $\{Z_t, 0 \le t \le 1\}$ is given by Itô's martingales representation theorem applied to the square integrable random variable $\xi + \int_0^1 f(s)ds$, that is

$$\xi + \int_0^1 f(s)ds \;=\; \mathbb{E}\left(\xi + \int_0^1 f(s)ds\right) + \int_0^1 Z_s dW_s.$$

Taking the conditional expectation with respect to $\mathcal{F}_t$, we deduce that

$$Y_t \;=\; \xi + \int_t^1 f(s)ds - \int_t^1 Z_s dW_s, \qquad 0 \le t \le 1,$$

i.e. (Y, Z) is a solution of our BSDE.
Let us define the following sequence $(Y^n, Z^n)_{n\in I\!N}$ such that $Y^0 = Z^0 = 0$ and $\left(Y^{n+1}, Z^{n+1}\right)$ is the unique solution of the BSDE

$$(1) \quad Z^{n+1} \text{ is a predictable process and } \mathbb{E}\left(\int_0^1 \mid Z_t^{n+1} \mid^2 dt\right) < +\infty,$$

$$(2) \qquad Y_t^{n+1} = \xi + \int_t^1 f(s, Y_s^n, Z_s^n)ds - \int_t^1 Z_s^{n+1} dW_s \;, 0 \le t \le 1.$$

We shall prove that the sequence (Y^n, Z^n) is Cauchy in the Banach space $\mathbb{L}$.
Using Itô's formula, we obtain for every $n > m$

$$\begin{aligned} e^{\alpha t}\left|Y_t^{n+1} - Y_t^{m+1}\right|^2 + \int_t^1 e^{\alpha s}\left|Z_s^{n+1} - Z_s^{m+1}\right|^2 ds + \alpha \int_t^1 e^{\alpha s}\left|Y_s^{n+1} - Y_s^{m+1}\right|^2 ds \\ = 2\int_t^1 e^{\alpha s}\left(Y_s^{n+1} - Y_s^{m+1}\right)^* \left[f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m)\right] ds \\ + 2\int_t^1 e^{\alpha s}\left(Y_s^{n+1} - Y_s^{m+1}\right)^* \left(Z_s^{n+1} - Z_s^{m+1}\right) dW_s, \end{aligned}$$

and then,

$$\begin{aligned} \mathbb{E}e^{\alpha t}\left|Y_t^{n+1} - Y_t^{m+1}\right|^2 + \mathbb{E}\int_t^1 e^{\alpha s}\left|Z_s^{n+1} - Z_s^{m+1}\right|^2 ds + \alpha\mathbb{E}\int_t^1 e^{\alpha s}\left|Y_s^{n+1} - Y_s^{m+1}\right|^2 ds \\ \le 2K\mathbb{E}\int_t^1 e^{\alpha s}\left|Y_s^{n+1} - Y_s^{m+1}\right|\Big(\left|Y_s^n - Y_s^m\right| + \left|Z_s^n - Z_s^m\right|\Big)ds, \end{aligned}$$

which implies

$$\mathbb{E}e^{\alpha t}\left|Y_t^{n+1}-Y_t^{m+1}\right|^2+\mathbb{E}\int_t^1 e^{\alpha s}\left|Z_s^{n+1}-Z_s^{m+1}\right|^2 ds$$

$$\le (K^2\beta^2-\alpha)\mathbb{E}\int_t^1 e^{\alpha s}\left|Y_s^{n+1}-Y_s^{m+1}\right|^2 ds+\frac{2}{\beta^2}\mathbb{E}\int_t^1 e^{\alpha s}\left|Y_s^{n}-Y_s^{m}\right|^2 ds$$
$$+\frac{2}{\beta^2}\mathbb{E}\int_t^1 e^{\alpha s}\left|Z_s^{n}-Z_s^{m}\right|^2 ds.$$

Choosing α and β such that $\dfrac{2}{\beta^2}=\dfrac{1}{2}$ and $\alpha-4K^2=1$, then

$$\mathbb{E}e^{\alpha t}\left|Y_t^{n+1}-Y_t^{m+1}\right|^2+\mathbb{E}\int_t^1 e^{\alpha s}\left|Z_s^{n+1}-Z_s^{m+1}\right|^2 ds$$
$$\le\frac{1}{2}\left(\mathbb{E}\int_t^1 e^{\alpha s}\left|Y_t^{n}-Y_t^{m}\right|^2 ds+\mathbb{E}\int_t^1 e^{\alpha s}\left|Z_s^{n}-Z_s^{m}\right|^2 ds\right).$$

It follows immediately that

$$\mathbb{E}\int_0^1 e^{\alpha s}\left|Y_s^{n}-Y_s^{m}\right|^2+\mathbb{E}\int_0^1 e^{\alpha s}\left|Z_s^{n}-Z_s^{m}\right|^2 ds\le\frac{C}{2^n}.$$

Consequently, $(Y^n,Z^n)_{n\in I\!N}$ is a Cauchy sequence in the Banach space of progressively measurable processes $\mathbb{L}$.
Let

$$Y=\lim_{n\longrightarrow\infty}Y^n,\quad \textit{and}\quad Z=\lim_{n\longrightarrow+\infty}Z^n.$$

It is easy to see that (Y,Z) is a solution of our BSDE.
Uniqueness. Let $\{(Y_t,Z_t,);\ 0\le t\le 1\}$ and $\{(Y'_t,Z'_t,);\ 0\le t\le 1\}$ denote two solutions of our BSDE, and define

$$\{(\Delta Y_t,\Delta Z_t)\ 0\le t\le 1\}=\left\{Y_t-Y'_t,Z_t-Z'_t;\ \ 0\le t\le 1\right\}.$$

It follows from Itô's formula that

$$\mathbb{E}\left[|\Delta Y_t|^2+\int_t^1|\Delta Z_s|^2 ds\right]=2\mathbb{E}\int_t^1\left\langle\Delta Y_s,f(s,Y_s,Z_s)-f(s,Y'_s,Z'_s)\right\rangle ds.$$

Hence

$$\mathbb{E}\left[|\Delta Y_t|^2+\int_t^1|\Delta Z_s|^2 ds\right]\ \le\ C\mathbb{E}\int_t^1|\Delta Y_s|^2 ds+\frac{1}{2}\mathbb{E}\int_t^1|\Delta Z_s|^2 ds,$$

the result follows from Gronwall's lemma. ■

Remark 1.4. *It should be stressed that restrictions on the integrability of the solutions are necessary to guarantee the uniqueness property.*

Indeed, from the papers of Dudley [25], for any time t, there exists a stochastic integral $I_t=\int_0^t\phi_s^* dW_s$ such that $I_0=1$ and $I_1=0$, and $\int_0^1\mid\phi_s\mid^2 ds<+\infty$, $\mathbb{P}$- a.s. This last property does not hold in expectation. Now, Consider the elementary BSDE

$$Y_t=\xi-\int_t^1 Z_s^* dW_s,$$

the square integrable solution (Y, Z) is given by the continuous martingale $\{Y_t = \mathbb{E}(\xi/\mathcal{F}_t)\}$, and the process Z is given by the martingale representation theorem. The processes $(Y + \lambda I, Z + \lambda\phi)$ are also solutions of the BSDE, but the square integrability condition is not satisfied by these solutions.

The following corollary shows, in particular, the existence and uniqueness result for linear backward stochastic differential equation. The solution of such equation is well known in mathematical finance as the pricing and hedging strategy of the contingent claim ξ (see El Karoui et *al* [27]).

Corollary 1.5. *Let (β, γ) be a bounded progressively measurable process, φ be a predictable and square integrable process on $\Omega \times [0, 1]$. Then the linear BSDE*

$$dY_t = (\varphi_t + Y_t\beta_t + Z_t^*\gamma_t)dt - Z_t^*dW_t; \quad Y_1 = \xi \tag{1.3}$$

has a unique solution (Y, Z) in $\mathbb{L}$ given explicitly by:

$$\Gamma_t Y_t = \mathbb{E}\left[\xi\Gamma_1 + \int_t^1 \Gamma_s\varphi_s ds/\mathcal{F}_t\right], \tag{1.4}$$

where Γ_t is the adjoint process defined by the forward linear BSDE

$$d\Gamma_s = \Gamma_s\left[\beta_s ds + \gamma_s^* dW_s\right], \quad \Gamma_0 = 1. \tag{1.5}$$

In particular if ξ and φ are non-negative, the process Y is also non-negative. If in addition $Y_0 = 0$, then for any t, $Y_t = 0$ a.s., $\xi = 0$ a.s. and $\varphi = 0$ $dt \otimes d\mathbb{P}$-a.s.

Proof . From Theorem 1.3, there exists a unique solution to the BSDE (1.3). Using Itô's formula we deduce

$$\Gamma_t Y_t + \int_0^t \Gamma_s\varphi_s ds = Y_0 + \int_0^t \Gamma_s Y_s\gamma_s^* dW_s + \int_0^t \Gamma_s Y_s Z_s^* dW_s.$$

Since $\sup_{s\leq 1} \mid Y_s \mid$ and $\sup_{s\leq 1} \mid \Gamma_s \mid$ are square integrable, the local martingale $\{\Gamma_t Y_t + \int_0^t \Gamma_s\varphi_s ds;$ $0 \leq s \leq 1\}$ is a uniformly integrable martingale, whose t-time value is the $\mathcal{F}_t$–conditional expectation of its terminal value. Hence, relation (1.4) is proved. In particular, if ξ and φ are non-negative, Y_t is also non-negative. If in addition $Y_0 = 0$, the expectation of the non-negative variable $\Gamma_1\xi + \int_0^1 \Gamma_s\varphi_s ds$ is equal to 0. Then, $Y_t = 0$ a.s., $\xi = 0$ a.s. and $\varphi = 0$ $dt \otimes d\mathbb{P}$-a.s. ■

For some given $t_0 \in [0, 1]$, we set

$$\mathcal{F}_t^{t_0} = \sigma\left\{W_s - W_{t_0}; t_0 \leq s \leq t\right\}, \ t \in [t_0, 1].$$

The following proposition, which is very important in PDEs, is an easy consequence of the uniqueness of BSDE (1.1).

Proposition 1.6. *We make the same assumptions as in Theorem 1.3. Furthermore we assume that, for some given $t_0 \in [0, 1]$, $f(., y, z)$ is $\mathcal{F}_t^{t_0}$-adapted on the interval $[t_0, 1]$ and ξ is $\mathcal{F}_1^{t_0}$- measurable. Let (Y, Z) be the solution of BSDE (1.1). Then (Y, Z) is $\mathcal{F}_t^{t_0}$-adapted on $[t_0, 1]$. In particular (Y_{t_0}, Z_{t_0}) is a.s. constant.*

Proof . We define a process (Y', Z') on the interval $[t_0, 1]$ as the $\mathcal{F}_t^{t_0}-$ adapted solution of the BSDE

$$Y'_t = \xi + \int_t^1 f(s, Y'_s, Z'_s)ds - \int_t^1 Z'_s dW^0_s, \quad 0 \leq t \leq 1,$$

where $W^0_t = W_t - W_{t_0}$. Obviously $\left(W^0_t\right)_{t_0 \leq t \leq 1}$ is an $\mathcal{F}_t^{t_0}$- Brownian motion on $[t_0, 1]$. But $(Y', Z')_{t_0 \leq t \leq 1}$ are also $\mathcal{F}_t-$adapted and

$$\int_t^1 Z'_s dW^0_s = \int_t^1 Z'_s dW_s.$$

It follows that, (Y', Z') coincides with the solution (Y, Z) on $[t_0, 1]$. Consequently, (Y, Z) is $\mathcal{F}_t^{t_0}-$ adapted on $[t_0, 1]$. ■

Now estimate the difference between two solutions in terms of the difference between the data. Given two final conditions ξ and $\xi' \in L^2(\Omega, \mathcal{F}_1, \mathbb{P})$ and two coefficient processes f and f' both satisfying the conditions above. Let $\{(Y_t, Z_t);\ 0 \leq t \leq 1\}$ (resp. $\{(Y'_t, Z'_t);\ 0 \leq t \leq 1\}$) be the solution of the BSDE (ξ, f) (resp. BSDE (ξ', f')). We have the following estimate for the difference of the above solutions.

Theorem 1.7. *There exists a constant C, which depends upon the Lipschitz constant of f', such that*

$$\mathbb{E} \sup_{0 \leq t \leq 1} \left|Y_t - Y'_t\right|^2 + \mathbb{E}\int_0^1 |\ Z_t - Z'_t\ |^2\ dt$$
$$\leq C\mathbb{E}\left(|\ \xi - \xi'\ |^2 + \int_0^1 |\ f(t, Y_t, Z_t) - f'(t, Y_t, Z_t)\ |^2\ dt\right).$$

Proof . Using Itô's formula for $|\ Y_t - Y'_t\ |^2$, yielding

$$\left|Y_t - Y'_t\right|^2 + \int_t^1 \left|Z_s - Z'_s\right|^2 ds$$
$$= \left|\xi - \xi'\right|^2 + 2\int_t^1 (Y_s - Y'_s)^*(f(s, Y_s, Z_s) - f'(s, Y'_s, Z'_s))ds$$
$$- 2\int_t^1 \langle Y_s - Y'_s, (Z_s - Z'_s)dW_s\rangle.$$

Taking expectation and using the fact that f' is K'-Lipschitz we obtain

$$\mathbb{E}\left|Y_t - Y'_t\right|^2 + \mathbb{E}\int_t^1 \left|Z_s - Z'_s\right|^2 ds$$
$$= \mathbb{E}\left|\xi - \xi'\right|^2 + \mathbb{E}\int_t^1 |\ Y_s - Y'_s\ |^2\ ds + \mathbb{E}\int_t^1 |\ f(s, Y_s, Z_s) - f'(s, Y_s, Z_s)\ |^2\ ds$$
$$+2\int_t^1 |\ Y_s - Y'_s\ |\ \big(K'\ |\ Y_s - Y'_s\ | + K'\ |\ Z_s - Z'_s\ |\ \big)ds.$$

By a standard arguments we get

$$\mathbb{E}\left|Y_t - Y'_t\right|^2 + \mathbb{E}\int_t^1 \left|Z_s - Z'_s\right|^2 ds$$
$$= \mathbb{E}\left|\xi - \xi'\right|^2 + (1 + 2K' + 2K'^2)\mathbb{E}\int_t^1 |\ Y_s - Y'_s\ |^2\ ds$$
$$+ \mathbb{E}\int_t^1 |\ f(s, Y_s, Z_s) - f'(s, Y_s, Z_s)\ |^2\ ds.$$

Using Gronwall's lemma and Burkholder-Davis-Gundy inequality we get the desired result. ■

Another important think in the viscosity solutions of PDEs is the comparison theorem and the non confluent property of solutions of BSDE. Also, we recall that this theorem gives a sufficient condition for the wealth process to be nonnegative and yields the classical properties of utilities. We restrict ourselves to the case $k = 1$ and we prove the following result:

Theorem 1.8. *Suppose that* $k = 1, \xi \leq \xi'$ *a.s. and* $f(t, y, z) \leq f'(t, y, z)\ dt \times d\mathbb{P}$ *a.e. Then* $Y_t \leq Y_t'$, $0 \leq t \leq 1$, *a.s. Moreover if* $Y_0' = Y_0$, *then* $Y_t' = Y_t$, $0 \leq t \leq 1$, *a.s. In particular, whenever either* $\mathbb{P}(\xi < \xi') > 0$ *or* $f(t, y, z) < f'(t, y, z)$, $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, *on a set of positive* $dt \times d\mathbb{P}$ *measure, then* $Y_0 < Y_0'$.

Proof . Define

$$\overline{Y}_t = Y_t - Y_t', \quad \overline{Z}_t = Z_t - Z_t', \ \ \overline{\xi} = \xi - \xi', \text{and} \quad U_t = f(t, Y_t', Z_t') - f'(t, Y_t', Z_t').$$

We can write

$$\overline{Y}_t = \overline{\xi} + \int_t^1 \left(\alpha_s \overline{Y}_s + \beta_s \overline{Z}_s + U_s\right) ds - \int_t^1 \overline{Z}_s dW_s, \quad 0 \leq t \leq 1,$$

where $\{\alpha_t; \quad 0 \leq t \leq 1\}$ is defined by

$$\alpha_t = \begin{cases} (f(t, Y_t, Z_t) - f(t, Y_t', Z_t))\,(Y_t - Y_t')^{-1} & \text{if} \quad Y_t \neq Y_t' \\ 0 & \text{if} \quad Y_t = Y_t', \end{cases}$$

and the $\mathbb{R}^n$ valued process $\{\beta_t; \quad 0 \leq t \leq 1\}$ as follows. For $1 \leq i \leq n$, let $Z_t^{(i)}$ denote the n-dimensional vector whose components are equal to those of $Z_t^{'}$, and whose $n - i$ last components are equal to those of Z_t. With this notation, we define for each $1 \leq i \leq n$,

$$\beta_t^i = \begin{cases} \left(ft, Y_t', Z_t^{(i)}) - f(t, Y_t', Z_t^{i-1})\right)(Z_t'^i - Z_t^i)^{-1} & \text{if} \quad Z_t^i \neq Z_t'^i \\ 0 & \text{if} \quad Z_t^i = Z_t'^i. \end{cases}$$

Since f is a Lipschitz function, α and β are bounded processes, for $0 \leq s \leq t \leq 1$, let

$$\Gamma_{s,t} = \exp\left[\int_s^t \langle \beta_r, dW_r \rangle + \int_s^t \left(\alpha_r - \frac{|\ \beta_r\ |^2}{2}\right) dr\right].$$

It is easy to see that for $0 \leq s \leq t \leq 1$,

$$\overline{Y_s} = \Gamma_{s,t}\overline{Y}_t + \int_s^t \Gamma_{s,r} U_r dr - \int_s^t \Gamma_{s,r}\left(\overline{Z}_r + \beta_r \overline{Y}_r\right) dW_r.$$

Hence

$$\overline{Y_s} = \mathbb{E}\left(\Gamma_{s,t}\overline{Y}_t + \int_s^t \Gamma_{s,r} U_r dr / \mathcal{F}_s\right).$$

The result follows from this formula and the negativity of $\overline{\xi}$ and U. ∎

Remark 1.9. *(see Lepeltier and San Martin [49])*
It should be noticed that there is an existence but not uniqueness result for one dimensional BSDE with continuous generator and with linear growth i.e. There exists $k > 0$ *such that for all* $y \in \mathbb{R}$, $z \in \mathbb{R}$ *and* $t \in [0, 1]$

$$|f(t, y, z)| \leq k\,(1 + |y| + |z|)\,.$$

1.2 BSDE related to SDE of Itô's type

From now on, we consider the Markovian case. So, we introduce a class of diffusion processes. Let $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ be functions such that

$$| b(t,x) - b(t,x^{'}) | + | \sigma(t,x) - \sigma(t,x^{'}) | \leq L | x - x^{'} |,$$

and

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|),$$

for some constant $L > 0$ and $C > 0$.
For each $(t,x) \in [0,T] \times \mathbb{R}$, let $\left\{ \left(X_s^{t,x} \right) ;\ s \in [0,T] \right\}$ be the unique solution of the stochastic differential equation

$$X_s^{t,x} = x + \int_t^{t\vee s} b\left(r, X_r^{t,x}\right) dr + \int_t^{t\vee s} \sigma\left(r, X_r^{t,x}\right) dB_r.$$

Let us state some properties of the process $\left\{ X_s^{t,x},\ s \in [0,T] \right\}$ which can be found in Kunita's book [46].

Proposition 1.10. *For each $t \geq 0$ there exists a version of $\left\{ X_s^{t,x}, s \geq t, x \in \mathbb{R}^d \right\}$ such that $s \rightarrow X_s^t$ is a $C\left(\mathbb{R}^d\right)-$valued continuous process. Moreover,*
(i) X_s^t and X_{s-t}^0 have the same distribution, $0 \leq t \leq s$;
(ii) $X_{t_1}^{t_0}, X_{t_2}^{t_1}, ..., X_{t_n}^{t_{n-1}}$ are independent, for all $n \in I\!N, 0 \leq t_0 < t_1 < ..... < t_n$;
(iii) $X_r^t = X_r^s \circ X_s^t,\ 0 \leq t < s < r$.
Furthermore, for all $p \geq 2$ there exists a real C_p such that for all $0 \leq t < s,\ \ x, x' \in \mathbb{R}^d$
(iv) $\mathbb{E}\left(\sup\limits_{t\leq r\leq s} \left| X_r^{t,x} - x \right|^p \right) \leq C_p (s-t)(1+|x|^p)$

$$\mathbb{E}\left(\sup_{t\leq r\leq s} \left| X_r^{t,x} - X_r^{t,x^{'}} - \left(x - x^{'}\right) \right|^p \right) \leq C_p (s-t)\left(\left| x - x^{'} \right|^p \right).$$

In the sequel, we assume $k = 1$ and consider the BSDE with data (ξ, f) where

$$\xi(\omega) = g\left(X_T^{t,x}(\omega) \right),$$

$$f(\omega, s, y, z) = f\left(s, X_s^{t,x}(\omega), y, z \right),$$

with $g : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d\times 1} \longrightarrow \mathbb{R}$ some functions such that

$$\begin{aligned} |g(x)| &\leq C(1+|x|^p), \\ |f(t,x,0,0)| &\leq C(1+|x|^p), \end{aligned}$$

for some $C, p > 0$, f is globally Lipschitz in (y,z) uniformly in (t,x).
Under our assumptions; Theorem 1.3 implies that for each $(t,x) \in [0,T] \times \mathbb{R}^d$ there exists a unique $\mathcal{F}_s^t-$progressively measurable process $\left(Y^{t,x}, Z^{t,x}\right)$ such that

$$Y_s^{t,x} = g\left(X_T^{t,x} \right) + \int_s^T f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr - \int_s^T Z_r^{t,x} dW_r.$$

We extend $\left(Y_s^{t,x}, Z_s^{t,x}\right)$ for $s \in [0,T]$ by putting

$$Y_s^{t,x} = Y_t^{t,x},\ Z_s^{t,x} = 0 \text{ for } s \in [0,t]\,.$$

It follows that

$$Y_s^{t,x} = g\left(X_T^{t,x}\right) + \int_s^T 1_{[t,T]}(r)\, f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr - \int_s^T Z_r^{t,x} dW_r.$$

Proposition 1.11. *For all* $t, t' \in [0,T]\,,\ x, x' \in \mathbb{R}^d$, *we have*

i) $\mathbb{E}(\sup\limits_{0\le s\le T} \left|Y_s^{t,x}\right|^2) \le C\left(1 + |x|^{2p}\right)$,
and

$$ii)\ \mathbb{E}(\sup_{0\le s\le T} |Y_s^{t,x} - Y_s^{t',x'}|^2) \le C\Big[\mathbb{E} \mid g(X_T^{t,x}) - g(X_T^{t',x'}) \mid^2$$
$$+ \mathbb{E}\Big(\int_0^T \mid 1_{[t,T]}(r) f(r, Y_r^{t,x}, Z_r^{t,x}) - 1_{[t',T]}(r) f(r, X_r^{t',x'}, Y_r^{t',x'}, Z_r^{t',x'}) \mid^2 dr\Big)\Big],$$

where $p \in \mathbb{N}$ *and* $C > 0$ *is a constant independent of* t, t', x *and* x'.

Proof . From Proposition 1.2, we have

$$\mathbb{E}\left(\left|Y_s^{t,x}\right|^2\right) \le CE\left(\left|g\left(X_T^{t,x}\right)\right|^2 + \int_t^T |f(r,0,0)|^2 dr\right).$$

Since $f(r,0,0) = f\left(r, X_r^{t,x}, 0, 0\right)$, by virtue of assumptions on f and g, we deduce that

$$\mathbb{E}\left(\left|Y_s^{t,x}\right|^2\right) \le \mathbb{E}\left(C(1 + \left|X_r^{t,x}\right|^p)^2 + C\int_t^T \left(1 + \left|X_r^{t,x}\right|^p\right)^2 dr\right),$$

and (i) follows by using Proposition 1.10. Now, (ii) follows from Theorem 1.7. ∎

Corollary 1.12. *Under the above assumptions, the deterministic function* $u(t,x) := Y_t^{t,x} : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *is continuous in* (t,x) *and for some real* C *and* $p, |u(t,x)| \le C\left(1 + |x|^p\right), (t,x) \in [0,T] \times \mathbb{R}^d$.

1.3 Viscosity solution of partial differential equations

The notion of viscosity solution for nonlinear degenerate parabolic PDEs, is notions of solutions which are not necessarily smooth enough to satisfy the equation in a classical sense.

It was introduced in Crandall and Lions [20] in order to solve first order Hamilton-Jacobi equations and then extended to second order equations in Lions [50, 51].

Now, let us introduce the system of parabolic PDEs, for which u will be a solution. First, we make a restriction, which is due to the fact that we want to consider viscosity solutions of our system of PDEs. We assume that for each $1 \le i \le k,\ f_i$, the i–th coordinate of f, depends only on the i–th row of the matrix z, and not on the other rows of z. Consider the system of semi-linear PDEs :

$$\begin{cases} \forall\ (t,x) \in [0,T] \times \in \mathbb{R}^d, \\ (\dfrac{\partial u_i}{\partial t} + Lu_i)(t,x) + f_i(t,x,u(t,x),(\nabla u\sigma)(t,x)) = 0,\ \ 1 \le i \le k, \\ u(T,x) = h(x),\ x \in \mathbb{R}^d \end{cases} \tag{1.6}$$

where

$$L = \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^*)_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial}{\partial x_i}.$$

First of all, we give a generalization of the Feynman-Kac formula stated by Pardoux and Peng [67].

Proposition 1.13. *Let $u \in C^{1,2}([0,1]\times\mathbb{R}^d,\mathbb{R}^k)$ be a classical solution of (1.6) and assume that there exists a constant C such that, for each (s,x),*

$$\mid u(s,x)\mid + \mid \nabla u(s,x)\sigma(s,x)\mid \leq C(1+\mid x\mid).$$

Then, for each (s,x), $(Y_s^{t,x} = u(s,X_s^{t,x}), Z_s^{t,x} = \nabla u(s,X_s)\sigma(s,X_s))$, a.s., where $(Y_s^{t,x}, Z_s^{t,x})$ is the unique solution of BSDE (1.1).

Proof . For the sake of simplicity, we assume that $k=1$. By applying Itô's formula to $u(s,X_s^{t,x})$, we obtain

$$du(s,X_s^{t,x}) = (\frac{\partial u}{\partial t}(s,X_s^{t,x}) + Lu(s,X_s^{t,x}))ds + \nabla u(s,X_s^{t,x})\sigma(s,X_s^{t,x})dW_s.$$

Since u solves equation (1.6), it follows that

$$du(s,X_s^{t,x}) = f(s,u(s,X_s^{t,x}),\nabla u(s,X_s^{t,x})\sigma(s,X_s^{t,x}))ds - \nabla u(s,X_s^{t,x})\sigma(s,X_s^{t,x})dW_s.$$

Hence, for each (s,x), $(u(s,X_s^{t,x}),\nabla u(s,X_s)\sigma(s,X_s))$ is a solution to the BSDE (1.1) and the result follows from uniqueness of BSDE (1.1). ■

Now we explain what we mean by a viscosity solution of PDEs. For a complete presentation of this notion of solution, we refer the reader to Crandall, Ichii and Lions [21].

Definition 1.14. *(a) $u \in C([0,T]\times\mathbb{R}^d,\mathbb{R}^k)$ is called a viscosity subsolution of (1.6) if $u_i(T,x)\leq h_i(x)$, $x\in\mathbb{R}^d$, $1\leq i\leq k$, and moreover for any $1\leq i\leq k$, $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^d)$, whenever $(t,x)\in[0,T]\times\mathbb{R}^d$ is a local maximum of $u_i-\varphi$, then*

$$-\frac{\partial\varphi}{\partial t}(t,x) - L\varphi(t,x) - f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)) \leq 0, \text{ if } x\in\mathbb{R}^d.$$

(b) $u \in C([0,T]\times\mathbb{R}^d,\mathbb{R}^k)$ is called a viscosity supersolution of (1.6) if $u_i(T,x)\geq h_i(x)$, $x\in\mathbb{R}^d$, $1\leq i\leq k$, and moreover for any $1\leq i\leq k$, $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^d)$, whenever $(t,x)\in[0,T]\times\mathbb{R}^d$ is a local minimum of $u_i-\varphi$, then

$$-\frac{\partial\varphi}{\partial t}(t,x) - L\varphi(t,x) - f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)) \geq 0, \text{ if } x\in\mathbb{R}^d.$$

(c) $u \in C([0,T]\times\mathbb{R}^d,\mathbb{R}^k)$ is called a viscosity solution of (1.6) if it is both a viscosity subsolution and supersolution of 1.6.

It can be deduced from the uniqueness theorem for BSDEs that

$$Y_{t+h}^{t,x} = Y_{t+h}^{t+h,X_{t+h}^{t,x}}, \quad h>0.$$

This implies that $Y_s^{t,x} = u(s,X_s^{t,x})$, $t\leq s\leq T$.

Now we can prove the following theorem:

Theorem 1.15. *u defined by Corollary 1.12, is a viscosity solution of the system of parabolic PDEs (1.6).*

Proof . Let us prove that u is a viscosity subsolution to the equation (1.6). Let $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^d)$ and $(t,x)\in[0,T]\times\mathbb{R}^d$ such that (t,x) is a point of local maximum of $u-\varphi$. We assume without loss of generality that

$$u(t,x)=\varphi(t,x).$$

We suppose that

$$\frac{\partial u}{\partial t}(t,x)+L\varphi(t,x)-f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x))<0,$$

and we will find a contradiction.
Let $0<\alpha\leq T-t$ and $x,y\in\mathbb{R}^d$ are such that for all $t\leq s\leq t+\alpha$, $\mid y-x\mid\leq\alpha$,

$$u_i(s,y)\leq\varphi(s,y),$$

$$\frac{\partial u}{\partial s}(s,y)+L\varphi(s,y)-f_i(s,y,u(s,y),(\nabla\varphi\sigma)(s,y))<0,$$

and define

$$\tau=inf\{s\geq t;\mid X_s^{t,x}-x\mid\geq\alpha\}\wedge(t+\alpha).$$

Let now

$$(\overline{Y}_s,\overline{Z}_s)=((Y_{s\wedge\tau}^{t,x})^i,\mathbf{1}_{[0,\tau]}(s)(Z_s)^i),\quad t\leq s\leq t+\alpha.$$

$(\overline{Y},\overline{Z})$ solves the one dimensional BSDE

$$\overline{Y}_s=u_i(\tau,X_\tau^{t,x})+\int_s^{t+\alpha}\mathbf{1}_{[0,\tau]}(s)f_i(X_r^{t,x},u(r,X_r^{t,x}),\overline{Z}_r)dr-\int_s^{t+\alpha}\overline{Z}_rdW_r,\ t\leq s\leq t+\alpha.$$

On the other hand, it follows from Itô's formula that

$$(\widehat{Y}_s,\widehat{Z}_s)=(\varphi(s,X_{s\wedge\tau}^{t,x}),\mathbf{1}_{[0,\tau]}(s)(\nabla\varphi\sigma)(s,X_s^{t,x})),$$

solves the BSDE

$$\widehat{Y}_s:=\varphi(s\wedge\tau,X_r^{t,x})-\int_s^{t+\alpha}\mathbf{1}_{[0,\tau]}(s)(\frac{\partial\varphi}{\partial t}+\overline{L}\varphi)(r,X_r^{t,x})dr-\int_s^{t+\alpha}\widehat{Z}_sdW_r.$$

From $u_i\leq\varphi$, and the choice of α and τ, we deduce with the help of the comparison theorem (see Theorem 1.8) that $\overline{Y}_t\leq\widehat{Y}_t$, and then $u_i(t,x)<\varphi(t,x)$, which contradicts our assumptions. ■

Remark 1.16. *Consider the linear parabolic partial differential equation*

$$\frac{\partial u}{\partial t}(t,x)+(Lu)(t,x)+c(t,x)u(t,x)+h(t,x)=0,\ 0<t<1,\ x\in\mathbb{R}^d$$
$$u(1,x)=g(x),\ x\in\mathbb{R}^d,$$

where L is the infinitesimal generator of a time-homogeneous diffusion process $\{X_t;t\geq0\}$, and $c,g\in C_b(\mathbb{R}^d)$. The solution of this equation is given by the well known Feynman-Kac formula

$$\mathbb{E}\Big[g(X_1^x)\exp\Big(\int_t^1c(r,X_r^{t,x})dr\Big)+\int_t^1h(s,X_s^{t,x})\exp\Big(\int_t^sc(r,X_r^{t,x})dr\Big)ds\Big].$$

Example 1.17. *Suppose $k = 1$ and $f(t, x, y, z) = c(t, x)y + h(t, x)$.*
In this case the corresponding BSDE is linear and has the form

$$Y_s^{t,x} = g(X_1^{t,x}) + \int_s^1 \left[c(r, X_r^{t,x})Y_r^{t,x} + h(r, X_r^{t,x})\right] dr - \int_s^1 Z_r^{t,x} dW_r.$$

By the same argument as in the proof of Theorem 1.8 this equation has an explicit solution

$$Y_s^{t,x} = g(X_1^{t,x}) \exp\left(\int_s^1 c(r, X_r^{t,x})dr\right) + \int_s^1 h(r, X_r^{t,x}) \exp\left(\int_s^r c(p, X_p^{t,x})dp\right) dr$$
$$- \int_s^1 \exp\left(\int_s^r c(p, X_p^{t,x})dp\right) Z_r^{t,x} dW_r.$$

Since $Y_t^{t,x} = \mathbb{E}(Y_t^{t,x})$, we obtain

$$Y_t^{t,x} = \mathbb{E}\left[g(X_1^x) \exp\left(\int_t^1 c(r, X_r^{t,x})dr\right) + \int_t^1 h(s, X_s^{t,x}) \exp\left(\int_t^s c(r, X_r^{t,x})dr\right) ds\right],$$

which is the classical Feynman-Kac formula (see Remark 1.16).

Remark 1.18. *Theorem 1.15 can be considered as a nonlinear extension of the Feynman-Kac formula.*

Now we introduce another definition of viscosity solution. To do this we need some preliminary definitions. Let $\mathcal{M}_{d\times d}$ stands for the set of $d \times d$ symmetric nonnegative matrices.

1.4 Reflected BSDE and viscosity solution of multivalued partial differential equations

Here we briefly outline some results, which extend the results of Sections 1.2 and 1.3, and therefore we just formulate them without proof. For more details and complete proofs the reader may turn to the paper of Pardoux and Rascanu [70] (see also Ouknine and N'zi [60]).

1.4.1 Reflected backward stochastic differential equation

In this subsection, the assumptions on f and ξ are exactly those in Section 1.2. Let ϕ : $\mathbb{R}^d \longrightarrow \mathbb{R}$ be a function which satisfies
(A.3) i) ϕ is a proper ($\phi \neq +\infty$), lower semi-continuous and convex function.
ii) $\mathbb{E} \mid \phi(\xi) \mid < +\infty$
In the sequel, we assume without loss of generality that $\phi(y) \geq \phi(0) = 0$.
Now, let us recall some properties of a Yosida approximation of subdifferential operator. We put

$$\begin{aligned}
Dom(\phi) &= \{u \in \mathbb{R}^k : \phi(u) < +\infty\} \\
\partial\phi(u) &= \{u^* \in \mathbb{R}^k :< u^*, v - u > +\phi(u) \leq \phi(v), \forall v \in \mathbb{R}^k\} \\
Dom(\partial(\phi)) &= \{u \in \mathbb{R}^k : \partial(\phi) \neq \emptyset\} \\
Gr(\partial\phi) &= \{(u, u^*) \in \mathbb{R}^k \times \mathbb{R}^k : u \in Dom(\partial(\phi)) \quad \text{and} \quad u^* \in \partial\phi(u)\}.
\end{aligned}$$

By virtue of Ouknine and N'zi [60] (see also Pardoux and Rascanu [70]), we have the following theorem

Theorem 1.19. *Let assumption **(A.1)**–**(A.3)** hold. Then, for each* $(t,x) \in [0,1] \times \mathbb{R}^d$*, there exists a unique triple* $(Y^{t,x}, Z^{t,x}, U^{t,x})$ *which solves the reflected backward stochastic differential equation with data* (ξ, f, ϕ)*, that is*

$$a)\ \mathbb{E}\int_0^1 \phi(Y_r^{t,x})dr < +\infty,$$

$$b)\ \mathbb{E}(\phi(Y_s^{t,x})) < +\infty, \quad \forall s \in [t,1],$$

$$c)\ (Y_s^{t,x}, U_s^{t,x}) \in \partial\phi,\ d\mathbb{P} \times dt \ a.e.\ on\ \Omega \times [0,1],$$

$$d)\ Y_s^{t,x} + \int_s^1 U_r^{t,x} dr = \xi + \int_s^1 f(r, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^1 Z_r^{t,x} dW_r,\ \forall s \in [t,1] \quad a.s.$$

We shall extend $Y_s^{t,x}$, $Z_s^{t,x}$, $U_s^{t,x}$, for all $s \in [0,1]$ by choosing $Y_s^{t,x} = Y_t^{t,x}$, $Z_s^{t,x} = 0$, $U_s^{t,x} = 0$, for all $s \in [0,t]$.

1.4.2 Viscosity solution to variational inequalities

We deal with the connection between the reflected BSDE studied in the Markovian framework and the following multivalued parabolic partial differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),(\bigtriangledown u\sigma)(t,x)) \in \partial\phi(u(t,x)) \\ t \in [0,T],\ x \in \mathbb{R}^d \\ u(T,x) = g(x),\ \ x \in \mathbb{R}^d. \end{cases} \tag{1.7}$$

Definition 1.20. *Let* $u \in \mathcal{C}([0,T] \times \mathbb{R})$ *and* $(t,x) \in [0,T] \times \mathbb{R}$*. We denote by* $\mathcal{P}^{2+}u(t,x)$ *(the parabolic superjet of u at* (t,x)*) the set of triple* $(p,q,X) \in \mathbb{R} \times \mathbb{R}^d \times \mathcal{M}_{d\times d}$ *which are such that*

$$u(s,y) \ \ \leq \ \ u(t,x) + p(s-t) + \langle q, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + o\left(|t-s| + |y-x|^2\right)$$

$\mathcal{P}^{2-}u(t,x)$ *(the parabolic subjet of u at* (t,x)*) is defined similarly as the set of triples* $(p,q,X) \in \mathbb{R} \times \mathbb{R}^d \times \mathcal{M}_{d\times d}$ *which are such that*

$$u(s,y) \ \ \geq \ \ u(t,x) + p(s-t) + \langle q, y-x\rangle + \frac{1}{2}\langle X(y-x), y-x\rangle + o\left(|t-s| + |y-x|^2\right).$$

Example 1.21. *Suppose that* $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$*. If* $u - \varphi$ *has a local maximum at* (t,x)*, then*

$$\left(\frac{\partial\varphi}{\partial t}(t,x), \nabla_x\varphi(t,x), \frac{\partial^2\varphi}{\partial x^2}(t,x)\right) \in \mathcal{P}^{2+}u(t,x).$$

If $u - \varphi$ *has a local minimum at* (t,x)*, then*

$$\left(\frac{\partial\varphi}{\partial t}(t,x), \nabla_x\varphi(t,x), \frac{\partial^2\varphi}{\partial x^2}(t,x)\right) \in \mathcal{P}^{2-}u(t,x).$$

Definition 1.22. *Let* $u \in \mathcal{C}\left([0,T] \times \mathbb{R}^d\right)$ *which satisfies* $u(T,x) = g(x)$.
a) u is a viscosity subsolution of (1.7) if

$$u(t,x) \in Dom(\phi),\ \forall (t,x) \in [0,T] \times \mathbb{R}^d,$$

and at any point $(t,x) \in [0,T] \times \mathbb{R}^d$*, for any* $(p,q,X) \in \mathcal{P}^{2+}u(t,x)$

$$\begin{aligned} &-p - \frac{1}{2}Tr\left((\sigma\sigma^*)(t,x)X\right) - \langle b(t,x), q\rangle \\ &-f(t,x,u(t,x),q\sigma(t,x)) \leq -\phi'_-(u(t,x)). \end{aligned} \tag{1.8}$$

b) u is a viscosity supersolution of (1.7) if

$$u(t,x) \in Dom(\phi),\ \forall (t,x) \in [0,T] \times \mathbb{R}^d$$

and at any point $(t,x) \in [0,T] \times \mathbb{R}^d$, for any $(p,q,X) \in \mathcal{P}^{2-}u(t,x)$

$$\begin{aligned} &-p - \frac{1}{2} Tr\left(\left(\sigma\sigma^* \right) (t,x) X \right) - \langle b(t,x), q \rangle \\ &-f(t,x,u(t,x), q\sigma(t,x)) \geq -\phi'_+ (u(t,x)) . \end{aligned} \tag{1.9}$$

c) u is a viscosity solution of (1.7) if it is both a viscosity sub- and super-solution.

We define

$$u(t,x) = Y_t^{t,x}, \quad (t,x) \in [0,T] \times \mathbb{R}^d, \tag{1.10}$$

which is a deterministic quantity since $Y_t^{t,x}$ is $\mathcal{F}_t$-adapted, and $\mathcal{F}_t$ is trivial σ-algebra.

Theorem 1.23. *The function defined by (1.10) is a viscosity solution to equation (1.7). Furthermore, if we suppose in addition that, for each $R > 0$, there exists a continuous function $\psi_R : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\psi_R(0) = 0$ and*

$$| f(t,x,y,z) - f(t,x',y,z) | \leq \psi_R(| x - x' | (1+ | z |)),$$

for all $t \in [0,T]$, $| x |$, $| x' | \leq R$, $| z | \leq R$, $z \in \mathbb{R}^n$, then u is the unique viscosity solution of PDE (1.7).

Corollary 1.24. *The function u satisfies:*

a) $u(t,x) \in Dom(\phi), \forall (t,x) \in [0,T] \times \mathbb{R}^d$,

b) $\sup_{x \in \mathbb{R}} |u(t,x)| \leq C, \forall t \in [0,T]$,

c) $u \in \mathcal{C}\left([0,T] \times \mathbb{R}^d \right)$,

where $C > 0$ is a constant independent of t and x.

1.5 Meyer and Zheng Tightness Criterion

In this section, we introduce the notions of pseudo-path topology and quasi-martingales (see Meyer and Zheng [56]). We put
• $\mathbb{D}([0,t], \mathbb{R}^k)$ the space of càdlàg functions of $[0,t]$ with values in $\mathbb{R}^k$.
• $\mathbb{L}^0([0,T], \mathbb{R}^k)$ the space of (equivalence classes of) Borel measurable functions.

To begin with, note that $\mathbb{D} \subset L^0$. For any $u \in L^0$, we define the pseudo-path of u to be a probability measure on $[0,t] \times \overline{\mathbb{R}}$:

$$P^u := \frac{1}{T} \int_0^T 1_A(t, u(t)) dt, \quad \forall A \in \mathcal{B}([0,t] \times \overline{\mathbb{R}}).$$

It can be shown that the mapping $\psi : u \longrightarrow P^u$ is one to one. Thus we can identify all $u \in L^0$ with the pseudo-path; and we denote all pseudo-path by $\mathcal{M}$. In particular, using the mapping ψ, the space $\mathbb{D}$ can then be embedded into the compact space $\overline{\mathcal{P}}$ of all probability laws on the compact space $[0,t] \times \overline{\mathbb{R}}$ (with the Prohorov metric). Clearly

$$\mathbb{D} \subset \mathcal{M} \subset \overline{\mathcal{P}}.$$

The induced topology on $\mathcal{M}$ and $\mathbb{D}$ are known as the *pseudo-path topology* or *Meyer and Zheng toplogy.* We have the following

Lemma 1.25. *(see Meyer and Zheng [56]) The pseudo-path topology on $\mathcal{M}$ is equivalent to the convergence in measure.*

The most significant application of the Meyer and Zheng topology is a tightness result for quasi-martingales, which we now briefly described. Let Y be an $\mathcal{F} := \{\mathcal{F}_t, t \geq 0\}$–adapted, càdlàg process defined on $[0,t]$, such that $\mathbb{E} \mid Y_t \mid < \infty$ for all $t \geq 0$. Let us define

$$V_t(Y) = \mathbb{E}(\sum_i \mid \mathbb{E}\big(Y_{t_{i+1}} - Y_{t_i}/\mathcal{F}_{t_i}\big) \mid \big),$$

and define

$$CV_t(Y) = \sup \mathbb{E}(\sum_i \mid \mathbb{E}\big(Y_{t_{i+1}} - Y_{t_i}/\mathcal{F}_{t_i}\big) \mid \big),$$

with "sup" meaning that the supremum is taken over all partitions of the interval $[0,t]$. If $CV_t(Y) < \infty$, then Y is called a quasi-martingale. We have the following

Theorem 1.26. *(See Meyer-Zheng [56] or Kurtz [47]).*
The sequence of quasi-martingale $\{V_s^n; 0 \leq s \leq t\}$ defined on the filtred probability space $\{\Omega; \mathcal{F}_s, 0 \leq s \leq t; \mathbb{P}\}$ is tight on $\mathbb{D}$ whenever

$$\sup_n \big(\sup_{0 \leq s \leq t} \mathbb{E} \mid V_s^n \mid + CV_t(V^n)\big) < +\infty.$$

Chapter 2

Reflected Backward Stochastic Differential Equations with Jumps and Locally Lipschitz Coefficients

The chapter is organized as follows. In Section 2.1, we prove an existence and uniqueness result for reflected BSDE with locally Lipschitz coefficient. In Section 2.2 we deal with a reflected BSDE with monotone condition with respect to the state variable Y and locally Lipschitz with respect to the variable Z. In Section 2.3, we study the continuous dependence result, or the stability property, for reflected BSDE's. Existence and uniqueness of reflected BSDE, under super-linear growth, is given in Section 2.4.

2.1 Reflected BSDE with jumps and locally Lipschitz coefficient

In this section, we prove existence and uniqueness results of solution for reflected multidimensional backward stochastic differential equation with jumps in d-dimensional convex region. Our contribution in this topic is to weaken the Lipschitz assumption on the data (ξ, f). This is done with locally Lipschitz coefficient f and an only square integrable terminal condition ξ.

2.1.1 Preliminaries

Let $(\Omega, F, I\!P, \mathcal{F}_t, W_t, \mu_t, t \in [0,1])$ be a complete Wiener-Poisson space in $I\!R^n \times I\!R^m \backslash \{0\}$, with Lévy measure λ, i.e. $(\Omega, F, I\!P)$ is a complete probability space, $(\mathcal{F}_t, t \in [0,1])$ is a right continuous increasing family of complete sub $\sigma-$algebras of F, $(W_t, t \in [0,1])$ is a standard Wiener process in $I\!R^n$ with respect to $(\mathcal{F}_t, t \in [0,1])$, and $(\mu_t, t \in [0,1])$ is a martingale measure in $I\!R^m \backslash \{0\}$ independent of $(W_t, t \in [0,1])$, corresponding to a standard Poisson random measure $p(t, A)$, namely, for any Borel measurable subset A of $I\!R^m \backslash \{0\}$ such that $\lambda(A) < \infty$, it holds :

$$\mu_t(A) = p(t, A) - t\lambda(A),$$

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where

$$I\!\!E\left(p(t,A)\right) = t\lambda\left(A\right)$$

λ is assumed to be a $\sigma-$finite measure on $I\!\!R^m\backslash\{0\}$ with its Borel field, satisfying

$$\int_{I\!\!R^m\backslash\{0\}}\left(1\wedge|x|^2\right)\lambda\left(dx\right) < +\infty.$$

In the sequel U denotes $I\!\!R^m\backslash\{0\}$ and $\mathcal{U}$ its Borel field. We assume that

$$\mathcal{F}_t = \sigma\left[\int_{A\times(0,s]} p(ds,dx); s\leq t, A\in\mathcal{U}\right]\vee\sigma\left[W_s, s\leq t\right]\vee\mathcal{N},$$

where $\mathcal{N}$ denotes the totality of $I\!\!P$-null sets and $\sigma_1\vee\sigma_2$ denotes the σ-field generated by $\sigma_1\cup\sigma_2$.

We denote by $\mathbb{L}^0$ the set of $\mathbb{R}^d\times\mathbb{R}^{d\times n}\times L^2\left(U,\mathcal{U},\lambda;I\!\!R^d\right)$–valued processes (Y,Z,U) defined on $\mathbb{R}_+\times\Omega$ which are $\mathcal{F}_t$–adapted and such that:

$$\|(Y,Z,U)\|_0^2 = \mathbb{E}\left(\sup_{0\leq t\leq 1}|Y_t|^2 + \int_0^1|Z_s|^2ds + \int_0^1\int_U|\,U_s(e)\,|^2\,\lambda\left(de\right)ds\right) < +\infty.$$

The couple $(\mathbb{L}^0, \|.\|_0)$ is then a Banach space. If $U=0$, $\mathbb{L}$ stands for $\mathbb{L}^0$

Let us introduce our reflected BSDE with jumps: Given a data (f,ξ) we want to solve the following backward stochastic differential equation:

$$Y_t = \xi + \int_t^1 f(s,Y_s,Z_s)ds - \int_t^1 Z_sdW_s + K_1 - K_t - \int_t^1\int_U U_s\left(e\right)\mu\left(de,ds\right). \tag{2.1}$$

Definition 2.1. *A solution of reflected BSDE with jumps (2.1) is a quadruple* $\left(Y_t,Z_t,K_t,U_t\right),\ 0\leq t\leq 1$ *of progressively measurable processes taking values in* $\mathbb{R}^d\times\mathbb{R}^{d\times n}\times\mathbb{R}^d\times L^2\left(U,\mathcal{U},\lambda;I\!\!R^d\right)$ *and satisfying:*

$$\begin{cases}(1)\,(Y,Z,U)\in\mathbb{L}^0\\ (2)\ Y_t=\xi+\int_t^1 f(s,Y_s,Z_s,U_s)ds-\int_t^1 Z_sdW_s-\int_t^1\int_U U_s\left(e\right)\mu\left(de,ds\right)+K_1-K_t\\ (3)\ \textit{the process } Y \textit{ is right continuous having left-hand limits (càdlàg)}\\ (4)\,K \textit{ is absolutely continuous, } K_0=0, \textit{ and } \int_0^{\cdot}\left(Y_t-\alpha_t\right)dK_t\leq 0\\ \textit{for every } \alpha_t \textit{ progressively measurable process which is right continuous having}\\ \textit{left-hand limits and takes values into } \overline{\Theta}\\ (5)\ Y_t\in\overline{\Theta},\ 0\leq t\leq 1\ .a.s.\end{cases}$$

We define the following three objects:

A.1 A terminal value $\xi\in L^2\left(\Omega,F_1,I\!\!P\right)$.

A.2 A function process f, which is a map:

$$f:\Omega\times[0,1]\times I\!\!R^d\times I\!\!R^{d\times n}\times L^2\left(U,\mathcal{U},\lambda;I\!\!R^d\right)\longrightarrow I\!\!R^d,$$

such that

(i) f is continuous in (y,z,u) for almost all (t,ω).

(ii) There exist $K>0$ and $0\leq\alpha<1$ such that $|\,f(t,\omega,y,z,u)\,|\leq K\big(1+|\,y\,|+|\,z\,|+\|u\|\big)^{\alpha}$.

A.3 A open subset Θ of $I\!R^d$ convex. We assume that $\xi \in \Theta$.

We denote by Lip_{loc} (resp. Lip) the set of processes f satisfying (i) and which are locally Lipschitz (resp. globally Lipschitz) with respect to (y, z, u).
Let $Lip_{loc,\alpha}$ (resp. Lip_α) denote the subset of processes which belong to Lip_{loc} (resp. Lip) and satisfy **A.2**(ii).
In the sequel L_N denotes the Lipschitz constant of the restriction of f to the ball of $\mathbb{R}^d \times \mathbb{R}^{d\times n} \times L^2\left(U,\mathcal{U},\lambda; I\!R^d\right)$ of radius N.
When the assumptions **A.2**(i), **A.2**(ii), are satisfied, we can define the family of semi norms $(\rho_n(f))_n$

$$\rho_n(f) = (\mathbb{E}\int_0^1 \sup_{|y|,|z|,\|u\|\leq n} \mid f(s,y,z,u) \mid^2 ds)^{\frac{1}{2}}.$$

2.1.2 Existence and uniqueness results

The main results are the following.

Theorem 2.2. *Let $f \in Lip_{loc,\alpha}$ and ξ be a square integrable random variable. Assume moreover that L_N satisfies*

$$\lim_{N\to+\infty} \frac{\exp(L_N^2 + 2L_N)}{(L_N^2 + 2L_N)N^{2(1-\alpha)}} = 0. \tag{2.2}$$

Then the reflected BSDE with jumps has one and only one solution $\{(Y_t, Z_t, U_t, K_t); 0 \leq t \leq 1\}$. In particular, if $L_N \leq \sqrt{(1-\alpha)\log(N)}$, the reflected BSDE with jumps has also a unique solution.

Remark 2.3. *It is well known that if the generator f is uniformly Lipschitz with bounded Lipschitz constants, then there exists a unique solution of BSDE (see [67] or [61]). This last assumption is usually not satisfied in many problems. For example, the classical pricing problem is equivalent to solve a one dimensional linear BSDE*

$$dY_t = (r_t Y_t + Z_t^* \theta_t)dt - Z_t^* dW_t; \quad Y_1 = \xi,$$

where ξ is the contingent claim to price and to hedge. In this model, r is the short rate of the interest and θ is the risk premium vector. To suppose that the short rate r is uniformly bounded is an assumption rarely satisfied in a market. The same remark for the risk premium vector.

The following corollary shows, in particular, that our result can allow the linear growth to the coefficient f (i.e $\alpha = 1$) and hence it cover the classical globally Lipschitz case.

Corollary 2.4. *Assume that $\alpha \leq 1$ and there exists a positive constant L such that, $L_N \leq L + \sqrt{(1-\alpha)\log(N)}$. Then our reflected BSDE has a unique solution.*

Proof of Corollary 2.4. If $\alpha = 1$ it is the Lipschitzian case and the result follows from Pardoux and Peng [67]. The case $\alpha < 1$, can be proved by the technics which will be developed in the proof of Theorem 2.2 below.

Remark 2.5. *As a result, there is existence and uniqueness of a solution if we replace Θ with $Domain(\phi)$, where ϕ is a convex, lower semi-continuous and proper function.*

It should be noted that the jump part behaves as the Brownian stochastic integral part (see Ouknine [61]), therefore for the proof of Theorem 2.2 we will deal, for the simplicity, with reflected BSDE without jump part.

Now, let us introduce our reflected BSDE. The solution is a triplet $(Y_t, Z_t, K_t)\,,\ 0 \le t \le 1$ of progressively measurable processes taking values in $\mathbb{R}^d \times \mathbb{R}^{d\times n} \times \mathbb{R}^d$ and satisfying:

$$\begin{cases} (1r)\,(Y,Z) \in \mathbb{L} \\ (2r)\,Y_t = \xi + \displaystyle\int_t^1 f(s,Y_s,Z_s)ds - \int_t^1 Z_s dW_s + K_1 - K_t, \quad 0 \le t \le 1 \\ (3r)\,\text{the process } Y \text{ is continuous} \\ (4r)\,\text{K is absolutely continuous, } K_0 = 0, \text{ and } \displaystyle\int_0^{\cdot} (Y_t - \alpha_t)\,dK_t \le 0 \\ \quad\text{for every } \alpha_t \text{ progressively measurable process which is continuous} \\ \quad\text{and takes values into } \overline{\Theta} \\ (5r)\,Y_t \in \overline{\Theta}, \quad 0 \le t \le 1\,.a.s. \end{cases}$$

In order to prove Theorem 2.2, we need the following auxiliary lemmas.

Lemma 2.6. *Let f be a process which belongs to $Lip_{loc,\alpha}$ and satisfies the assumptions **A.2**(i), **A.2**(ii). Then there exists a sequence of processes f_n such that*
(a) For each n, $f_n \in Lip_\alpha$.
(b) For every p, $\rho_p(f_n, f) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof . Let ψ_n be a sequence of smooth functions with support in the ball $B(0, n+1)$ and such that $\psi_n = 1$ in the ball $B(0,n)$. It is not difficult to see that the sequence (f_n) of truncated functions, defined by $f_n = f\psi_n$, satisfies all the properties quoted in Lemma 2.6.

■

Let (f_n) be the sequence of processes associated to f by Lemma 2.6. We get from Ouknine [61] that there exists a unique triplet $\{(Y_t^n, Z_t^n, K_t^n; 0 \le t \le 1)\}$ of progressively measurable processes taking values in $\mathbb{R}^d \times \mathbb{R}^{d\times n} \times \mathbb{R}^d$ and satisfying:

$$\begin{cases} (1')\ Z^n \text{ is a predictable process and } \mathbb{E}\displaystyle\int_0^1 \mid Z_t^n \mid^2 dt < +\infty \\ (2')\ Y_t^n = \xi + \displaystyle\int_t^1 f_n(s, Y_s^n, Z_s^n)ds - \int_t^1 Z_s^n dW_s + K_1^n - K_t^n \\ (3')\ \text{the process } Y^n \text{ is continuous} \\ (4')\ K^n \text{ is absolutely continuous, } K_0^n = 0, \text{ and } \displaystyle\int_0^{\cdot} (Y_t^n - \alpha_t)\,dK_t^n \le 0 \\ \quad\text{for every } \alpha_t \text{ progressively measurable process which is continuous} \\ \quad\text{and takes values into } \overline{\Theta} \\ (5')\ Y_t^n \in \overline{\Theta},\ 0 \le t \le 1.\,a.s. \end{cases}$$

We formulate uniform estimates for the processes (Y^n, Z^n, K^n) in the following way.

Lemma 2.7. *Let assumptions **A.1**, **A.2** hold. Then, there exists a constant C depending only in K and $\mathbb{E} \mid \xi \mid^2$, such that*

$$\mathbb{E}(\sup_{0\le t\le 1} |Y_t^n|^2 + \int_0^1 |Z_s^n|^2\,ds + \mid K_1^n \mid^2) \le C,\ \forall\, n \in \mathbb{N}^*.$$

Proof : Using Itô's formula we obtain,

$$|Y_t^n|^2 + \int_t^1 |Z_s^n|^2\,ds = |\,\xi\,|^2 + 2\int_t^1 f_n(s, Y_s^n, Z_s^n) Y_s^n ds - 2\int_t^1 Z_s^n Y_s^n dW_s + 2\int_t^1 Y_s^n dK_s^n,$$

Note that we can assume, without loss of generality, that $0 \in \Theta$. Hence, by relation (4'), we have

$$\int_t^1 Y_s^n dK_s^n \leq 0.$$

Taking expectation in both sides in the above equation, we get

$$\mathbb{E}\,|Y_t^n|^2 + \mathbb{E}\int_t^1 |Z_s^n|^2\,ds \leq \mathbb{E}\,|\,\xi\,|^2 + 2\mathbb{E}\int_t^1 f_n(s, Y_s^n, Z_s^n) Y_s^n ds.$$

Hence, using the elementary inequality $2ab \leq \beta^2 a^2 + \frac{1}{\beta^2} b^2$ and the fact that $f_n \in Lip_\alpha$, we have

$$\begin{aligned}
&\mathbb{E}\,|Y_t^n|^2 + \mathbb{E}\int_t^1 |Z_s^n|^2\,ds \\
&\leq \mathbb{E}\,|\,\xi\,|^2 + \beta^2\mathbb{E}\int_t^1 |Y_s^n|^2\,ds + \frac{1}{\beta^2}\mathbb{E}\int_t^1 |\,f_n(s, Y_s^n, Z_s^n)\,|^2\,ds \\
&\leq \mathbb{E}\,|\,\xi\,|^2 + \beta^2\mathbb{E}\int_t^1 |Y_s^n|^2\,ds + \frac{K}{\beta^2}\mathbb{E}\int_t^1 (1+ |\,Y_s^n\,| + |\,Z_s^n\,|)^{2\alpha} ds \\
&\leq \mathbb{E}\,|\,\xi\,|^2 + \beta^2\mathbb{E}\int_t^1 |Y_s^n|^2\,ds + \frac{C}{\beta^2} + \frac{C}{\beta^2}\mathbb{E}\int_t^1 |Y_s^n|^2\,ds + \frac{C}{\beta^2}\mathbb{E}\int_t^1 |Z_s^n|^2\,ds,
\end{aligned}$$

where C is a constant which can be changed from line to line. Choosing $\frac{C}{\beta^2} = \frac{1}{2}$, we obtain

$$\mathbb{E}\,|Y_t^n|^2 + \mathbb{E}\int_t^1 |Z_s^n|^2\,ds \leq C(1 + \mathbb{E}\int_t^1 |Y_s^n|^2\,ds), \tag{2.3}$$

Gronwall's lemma applied to Y^n gives

$$\sup_{0\leq t\leq 1} \mathbb{E}\,|Y_t^n|^2 \leq C, \tag{2.4}$$

from this last inequality and (2.3), we obtain

$$\sup_n (\mathbb{E}\int_t^1 |Z_s^n|^2\,ds) < \infty. \tag{2.5}$$

Now, from (2')

$$K_1^n - K_t^n = Y_t^n - \xi - \int_t^1 f_n(s, Y_s^n, Z_s^n) ds + \int_t^1 Z_s^n dW_s,$$

then

$$\mathbb{E}\,|\,K_1^n - K_t^n\,|^2 \leq C(\mathbb{E}\,|\,\xi\,|^2 + \mathbb{E}\,|Y_t^n|^2 + 1 + \mathbb{E}\int_t^1 |\,Y_s^n\,|^2\,ds + \mathbb{E}\int_t^1 |Z_s^n|^2\,ds),$$

and from (2.4) and (2.5), we deduce that

$$\sup_n \mathbb{E}\,|\,K_1^n\,|^2 \leq C.$$

Using the Burkholder-Davis-Gundy inequality, we obtain

$$\mathbb{E}\left(\sup_{0\leq t\leq 1} |Y_t^n|^2 + \int_0^1 |Z_s^n|^2\,ds + |\,K_1^n\,|^2\right) \leq C,\ \forall\, n \in \mathbb{N}^*.$$

Hence, Lemma 2.7 is proved. ■

We shall prove the convergence of the sequence $(Y^n, Z^n, K^n)_n$, $n \in \mathbb{N}^*$.

Lemma 2.8. *Under assumptions of Theorem 2.2, there exist (Y,Z,K) such that*

$$\lim_{n\to\infty} \mathbb{E}\{\sup_{0\le t\le 1} \mid Y_1^n - Y_t \mid^2 + \sup_{0\le t\le 1} \mid K_1^n - K_t \mid^2 + \int_0^1 |Z_s^n - Z_t|^2\, ds\} = 0.$$

Proof : It follows from Itô's formula that

$$\begin{aligned}
&|Y_t^n - Y_t^m|^2 + \int_t^1 |Z_s^n - Z_s^m|^2\, ds \\
&= 2\int_t^1 (Y_s^n - Y_s^m)^* (f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m))ds \\
&- 2\int_t^1 (Y_s^n - Y_s^m)^* (Z_s^n - Z_s^m)\, dW_s + 2\int_t^1 (Y_s^n - Y_s^m)\,(dK_s^n - dK_s^m).
\end{aligned}$$

Since $Y^n, Y^m \in \overline{\Theta}$ are progressively measurable and continuous processes, then from (4') it follows that

$$\int_t^1 (Y_s^n - Y_s^m)\, dK_s^n \le 0 \text{ and } \int_t^1 (Y_s^m - Y_s^n)\, dK_s^m \le 0,$$

then

$$\int_t^1 (Y_s^n - Y_s^m)\,(dK_s^n - dK_s^m) = \int_t^1 (Y_s^n - Y_s^m)\, dK_s^n + \int_t^1 (Y_s^m - Y_s^n)\, dK_s^m \le 0.$$

For an arbitrary number $N > 1$, let L_N be the Lipschitz constant of f in the ball $B(0, N)$. We put $A_{n,m}^N := \{(s, \omega); \mid Y_s^n \mid + \mid Z_s^n \mid + \mid Y_s^m \mid + \mid Z_s^m \mid \ge N\}$, $\overline{A}_{n,m}^N := \Omega \setminus A_{n,m}^N$.
Taking the expectation in the above equation, we deduce that

$$\begin{aligned}
&\mathbb{E} \mid Y_t^n - Y_t^m \mid^2 + \mathbb{E}\int_t^1 \mid Z_s^n - Z_s^m \mid^2\, ds \\
&\le 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)\rangle \mathbf{1}_{A_{n,m}^N} ds \\
&+ 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n)\rangle \mathbf{1}_{\overline{A}_{n,m}^N} ds \\
&+ 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m)\rangle \mathbf{1}_{\overline{A}_{n,m}^N} ds \\
&+ 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f(s, Y_s^m, Z_s^m) - f_m(s, Y_s^m, Z_s^m)\rangle \mathbf{1}_{\overline{A}_{n,m}^N} ds \\
&= J_1(n, m, N) + J_2(n, m, N) + J_3(n, m, N) + J_4(n, m, N)
\end{aligned}$$

It is not difficult to check that,

$$\begin{aligned}
J_2(n, m, N) &= 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n)\rangle \mathbf{1}_{\overline{A}_{n,m}^N} ds \\
&\le \mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 ds + \rho_N^2(f_n - f).
\end{aligned}$$

Likewise we show that,

$$\begin{aligned}
J_4(n, m, N) &= 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f(s, Y_s^m, Z_s^m) - f_m(s, Y_s^m, Z_s^m)\rangle \mathbf{1}_{\overline{A}_{n,m}^N} ds \\
&\le \mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 ds + \rho_N^2(f_m - f).
\end{aligned}$$

Now

$$J_1(n,m,N) = 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f_n(s,Y_s^n,Z_s^n) - f_m(s,Y_s^m,Z_s^m)\rangle \mathbf{1}_{A_{n,m}^N} ds$$
$$\leq \beta^2\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2\mathbf{1}_{A_{n,m}^N} ds + \frac{1}{\beta^2}\mathbb{E}\int_t^1 \mid f_n(s,Y_s^n,Z_s^n) - f_m(s,Y_s^m,Z_s^m)\mid^2 \mathbf{1}_{A_{n,m}^N} ds$$
$$\leq \beta^2\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2\mathbf{1}_{A_{n,m}^N} ds + \frac{2K}{\beta^2}\mathbb{E}\int_t^1 (1+\mid Y_s^n\mid + \mid Z_s^n\mid + \mid Y_s^m\mid + \mid Z_s^m\mid)^{2\alpha}\mathbf{1}_{A_{n,m}^N} ds$$
$$\leq \beta^2\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2\mathbf{1}_{A_{n,m}^N} ds + \frac{C_1(K,\xi)}{\beta^2 N^{2(1-\alpha)}}\mathbb{E}\int_t^1 (1+\mid Y_s^n\mid + \mid Z_s^n\mid + \mid Y_s^m\mid + \mid Z_s^m\mid)^{2} ds$$
$$\leq \beta^2\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2\mathbf{1}_{A_{n,m}^N} ds + \frac{C_2(K,\xi)}{\beta^2 N^{2(1-\alpha)}}.$$

Hence

$$J_1(n,m,N) \leq \beta^2\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2\mathbf{1}_{A_{n,m}^N} ds + \frac{C_2(K,\xi)}{\beta^2 N^{2(1-\alpha)}}.$$

Since f is L_N-locally Lipschitz we get

$$J_3(n,m,N) = 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f(s,Y_s^n,Z_s^n) - f(s,Y_s^m,Z_s^m)\rangle \mathbf{1}_{\overline{A}_{n,m}^N} ds$$
$$\leq (2L_N + \gamma^2)\mathbb{E}\int_t^1 |Y^n - Y^m|^2\mathbf{1}_{\overline{A}_{n,m}^N} ds + \frac{L_N^2}{\gamma^2}\mathbb{E}\int_t^1 |Z_s^n - Z_s^m)|^2 ds.$$

If we choose $\beta^2 = L_N^2 + 2L_N$ and $\gamma^2 = L_N^2$ then we use the above estimates we have

$$\mathbb{E}(|Y_t^n - Y_t^m|^2) + \mathbb{E}\int_t^1 |Z_s^n - Z_s^m|^2 ds \leq (L_N^2 + 2L_N + 2)\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 ds$$
$$+ [\rho_N^2(f_n - f) + \rho_N^2(f_m - f)] + \frac{C_3(K,\xi)}{(L_N^2 + 2L_N)N^{2(1-\alpha)}}.$$

It follows from Gronwall's lemma that, for every $t \in [0,1]$,

$$\mathbb{E}(|Y_t^n - Y_t^m|^2) \leq \left[[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)] + \frac{C_4(K,\xi)}{(L_N^2 + 2L_N)N^{2(1-\alpha)}}\right]\exp(L_N^2 + 2L_N + 2).$$

Using Burkholder-Davis-Gundy inequality, we show that there exists a universal positive constant C such that,

$$\mathbb{E}(\sup_{0\leq t\leq 1} \mid Y_t^n - Y_t^m\mid^2) \leq C\Bigg[[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)]$$
$$+ \frac{C_5(K,\xi)}{(L_N^2 + 2L_N)N^{2(1-\alpha)}}\Bigg]\exp(L_N^2 + 2L_N + 2).$$

$$\mathbb{E}\int_0^1 |Z_s^n - Z_s^m|^2\, ds \leq C\Bigg[[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)]$$
$$+ \frac{C_5(K,\xi)}{(L_N^2 + 2L_N)N^{2(1-\alpha)}}\Bigg]\exp(L_N^2 + 2L_N + 2).$$

Passing to the limit on n, m and on N, we show that $(Y^n, Z^n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in the Banach space of progressively measurable processes $\mathbb{L}$.
We set

$$Y = \lim_{n\longrightarrow+\infty} Y^n, \text{ and } Z = \lim_{n\longrightarrow+\infty} Z^n.$$

If we return to the equation satisfied by the triple $(Y^n, Z^n, K^n)_{n\in\mathbb{N}^*}$, we see that

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq 1} \mid K_t^n - K_t^m \mid^2 \leq C\Bigg[&\mathbb{E}\sup_{0\leq t\leq 1} \mid Y_t^n - Y_t^m \mid^2 \\ &+ \mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m) \mid^2 ds \\ &+ \mathbb{E}\int_0^1 \mid Z_s^n - Z_s^m \mid^2 ds\Bigg].\end{aligned}$$

We shall prove that the sequence of processes $f_n(., Y^n, Z^n)_n$ converges to $f(., Y, Z)$ in $L^2([0,1]\times\Omega)$

$$\begin{aligned}&\mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 ds \\ &\leq \mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 \mathbf{1}_{A_{n,m}^N} ds \\ &+ 2\mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n) \mid^2 \mathbf{1}_{\overline{A}_{n,m}^N} ds \\ &+ 2\mathbb{E}\int_0^1 \mid f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 \mathbf{1}_{\overline{A}_{n,m}^N} ds \\ &\leq \frac{C}{N^{2(1-\alpha)}}\sup_n \mathbb{E}\int_0^1 (1+ \mid Z_s^n \mid^2 + \mid Z_s \mid^2 + \mid Y_s^n \mid^2 + \mid Y_s \mid^2) ds \\ &+ 2\rho_N^2(f_n - f) + 2L_N^2 \mathbb{E}\int_0^1 \left(\mid Z_s^n - Z_s \mid^2 ds + \mid Y_s^n - Y_s \mid^2\right) ds.\end{aligned}$$

Hence

$$\begin{aligned}&\mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 ds \\ &\qquad \leq 2\rho_N^2(f_n - f) + 2L_N^2\left(\mathbb{E}\int_0^1 \mid Y_s^n - Y_s \mid^2 ds + \mathbb{E}\int_0^1 \mid Z_s^n - Z_s \mid^2 ds\right) + \frac{C(K,\xi)}{N^{2(1-\alpha)}}.\end{aligned}$$

Passing to the limit successively on n and N, we obtain

$$\mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 ds \longrightarrow 0, \text{ as } n \to \infty.$$

Now

$$\mathbb{E}\sup_{0\leq t\leq 1} \mid K_t^n - K_t^m \mid^2 \longrightarrow 0, \text{ as } n, m \to \infty.$$

Consequently there exists a progressively measurable process K such that

$$\mathbb{E}\sup_{0\leq t\leq 1} \mid K_t^n - K_t \mid^2 \longrightarrow 0, \text{ as } n \to \infty,$$

and clearly (K_t) is increasing (with $K_0 = 0$) and a continuous process. ∎

Proof of Theorem 2.2: Combining Lemmas 2.6, 2.7, 2.8 and passing to the limit in the RBSDE (2'), we show that the triplet $\{(Y_t, Z_t, K_t); 0 \leq t \leq 1\}$ is a solution of our RBSDE. In order to finish the proof of Theorem 2.2, it remains to check (1r), (4r) and (5r).
From Lemma 2.7, we have

$$\mathbb{E}\int_0^1 (\mid Y_s^n \mid^2 + \mid Z_s^n \mid^2) ds \leq C.$$

from which (1r) follows by using Lemma 2.8 and Fotou's Lemma.
Let α be a continuous process with values in $\overline{\Theta}$, it holds that

$$\langle Y^n(t) - \alpha(t), dK^n(t)\rangle \leq 0,$$

by Shaisho [75] (see also Lemma 3.5 of Chapter 3), we obtain

$$\langle Y(t) - \alpha(t), dK(t)\rangle \leq 0.$$

To finish the proof of our existence result, we shall show that,

$$\mathbb{P}\left\{Y_t \in \overline{\Theta}; 0 \leq t < +\infty\right\} = 1.$$

Since the process (Y_t) is continuous, it suffices to prove that

$$\mathbb{P}\left\{Y_t \in \overline{\Theta}\right\} = 1 \quad \forall\, t \geq 0.$$

Since, $Y^n \in \overline{\Theta}$, and Y^n converges to Y in L^2, there exists a subsequence Y^{n_k} such that $Y^{n_k} \longrightarrow Y$ a.s, hence $Y \in \overline{\Theta}$. ■

Uniqueness : Let $\{(Y_t, Z_t, K_t)\,;\ 0 \leq t \leq 1\}$ and $\{(Y'_t, Z'_t, K'_t)\,;\ 0 \leq t \leq 1\}$ be two solutions of our BSDE. Define

$$\{(\Delta Y_t, \Delta Z_t, \Delta K_t)\,;\ 0 \leq t \leq 1\} = \left\{\left(Y_t - Y'_t, Z_t - Z'_t, K_t - K'_t\right)\,;\ 0 \leq t \leq 1\right\}.$$

It follows from Itô's formula that

$$\mathbb{E}\left[|\Delta Y_t|^2 + \int_t^1 |\Delta Z_s|^2\, ds\right]$$

$$= 2\mathbb{E}\int_t^1 \left\langle \Delta Y_s, f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)\right\rangle ds + 2\mathbb{E}\int_t^1 \langle \Delta Y_s, d\Delta K_s\rangle\,.$$

By Shaisho [75], we get

$$\mathbb{E}\int_t^1 \langle \Delta Y_s, d\Delta K_s\rangle \leq 0.$$

Now, for $N > 1$, let L_N the Lipschitz constant of f in the balls $B(0, N)$, $A_N := \{(s, w); \mid Y_s \mid + \mid Y'_s \mid + \mid Z_s \mid + \mid Z'_s \mid \geq N\}$, $A_N^c := \Omega \setminus A_N$.

$$\begin{aligned}\mathbb{E}\left[|\Delta Y_t|^2 + \int_t^1 |\Delta Z_s|^2\, ds\right] &\leq \beta^2 \mathbb{E}\int_t^1 \mid \Delta Y_s \mid^2 1_{A_N} ds \\ &+ \frac{1}{\beta^2}\mathbb{E}\int_t^1 \mid f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s) \mid^2 1_{A_N} ds \\ &+ \mathbb{E}\int_t^1 \langle \Delta Y_s, \quad f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)\rangle 1_{A_N^c} ds,\end{aligned}$$

as in the proof of Lemma 2.8, we obtain that

$$\mathbb{E}\left[|\Delta Y_t|^2 + \int_t^1 |\Delta Z_s|^2\, ds\right] \leq \beta^2 \mathbb{E}\int_t^1 |\Delta Y_s|^2\, 1_{A_N} ds + \frac{C(K,\xi)}{\beta^2 N^{2(1-\alpha)}}$$
$$+ (2L_N + \gamma^2)\mathbb{E}\int_t^1 |\Delta Y_s|^2\, 1_{A_N^c} ds + \frac{L_N^2}{\gamma^2}\mathbb{E}\int_t^1 |\Delta Z_s|^2\, ds.$$

If we choose β and γ such that $\beta^2 = L_N^2 + 2L_N$ and $\gamma^2 = L_N^2$, and using Gronwall's and Burkholder-Davis-Gundy inequality, we get

$$\mathbb{E}\sup_{0\leq t\leq 1} |\Delta Y_t|^2 \leq \frac{C(K,\xi)}{(L_N^2 + 2L_N)N^{2(1-\alpha)}} e^{L_N^2 + 2L_N}.$$

and

$$\mathbb{E}\int_0^1 |\Delta Z_s|^2\, ds \leq \frac{C(K,\xi)}{(L_N^2 + 2L_N)N^{2(1-\alpha)}} e^{L_N^2 + 2L_N},$$

from which the uniqueness follows.
Now let us prove the second result. If $L_N = \sqrt{(1-\alpha)\log(N)}$, arguing as above and using that $2L_N \leq \frac{1}{4}L_N^2 + 16$, to show that

$$\mathbb{E}(\sup_{0\leq t\leq 1} \mid Y_t^n - Y_t^m \mid^2) \leq C\left[[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)]N^{2(1-\alpha)} + \frac{C(K,\xi)}{(L_N^2 + 2L_N)}\right]\exp(18).$$

$$\mathbb{E}\int_0^1 |Z_t^n - Z_t^m|^2\, ds \leq C\left[[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)]N^{2(1-\alpha)} + \frac{C(K,\xi)}{(L_N^2 + 2L_N)}\right]\exp(18),$$

from which the result follows. Theorem 2.2 is proved. ■

Remark 2.9. *Assume that the generator f is locally Lipschitz in Y with locally Lipschitz constant L_N and globally Lipschitz in Z. Then, if L_N behaves as $\log(N)$ or satisfies the following condition*

$$\lim_{N\to+\infty} \frac{\exp(2L_N)}{(2L_N)N^{2(1-\alpha)}} = 0.$$

Then, our reflected BSDE with jumps has a unique solution.

2.2 Monotone case

The aim of this section is to prove similar results in the case where the generator f is monotone on its Y-variable and locally lipschitz on its Z-variable. The existence and uniqueness results was established by Pardoux in [66] in the case where the generator is globally Lipschitz with respect to Z . Our results can be seen as a localization of the ones given by Pardoux in [66].
In the sequel, the following assumptions will be fulfilled:

(i) f is continuous in (y, z) for almost all (t, ω).
(ii) There exist $K > 0$ and $0 \leq \alpha < 1$ such that

$$\mid f(t,\omega,y,z) \mid \leq K(1+ \mid y \mid + \mid z \mid)^{\alpha}.$$

iii) For each $N > 0$, there exist L and L_N such that:

$$\langle y - y', f(t,y,z) - f(t,y',z)\rangle \leq L \mid y - y' \mid^2,$$
$$\mid f(t,y,z) - f(t,y,z') \mid \leq L_N \mid z - z' \mid; \mid y \mid, \mid z \mid, \mid z' \mid \leq N.$$

Theorem 2.10. *Let f as above and ξ be a square integrable random variable. Suppose that L_N satisfies*

$$\lim_{N\to+\infty} \frac{\exp(L_N^2)}{L_N^2 N^{2(1-\alpha)}} = 0. \tag{2.6}$$

Then the reflected BSDE has one and only one solution $\{(Y_t, Z_t, K_t); 0 \leq t \leq 1\}$.

Example 2.11. *If $L_N \leq \sqrt{(1-\alpha)\log(N)}$, then our equation has a unique solution.*

Proof of Theorem 2.10: Using the same approximating sequence, one can prove that the approximating solutions converge in mean square to the right solution . The only problem is to show that

$$\mathbb{E}\int_t^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 ds \longrightarrow 0, \text{ as } n \to \infty.$$

But this is a consequence of the convergence in probability and the uniform integrability of the sequence $f_n(s, Y_s^n, Z_s^n)$ (see Chapter 3). ■

Remark 2.12. *Theorem 2.2 and Theorem 2.10 remain true if* $L_N = \sqrt{(1-\alpha)\log(N) + \frac{1}{2}\log\log\log(N)}$.

Remark 2.13. *What we have shown is the link between the smoothness of the generator and its growth at infinity. Our results, on existence and uniqueness, remains true if we impose a weaker condition on the growth of f namely* $\dfrac{\mid f(t,\omega,y,z)\mid}{(1+\mid y \mid + \mid z \mid)}$ *converges to 0 when* $(\mid y \mid + \mid z \mid)$ *converges to ∞;* $(1-\alpha)\log(N)$ *must be replaced by* $\log\psi(N)$ *where,* $\psi(N) = \sup\limits_{|y|+|z|\geq N} \dfrac{\mid f(t,\omega,y,z)\mid}{(1+\mid y \mid + \mid z \mid)}$.

2.3 Stability result for reflected BSDE's

In this section, we prove a stability result for reflected backward stochastic differential equations under locally Lipschitz coefficient. Let f_n be a sequence of processes which satisfies **A.2**(ii) for each $n \in \mathbb{N}$ and ξ^n be a sequence of random variables such that $\mathbb{E} \mid \xi^n \mid^2 < +\infty$. Consider the following BSDE's

$$\begin{cases} (1n)\ Z^n \text{ is a predictable process and } \mathbb{E}\displaystyle\int_0^1 \mid Z_t^n \mid^2 dt < +\infty \\ (2n)\ Y_t^n = \xi^n + \displaystyle\int_t^1 f_n(s, Y_s^n, Z_s^n) ds - \int_t^1 Z_s^n dW_s + K_1^n - K_t^n \\ (3n) \text{ the process } Y^n \text{ is continuous} \\ (4n)\ K^n \text{ is absolutely continuous, } K_0^n = 0, \text{ and } \displaystyle\int_0^{\cdot} (Y_t^n - \alpha_t)\, dK_t^n \leq 0 \\ \quad \text{for every } \alpha_t \text{ progressively measurable process which is continuous} \\ \quad \text{and takes values into } \overline{\Theta} \\ (5n)\ Y_t^n \in \overline{\Theta},\ 0 \leq t \leq 1.\ a.s. \end{cases}$$

where Y^n, Z^n, K^n take values in $\mathbb{R}^d$, $\mathbb{R}^{d\times n}$ and $\mathbb{R}^d$ respectively.

Let consider the following assumptions:

A.4 for $N \in \mathbb{N}$, $\rho_N(f_n - f) \longrightarrow 0$,

A.5 $\mathbb{E} \mid \xi^n - \xi \mid^2 \longrightarrow 0$, as $n \to +\infty$.

We assume also that equation (1n)-(5n) has a solution.

Theorem 2.14. *Let **A.4**, **A.5** be satisfied and $f \in Lip_{loc,\alpha}$. Assume that L_N satisfies $L_N \le \sqrt{(1-\alpha)\log(N)}$. Then, we have the following strong convergence for all $t \in [0,1]$*

$$\mathbb{E} \mid Y_t^n - Y_t \mid^2 + \mathbb{E}\int_0^1 \mid Z_s^n - Z_s \mid^2 ds + \mathbb{E} \mid K_t^n - K_t \mid^2 \longrightarrow 0, \quad as \quad n \to +\infty.$$

Remark 2.15. *It should be noted that Theorem 2.14 remains true if L_N satisfies condition (2.2).*

Proof of Theorem 2.14: By Itô's formula we have

$$\begin{aligned}
&|Y_t^n - Y_t|^2 + \int_t^1 |Z_s^n - Z_s|^2 \, ds \\
&= |\xi^n - \xi|^2 + 2\int_t^1 (Y_s^n - Y_s)^* \left(f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)\right) ds \\
&- 2\int_t^1 (Y_s^n - Y_s)^* (Z_s^n - Z_s) \, dW_s + 2\int_t^1 (Y_s^n - Y_s) (dK_s^n - dK_s).
\end{aligned}$$

Since $Y^n, Y \in \overline{\Theta}$, progressively measurable and continuous processes, then by (4r) we have

$$\int_t^1 (Y_s^n - Y_s) \, dK_s^n \le 0 \text{ and } \int_t^1 (Y_s - Y_s^n) \, dK_s \le 0,$$

thus

$$\int_t^1 (Y_s^n - Y_s) (dK_s^n - dK_s) = \int_t^1 (Y_s^n - Y_s) \, dK_s^n + \int_t^1 (Y_s - Y_s^n) \, dK_s \le 0.$$

For an arbitrary number $N > 1$, let L_N be the Lipschitz constant of f in the ball $B(0, N)$. We put $B_{n,N} := \{(s, \omega); \mid Y_s^n \mid + \mid Z_s^n \mid + \mid Y_s \mid + \mid Z_s \mid \ge N\}$, $B_{n,N}^c := \Omega \setminus B_{n,N}$.

Taking the expectation in the above equation, we show that

$$\begin{aligned}
&\mathbb{E} \mid Y_t^n - Y_t \mid^2 + \mathbb{E}\int_t^1 \mid Z_s^n - Z_s \mid^2 ds \\
&\le \mathbb{E}\,|\xi^n - \xi|^2 + 2\mathbb{E}\int_t^1 (Y_s^n - Y_s)^* \left(f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)\right) 1_{B_{n,N}} ds \\
&+ 2\mathbb{E}\int_t^1 (Y_s^n - Y_s)^* \left(f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)\right) 1_{B_{n,N}^c} ds.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E} \mid Y_t^n - Y_t \mid^2 + \mathbb{E}\int_t^1 \mid Z_s^n - Z_s \mid^2 ds \\
&\le \mathbb{E}\,|\xi^n - \xi|^2 + 2\mathbb{E}\int_t^1 (Y_s^n - Y_s)^* \left(f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)\right) 1_{B_{n,N}} ds \\
&+ 2\mathbb{E}\int_t^1 (Y_s^n - Y_s)^* \left(f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n)\right) 1_{B_{n,N}^c} ds \\
&+ 2\mathbb{E}\int_t^1 (Y_s^n - Y_s)^* \left(f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)\right) 1_{B_{n,N}^c} ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathbb{E} \mid Y_t^n - Y_t \mid^2 + \mathbb{E}\int_t^1 \mid Z_s^n - Z_s \mid^2 ds \\
&\leq \mathbb{E} \mid \xi^n - \xi \mid^2 + \beta^2 \mathbb{E}\int_t^1 \mid Y_s^n - Y_s \mid^2 1_{B_{n,N}} ds + \frac{C(K,\xi)}{\beta^2 N^{2(1-\alpha)}} \\
&+ \mathbb{E}\int_t^1 \mid Y_s^n - Y_s \mid^2 ds + \rho_N^2(f_n - f) + \beta^2 \mathbb{E}\int_t^1 \mid Y_s^n - Y_s \mid^2 1_{B_{n,N}^c} ds \\
&+ \frac{2L_N^2}{\beta^2}\mathbb{E}\int_t^1 \mid Y_s^n - Y_s \mid^2 ds + \frac{2L_N^2}{\beta^2}\mathbb{E}\int_t^1 \mid Z_s^n - Z_s \mid^2 ds.
\end{aligned}$$

If we choose β such that $\frac{2L_N^2}{\beta^2} = 1$, we obtain

$$\begin{aligned}
&\mathbb{E}\,|Y_t^n - Y_t^m|^2 + \mathbb{E}\int_t^1 \mid Z_s^n - Z_s \mid^2 ds \\
&\qquad \leq \mathbb{E}\,|\xi^n - \xi|^2 + \frac{C(K,\xi)}{L_N^2 N^{2(1-\alpha)}} + \rho_N^2(f_n - f) + (2L_N^2 + 1)\mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m \mid^2 ds.
\end{aligned}$$

and thus, from Gronwall inequality, we get

$$\mathbb{E}\,|Y_t^n - Y_t^m|^2 + \mathbb{E}\int_t^1 \mid Z_s^n - Z_s \mid^2 ds \leq \big[\mathbb{E}\,|\xi^n - \xi|^2 + \frac{C(K,\xi)}{L_N^2 N^{2(1-\alpha)}} + \rho_N^2(f_n - f)\big] e^{(2L_N^2+1)}.$$

Using the fact that $L_N = \sqrt{(1-\alpha)\log(N)}$ and passing to the limit successively on n and N, we obtain

$$\mathbb{E}\,|Y_t^n - Y_t|^2 + \mathbb{E}\int_0^1 \mid Z_s^n - Z_s \mid^2 ds \longrightarrow 0, \quad \forall t \in [0,1].$$

If we return to the equation satisfied by the triple $(Y^n, Z^n, K^n)_{n\in\mathbb{N}^*}$, we see that

$$\begin{aligned}
\mathbb{E} \mid K_t^n - K_t \mid^2 \leq C\big[&\mathbb{E}\,|\xi^n - \xi|^2 + \mathbb{E} \mid Y_t^n - Y_t \mid^2 \\
&+ \mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 ds \\
&+ \mathbb{E}\int_0^1 \mid Z_s^n - Z_s \mid^2 ds\big].
\end{aligned}$$

We shall prove that the sequence of processes $f_n(., Y^n, Z^n)_n$ converges to $f(., Y, Z)$ in $L^2([0,1]\times\Omega)$.

$$\begin{aligned}
&\mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 ds \\
&\leq 2\mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n) \mid^2 ds + 2\mathbb{E}\int_0^1 \mid f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 ds \\
&\leq 2\rho_N^2(f_n - f) + \frac{C}{N^{2(1-\alpha)}} \sup_n \mathbb{E}\int_0^1 (1 + \mid Z_s^n \mid^2 + \mid Y_s^n \mid^2) ds \\
&+ 2L_N^2 \mathbb{E}\int_0^1 (\mid Z_s^n - Z_s \mid^2 ds + \mathbb{E}\int_0^1 \mid Y_s^n - Y_s \mid^2) ds \\
&+ \frac{C}{N^{2(1-\alpha)}} \sup_n \mathbb{E}\int_0^1 (1 + \mid Z_s^n \mid^2 + \mid Z_s \mid^2 + \mid Y_s^n \mid^2 + \mid Y_s \mid^2) ds.
\end{aligned}$$

Hence

$$\mathbb{E}\int_0^1 \mid f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)\mid^2 ds$$

$$\le 2\rho_N^2(f_n-f)+2L_N^2\left(\mathbb{E}\int_0^1\mid Y_s^n-Y_s\mid^2 ds+\mathbb{E}\int_0^1\mid Z_s^n-Z_s\mid^2 ds\right)+\frac{C}{N^{2(1-\alpha)}}.$$

Passing to the limit successively on n and N, we obtain

$$\mathbb{E}\int_0^1 \mid f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)\mid^2 ds \longrightarrow 0,\ \text{as}\, n\to\infty.$$

Now

$$\forall t\in[0,1],\ \mathbb{E}\mid K_t^n-K_t\mid^2\longrightarrow 0,\ \text{as}\, n,m\to\infty.$$

Theorem 2.14 is proved. ■

2.4 Reflected BSDE with super-linear growth

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space and $(W_t,t\in[0,1])$ be a n-dimensional Wiener process defined on it. Let $(\mathcal{F}_t,t\in[0,1])$ denote the natural filtration of (W_t) augmented with the $\mathbb{P}$-null sets of $\mathcal{F}$.

In this section, we are concerned with the existence and uniqueness results for reflected BSDE with super-linear growth of the following type: $C(1+\mid y\mid\sqrt{\mid\log\mid y\mid\mid})$, $C(1+\mid y\mid\sqrt{\mid\log\mid\log\mid y\mid\mid\mid})$... We state the following assumptions:
(A.1) A function process f, which is a map:

$$f:\Omega\times[0,1]\times I\!R^d\times I\!R^{d\times n}\longrightarrow I\!R^d,$$

such that

(i) f is continuous in (y,z) for almost all (t,ω).
(ii) There exists a constant $M>0$ such that,

$$<y,f(t,\omega,y,z)>\le M(1+|y|^2+|y||z|)\quad P-a.s.,\ a.e.\, t\in[0,1].$$

(iii) There exist $M>0$ and $\alpha\in[0,1]$ such that,

$$|f(t,\omega,y,z)|\le M(1+|y|\sqrt{|\log|y||}+|z|^\alpha)\quad P-a.s.,\ a.e.\, t\in[0,1].$$

(iv) There exists $M>0$ such that,

$$|f(t,\omega,y,z)|\le M(1+|y|\sqrt{|\log|y||}+|z|)\quad P-a.s.,\ a.e.\, t\in[0,1].$$

(v) For each $N>0$, there exists L_N such that:

$$\mid f(t,\omega,y,z)-f(t,\omega,y,z')\mid\le L_N\left(\mid y-y'\mid+\mid z-z'\mid\right)$$
$$\mid y\mid,\mid y'\mid,\mid z\mid,\mid z'\mid\le N\quad P-a.s.,\ a.e.\, t\in[0,1].$$

(vi) There exist $M>0$ and $\alpha\in[0,1]$ such that,

$$|f(t,\omega,y,z)|\le M(1+|y||\log|y||+|z|^\alpha)\quad P-a.s.,\ a.e.\, t\in[0,1].$$

(vii) There exists $M > 0$ such that,

$$|f(t,\omega,y,z)| \leq M(1+|y||\log|y|| + |z|) \quad P-a.s., \, a.e. \, t \in [0,1].$$

(A.2) A open and convex subset Θ of $\mathbb{R}^d$.

The main results are the following.

Theorem 2.16. *Let **(A.1)**$(i)-(iii)$, (v) and **(A.2)** be satisfied. Assume moreover that $E(|\xi|^5) < \infty$. Then the reflected BSDE (1r)-(5r) has one and only one solution $\{(Y_t, Z_t, K_t); 0 \leq t \leq 1\}$ if one of the following two conditions is satisfied:*
(a) $\alpha < 1$ *and* $\displaystyle\lim_{N\to\infty} \frac{1}{L_N^2}\left(\frac{1}{N} + \frac{1}{N^{2(1-\alpha)}} + \frac{1}{N^2}\right)\exp(2L_N^2) = 0$
(b) $\alpha \leq 1$ *and* $\exists L \geq 0, \quad 2L_N^2 \leq L + 2(1-\alpha)\log N.$

Theorem 2.17. *Let **(A.1)**$(i)-(ii)$, $(iv)-(v)$ and **(A.2)** be satisfied. Assume moreover that $E(|\xi|^4) < \infty$ and L_N satisfies the following relation,*

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \exp(2L_N^2) = 0.$$

Then the reflected BSDE (1r)-(5r) has one solution.

Arguing as in the prove of Theorem 2.2 one can show the

Remark 2.18. *The hypothesis (a) can be slightly relaxed in*
(a') $\alpha < 1$ *and* $\displaystyle\lim_{N\to\infty} \frac{1}{L_N^2 + 2L_N}\left(\frac{1}{N} + \frac{1}{N^{2(1-\alpha)}} + \frac{1}{N^2}\right)\exp(L_N^2 + 2L_N) = 0.$

In order to prove Theorem 2.16, we need the following auxiliary lemmas.

Lemma 2.19. *Let f be a process which satisfies assumptions of Theorem 2.16. Then there exists a sequence of processes (f_n) such that,*
-(i)- For each n, f_n is globally Lipschitz in (y,z) a.e. t and P-a.s. ω.
-(ii)- For every $N \in \mathbb{N}^*$, $|f_n(t,\omega,y,z) - f_n(t,\omega,y',z')| \leq L_{(N+\frac{1}{n})}(|y-y'| + |z-z'|)$, for n large enough and for each (y,y',z,z') such that $|y|, |y'| \leq N$, $|z| \leq N$, $|z'| \leq N$.
-(iii)- There exists a constant $K(M) > 0$ such that for each (y,z) and for n large enough,
$\quad < y, f_n(t,\omega,y,z) > \leq K(M)(1 + |y|^2 + |y||z|)$ *P-a.s. and a.e. $t \in [0,1]$.*
-(iv)- There exists a constant $K(M) > 0$ such that for each (y,z),
$\quad \sup_n(|f_n(t,\omega,y,z)|) \leq K(M)(1 + |y|\sqrt{|\log|y||} + |z|^\alpha)$ *P-a.s., a.e. $t \in [0,1]$.*
-(vi)- For every N, $\rho_N(f_n - f) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof . Let $\varphi_n : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ be a sequence of smooth functions with compact support which approximate the Dirac measure at 0 and which satisfy $\int \varphi_n(u)du = 1$. Let $\psi_n : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ be a sequence of smooth functions such that $0 \leq \psi_n \leq 1$, $\psi_n(u) = 1$ for $|u| \leq n$ and $\psi_n(u) = 0$ for $|u| \geq n+1$. Likewise we define the sequence ψ'_n from $\mathbb{R}^{d\times r}$ to $\mathbb{R}_+$. We put, $f_{q,n}(t,y,z) = \int f(t, y-u, z)\varphi_q(u)du\psi_n(y)\psi'_n(z)$. For $n \in \mathbb{N}^*$, let $q(n)$ be an integer such that $q(n) \geq M[4n^2 + 10n + 12]$. It is not difficult to see that the sequence $f_n := f_{q(n),n}$ satisfy all the assertions (i)-(vi). ■

Let (f_n) be the sequence of processes associated to f by Lemma 2.19. We get from Ouknine [61] that there exists a unique triplet $\{(Y_t^n, Z_t^n, K_t^n; 0 \leq t \leq 1)\}$ of progressively measurable processes taking values in $\mathbb{R}^d \times \mathbb{R}^{d\times n} \times \mathbb{R}^d$ and satisfying:

$$\begin{cases} (1')\ Z^n \text{ is a predictable process and } \mathbb{E}\displaystyle\int_0^1 \mid Z_t^n \mid^2 dt < +\infty \\ (2')\ Y_t^n = \xi + \displaystyle\int_t^1 f_n(s, Y_s^n, Z_s^n) ds - \int_t^1 Z_s^n dW_s + K_1^n - K_t^n \\ (3')\ \text{the process } Y^n \text{ is continuous} \\ (4')\ K^n \text{ is absolutely continuous, } K_0^n = 0, \text{ and } \displaystyle\int_0^{\cdot} (Y_t^n - \alpha_t)\, dK_t^n \leq 0 \\ \quad \text{for every } \alpha_t \text{ progressively measurable process which is continuous} \\ \quad \text{and takes values into } \overline{\Theta} \\ (5')\ Y_t^n \in \overline{\Theta},\ 0 \leq t \leq 1.\, a.s. \end{cases}$$

We formulate uniform estimates for the processes (Y^n, Z^n, K^n) as follows.

Lemma 2.20. *(a) Let assumptions of Theorem 2.16 hold. Then there exists a constant C depending only in M and ξ, such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq 1} |Y_t^n|^2 + \int_0^1 |Z_s^n|^2\, ds\right) \leq C,\ \forall\, n \in \mathbb{N}^*.$$

(b) Assume moreover that there exists an integer $p > 1$ such that $\mathbb{E} \mid \xi \mid^{2p} < \infty$. Then, there exists a constant C depending only in M, p and ξ, such that

$$\mathbb{E}\left(\sup_{0\leq t\leq 1} |Y_t^n|^{2p} + \mid K_1^n \mid^2\right) \leq C,,\ \forall\, n \in \mathbb{N}^*.$$

Proof . Assertion (a) follows from Itô's formula, assumption **(A.1)**(ii), Gronwall's lemma and Burkholder-Davis-Gundy inequality. Let us prove (b).
Using Itô's formula we obtain,

$$|Y_t^n|^2 + \int_t^1 |Z_s^n|^2\, ds = \mid \xi \mid^2 + 2\int_t^1 f_n(s, Y_s^n, Z_s^n) Y_s^n ds - 2\int_t^1 Z_s^n Y_s^n dW_s + 2\int_t^1 Y_s^n dK_s^n.$$

Without loss of generality we can assume that $0 \in \Theta$. Hence by relation (4') we have

$$\int_t^1 Y_s^n dK_s^n \leq 0.$$

We use **(A.1)**(ii) and the inequality $ab \leq \dfrac{a^2}{2} + \dfrac{b^2}{2}$ to obtain

$$|Y_t^n|^2 = \mid \xi \mid^2 + 2C + (2C + 2C^2)\int_t^1 |Y_s^n|^2\, ds - 2\int_t^1 Z_s^n Y_s^n dW_s.$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ in both sides we deduce

$$|Y_t^n|^2 \leq \mathbb{E}\left(\mid \xi \mid^2 + 2C + (2C + 2C^2)\int_t^1 |Y_s^n|^2\, ds / \mathcal{F}_t\right).$$

Jensen's inequality shows that for every $p > 1$,

$$\mathbb{E}\,|Y_t^n|^{2p} \leq C_p \left(\mathbb{E}\big[\,|\,\xi\,|^{2p}\,\big] + (2C)^p + (2C+2C^2)^p \mathbb{E}\big[\int_t^1 |\,Y_s^n\,|^{2p}\,ds\big]\right)$$
$$\leq C_p(1+\mathbb{E}\int_t^1 |Y_t^n|^{2p}\,ds).$$

Gronwall's lemma implies that

$$\sup_{0\leq t\leq 1} \mathbb{E}\,|\,Y_t^n\,|^{2p} < +\infty, \quad \forall n \in \mathbb{N}^*. \tag{2.7}$$

It follows from Doob's maximal inequality that

$$\mathbb{E}\sup_{0\leq t\leq 1} |Y_t^n|^{2p} < +\infty, \quad \forall n \in \mathbb{N}^*.$$

Now, from (2') we have

$$K_1^n - K_t^n = Y_t^n - \xi - \int_t^1 f_n(s, Y_s^n, Z_s^n)ds + \int_t^1 Z_s^n dW_s,$$

Thanks to assumption **(A.1)**(iii), we obtain

$$\mathbb{E}\,|\,K_1^n - K_t^n\,|^2$$
$$\leq C(\mathbb{E}\,|\,\xi\,|^2 + \mathbb{E}\,|Y_t^n|^2 + 1 + \mathbb{E}\int_t^1 |\,Y_s^n\,|^4\,ds + \mathbb{E}\int_t^1 |Z_s^n|^2\,ds)$$

and from assertion (a) and (2.4), we deduce that

$$\sup_n \mathbb{E}\,|\,K_1^n\,|^2 \leq C, \quad \text{for all } n \in \mathbb{N}^*.$$

Hence, (b) is proved. ■

We shall prove the convergence of the sequence $(Y^n, Z^n, K^n)_n$, $n \in \mathbb{N}^*$.

Lemma 2.21. *Under assumptions of Theorem 2.16, there exist (Y,Z,K) such that*

$$\lim_{n\to\infty} \mathbb{E}\{\sup_{0\leq t\leq 1} |\,Y_1^n - Y_t\,|^2 + \sup_{0\leq t\leq 1} |\,K_1^n - K_t\,|^2 + \int_0^1 |Z_s^n - Z_t|^2\,ds\} = 0.$$

Proof : It follows from Itô's formula that

$$|Y_t^n - Y_t^m|^2 + \int_t^1 |Z_s^n - Z_s^m|^2\,ds$$
$$= 2\int_t^1 (Y_s^n - Y_s^m)^* \,(f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m))ds$$
$$-\,2\int_t^1 (Y_s^n - Y_s^m)^* \,(Z_s^n - Z_s^m)\,dW_s + 2\int_t^1 (Y_s^n - Y_s^m)\,(dK_s^n - dK_s^m).$$

For an arbitrary number $N > 1$, let L_N be the Lipschitz constant of f in the ball $B(0, N)$. We put $A_{n,m}^N := \{(s,\omega); |\,Y_s^n\,|^2 + |\,Z_s^n\,|^2 + |\,Y_s^m\,|^2 + |\,Z_s^m\,|^2 \geq N^2\}$, $\overline{A}_{n,m}^N := \Omega \setminus A_{n,m}^N$.
As in Lemma 2.8, we deduce that

$$\mathbb{E}\,|\,Y_t^n - Y_t^m\,|^2 + \mathbb{E}\int_t^1 |\,Z_s^n - Z_s^m\,|^2\,ds$$
$$\leq \beta^2 \mathbb{E}\int_t^1 |\,Y_s^n - Y_s^m\,|^2\,ds + \frac{1}{\beta^2}\mathbb{E}\int_t^1 |\,f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)\,|^2\,ds.$$

Hence

$$\begin{aligned}
&\mathbb{E} \mid Y_t^n - Y_t^m \mid^2 + \mathbb{E}\int_t^1 \mid Z_s^n - Z_s^m \mid^2 ds \\
&\leq \beta^2 \mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m \mid^2 ds + \frac{1}{\beta^2}\mathbb{E}\int_t^1 \mid f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m) \mid^2 \mathbf{1}_{A_{n,m}^N} ds \\
&+ \frac{4}{\beta^2}\mathbb{E}\int_t^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n) \mid^2 \mathbf{1}_{\overline{A}_{n,m}^N} ds \\
&+ \frac{2}{\beta^2}\mathbb{E}\int_t^1 \mid f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m) \mid^2 \mathbf{1}_{\overline{A}_{n,m}^N} ds \\
&+ \frac{4}{\beta^2}\mathbb{E}\int_t^1 \mid f(s, Y_s^m, Z_s^m) - f_m(s, Y_s^n, Z_s^n) \mid^2 \mathbf{1}_{\overline{A}_{n,m}^N} ds.
\end{aligned}$$

Using the fact that f_n satisfies $\mathbf{(A.1)}(iii)$, Hölder inequality, Chebychef inequality and Lemma 2.20, we obtain

$$\begin{aligned}
&\mathbb{E}\int_t^1 \mid f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m) \mid^2 \mathbf{1}_{A_{n,m}^N} ds \\
&\leq \mathbb{E}\int_t^1 (1 + \mid Y_s^n \mid^2 + \mid Z_s^n \mid^\alpha + \mid Y_s^m \mid^2 + \mid Z_s^m \mid^\alpha)^2 \mathbf{1}_{A_{n,m}^N} ds \\
&\leq C(\xi, M)\left(\frac{1}{N^{2(1-\alpha)}} + \frac{1}{N} + \frac{1}{N^2}\right).
\end{aligned}$$

Since f is L_N-locally Lipschitz we get

$$\begin{aligned}
&\mathbb{E} \mid Y_t^n - Y_t^m \mid^2 + \mathbb{E}\int_t^1 \mid Z_s^n - Z_s^m \mid^2 ds \\
&\leq \beta^2 \mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m \mid^2 ds + \frac{C(\xi, M)}{\beta^2}\left(\frac{1}{N^{2(1-\alpha)}} + \frac{1}{N} + \frac{1}{N^2}\right) + \frac{4}{\beta^2}\rho_N^2(f_n - f) \\
&+ \frac{4}{\beta^2}\rho_N^2(f_m - f) + \frac{2L_N^2}{\beta^2}\mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m \mid^2 ds + \frac{2L_N^2}{\beta^2}\mathbb{E}\int_t^1 \mid Z_s^n - Z_s^m \mid^2 ds.
\end{aligned}$$

If we choose β such that $\frac{2L_N^2}{\beta^2} = 1$, we obtain

$$\begin{aligned}
\mathbb{E}\,|Y_t^n - Y_t^m|^2 \leq \frac{4}{\beta^2}\left(\rho_N^2(f_n - f) + \rho_N^2(f_m - f)\right) + \frac{C(\xi, M)}{\beta^2}\left(\frac{1}{N^{2(1-\alpha)}} + \frac{1}{N} + \frac{1}{N^2}\right) \\
+ (1 + \beta^2)\mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m \mid^2 ds.
\end{aligned}$$

It follows from Gronwall lemma that, for every $t \in [0, 1]$,

$$\begin{aligned}
&\sup_{0\leq t\leq 1} \mathbb{E}\,|Y_t^n - Y_t^m|^2 \\
&\leq \left(\frac{2}{L_N^2}(\rho_N^2(f_n - f) + \rho_N^2(f_m - f)) + \frac{C(\xi, M)}{2L_N^2}\left(\frac{1}{N^{2(1-\alpha)}} + \frac{1}{N} + \frac{1}{N^2}\right)\right)\exp(2L_N^2 + 1).
\end{aligned}$$

Using Burkholder-Davis-Gundy inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq 1} |Y_t^n - Y_t^m|^2 \\
&\leq C\left(\frac{2}{L_N^2}(\rho_N^2(f_n - f) + \rho_N^2(f_m - f)) + \frac{1}{2L_N^2}\left(\frac{1}{N^{2(1-\alpha)}} + \frac{1}{N} + \frac{1}{N^2}\right)\right)\exp(2L_N^2 + 1)
\end{aligned}$$

$$\mathbb{E}\sup_{0\le t\le 1}\int_0^1 |Z_t^n - Z_t^m|^2$$
$$\le C\left(\frac{2}{L_N^2}(\rho_N^2(f_n-f)+\rho_N^2(f_m-f))+\frac{1}{2L_N^2}(\frac{1}{N^{2(1-\alpha)}}+\frac{1}{N}+\frac{1}{N^2})\right)\exp(2L_N^2+1).$$

Passing to the limit on n, m and on N, we show that $(Y^n, Z^n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in the Banach space of progressively measurable processes $\mathbb{L}$, which is defined as above.
We set

$$Y = \lim_{n\longrightarrow+\infty} Y^n \quad\text{and}\quad Z = \lim_{n\longrightarrow+\infty} Z^n.$$

If we return to the equation satisfied by the triple $(Y^n, Z^n, K^n)_{n\in\mathbb{N}^*}$, we see that

$$\begin{aligned}\mathbb{E}\sup_{0\le t\le 1}\mid K_t^n - K_t^m\mid^2 \le C\big[&\mathbb{E}\sup_{0\le t\le 1}\mid Y_t^n - Y_t^m\mid^2\\ &+\mathbb{E}\int_0^1 \mid f_n(s,Y_s^n,Z_s^n)-f_m(s,Y_s^m,Z_s^m)\mid^2 ds\\ &+\mathbb{E}\int_0^1\mid Z_s^n - Z_s^m\mid^2 ds\big].\end{aligned}$$

We shall prove that the sequence of processes $f_n(.,Y^n,Z^n)_n$ converges to $f(.,Y,Z)$ in $L^2([0,1]\times\Omega)$

$$\mathbb{E}\int_0^1\mid f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)\mid^2 ds$$
$$\le 2\mathbb{E}\int_0^1\mid f_n(s,Y_s^n,Z_s^n)-f(s,Y_s^n,Z_s^n)\mid^2 ds + 2\mathbb{E}\int_0^1\mid f(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)\mid^2 ds$$
$$\le 2\rho_N^2(f_n-f)+C(\xi,M)(\frac{1}{N^{2(1-\alpha)}}+\frac{1}{N}+\frac{1}{N^2})$$
$$+\,2L_N^2\mathbb{E}\int_0^1(\mid Z_s^n - Z_s\mid^2 ds+\int_0^1\mid Y_s^n - Y_s\mid^2)ds$$

Hence

$$\mathbb{E}\int_0^1\mid f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)\mid^2 ds$$
$$\le 2\rho_N^2(f_n-f)+2L_N^2\left(\mathbb{E}\int_0^1\mid Y_s^n-Y_s\mid^2 ds+\mathbb{E}\int_0^1\mid Z_s^n-Z_s\mid^2 ds\right)$$
$$+\,C(\xi,M)(\frac{1}{N^{2(1-\alpha)}}+\frac{1}{N}+\frac{1}{N^2}).$$

Passing to the limit successively on n and N, we obtain

$$\mathbb{E}\int_0^1\mid f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)\mid^2 ds\longrightarrow 0,\ \text{as}\, n\to\infty.$$

Now

$$\mathbb{E}\sup_{0\le t\le 1}\mid K_t^n - K_t^m\mid^2\longrightarrow 0,\ \text{as}\, n,m\to\infty.$$

Consequently there exists a progressively measurable process K such that

$$\mathbb{E}\sup_{0\le t\le 1}\mid K_t^n - K_t\mid^2\longrightarrow 0\,\text{as}\, n\to\infty,$$

and clearly (K_t) is increasing (with $K_0 = 0$) and a continuous process. ■

Proof of Theorem 2.16: Combining Lemmas 2.20, 2.21 and passing to the limit in the RBSDE (2'), we show that the triplet $\{(Y_t, Z_t, K_t); 0 \leq t \leq 1\}$ is a solution of our RBSDE. The sequel of the proof can be performed as that of Theorem 2.2. Theorem 2.16 is proved.■

Corollary 2.22. *Let **(A.1)**(i)-(ii), (vi) and **(A.2)** be satsfied. Assume moreover that the generator f is locally L_N–Lipschitz in Y and L-globally Lipschitz in Z and $\mathbb{E}(\mid \xi \mid^5) < \infty$. Then if L_N satisfies*

$$\lim_{N\to\infty}\left(\frac{1}{N^{2(1-\alpha)}}+\frac{1}{N^2}+\frac{1}{N}\right)\exp(2L_N)=0,$$

our reflected BSDE (1r)-(5r) has a unique solution.

Proof . The arguments used in the proof of Lemma 2.21 lead to

$$\begin{aligned}
&\mathbb{E}\mid Y_t^n - Y_t^m\mid^2 + \mathbb{E}\int_t^1 \mid Z_s^n - Z_s^m\mid^2 ds\\
&\leq \mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m\mid^2 \mathbf{1}_{A_{n,m}^N} ds + 2\mathbb{E}\int_t^1 \mid f_n(s,Y_s^n,Z_s^n) - f_m(s,Y_s^m,Z_s^m)\mid^2 \mathbf{1}_{A_{n,m}^N} ds\\
&+ \mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m\mid\mid f_n(s,Y_s^n,Z_s^n) - f_m(s,Y_s^m,Z_s^m)\mid \mathbf{1}_{\overline{A}_{n,m}^N} ds\\
&\leq \mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m\mid^2 \mathbf{1}_{A_{n,m}^N} ds + C(\xi,M)\left(\frac{1}{N^{2(1-\alpha)}}+\frac{1}{N}+\frac{1}{N^2}\right)\\
&+ 2\mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m\mid\mid f_n(s,Y_s^n,Z_s^n) - f(s,Y_s^n,Z_s^n)\mid \mathbf{1}_{\overline{A}_{n,m}^N} ds\\
&+ 2\mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m\mid\mid f(s,Y_s^n,Z_s^n) - f(s,Y_s^m,Z_s^n)\mid \mathbf{1}_{\overline{A}_{n,m}^N} ds\\
&+ 2\mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m\mid\mid f(s,Y_s^m,Z_s^n) - f(s,Y_s^m,Z_s^m)\mid \mathbf{1}_{\overline{A}_{n,m}^N} ds\\
&+ 2\mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m\mid\mid f(s,Y_s^m,Z_s^m) - f_m(s,Y_s^m,Z_s^m)\mid \mathbf{1}_{\overline{A}_{n,m}^N} ds,
\end{aligned}$$

hence

$$\begin{aligned}
&\mathbb{E}\mid Y_t^n - Y_t^m\mid^2 + \mathbb{E}\int_t^1 \mid Z_s^n - Z_s^m\mid^2 ds\\
&\leq \mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m\mid^2 ds + C(\xi,M)\left(\frac{1}{N^{2(1-\alpha)}}+\frac{1}{N}+\frac{1}{N^2}\right)\\
&+ 2\rho_N^2(f_n - f) + 2\rho_N^2(f_m - f) + \beta^2\mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m\mid^2 ds\\
&+ 2L_N\mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m\mid^2 ds + \frac{L^2}{\beta^2}\mathbb{E}\int_t^1 \mid Z_s^n - Z_s^m\mid^2 ds.
\end{aligned}$$

Choosing $\frac{L^2}{\beta^2} = 1$ then using Gronwall lemma and Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq 1}\mid Y_t^n - Y_t^m\mid^2\\
&\leq C\left(2\rho_N^2(f_n - f) + 2\rho_N^2(f_m - f) + \frac{1}{N^{2(1-\alpha)}}+\frac{1}{N}+\frac{1}{N^2}\right)\exp(2L_N^2 + L^2 + 1).
\end{aligned}$$

$$\mathbb{E}\int_0^1 \mid Z_s^n - Z_s^m \mid^2 ds$$
$$\leq C\left(2\rho_N^2(f_n - f) + 2\rho_N^2(f_m - f) + \frac{1}{N^{2(1-\alpha)}} + \frac{1}{N} + \frac{1}{N^2}\right)\exp(2L_N^2 + L^2 + 1).$$

Passing to the limit on n, m and N, we get the desired result. ■

Proof of Theorem 2.17. Arguing as in the proof of Theorem 2.16 we show that

$$\mathbb{E} \mid Y_t^n - Y_t^m \mid^2 + \mathbb{E}\int_t^1 \mid Z_s^n - Z_s^m \mid^2 ds$$
$$\leq 2\mathbb{E}\int_t^1 \langle Y_s^n - Y_s^m, f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)\rangle \mathbf{1}_{A_{n,m}^N} ds$$
$$+ \beta^2\mathbb{E}\int_t^1 \mid Y_s^n - Y_s^m \mid^2 \mathbf{1}_{\overline{A}_{n,m}^N} ds$$
$$+ \frac{4}{\beta^2}\mathbb{E}\int_t^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n) \mid^2 \mathbf{1}_{\overline{A}_{n,m}^N} ds$$
$$+ \frac{2}{\beta^2}\mathbb{E}\int_t^1 \mid f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m) \mid^2 \mathbf{1}_{\overline{A}_{n,m}^N} ds$$
$$+ \frac{4}{\beta^2}\mathbb{E}\int_t^1 \mid f(s, Y_s^m, Z_s^m) - f_m(s, Y_s^n, Z_s^n) \mid^2 \mathbf{1}_{\overline{A}_{n,m}^N} ds.$$

We use Hölder inequality, Chebychef inequality and Lemma 2.20 to show that

$$\mathbb{E}\int_t^1 \langle Y_s^n - Y_s^m, f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)\rangle \mathbf{1}_{A_{n,m}^N} ds$$
$$\leq 2E\int_t^1 |Y_s^n - Y_s^m||f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)|\mathbf{1}_{A_{n,m}^N} ds$$
$$\leq 2\big(E\int_t^1 |Y_s^n - Y_s^m|^2 \mathbf{1}_{A_{n,m}^N} ds\big)^{\frac{1}{2}}\big(E\int_t^1 |f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)|^2 ds\big)^{\frac{1}{2}}$$
$$\leq 2\big(E\int_t^1 |Y_s^n - Y_s^m|^4 ds\big)^{\frac{1}{4}}\big(E\int_t^1 \mathbf{1}_{A_{n,m}^N} ds\big)^{\frac{1}{2}} C(K, \xi)$$
$$\leq \frac{K(M, \xi)}{\sqrt{N}}.$$

Therefore

$$E(|Y_t^n - Y_t^m|^2) \leq \big[\frac{2}{L_N^2}\big(\rho_N^2(f_n - f) + \rho_N^2(f_m - f)\big) + \frac{K(M, \xi)}{\sqrt{N}}\big]\exp(2L_N^2).$$

Passing to the limit first on n, m and next on N then using the Burkholder-Davis-Gundy inequality, we show that (Y^n, Z^n) is a Cauchy sequence in the Banach space $(\mathbb{L}, ||.||)$. The sequel of the proof can be performed as that of Theorem 2.16. Theorem 2.17 is proved. ■

Corollary 2.23. *Let **(A.1)**(i)-(ii), (vii) and **(A.2)** be satisfied. Assume moreover that the generator f is locally L_N–Lipschitz in Y and L-globally Lipschitz in Z and $\mathbb{E}(\mid \xi \mid^4) < \infty$. Then if L_N satisfies*

$$\lim_{N\to\infty}\left(\frac{1}{\sqrt{N}}\right)\exp(2L_N) = 0,$$

our reflected BSDE (1r)-(5r) has a unique solution.

Proof . Arguing as in the proof of Corollary 2.22 we obtain

$$\mathbb{E} \sup_{0 \leq t \leq 1} \mid Y_t^n - Y_t^m \mid^2 \leq C \left(2\rho_N^2(f_n - f) + 2\rho_N^2(f_m - f) + \frac{1}{\sqrt{N}} \right) \exp(2L_N^2 + L^2 + 1).$$

$$\mathbb{E} \int_0^1 \mid Z_s^n - Z_s^m \mid^2 ds \leq C \left(2\rho_N^2(f_n - f) + 2\rho_N^2(f_m - f) + \frac{1}{\sqrt{N}} \right) \exp(2L_N^2 + L^2 + 1).$$

From which the result follows. ■

Let us give the following example

Example 2.24. *For* $i = 1, ..., d$ *let* $h_i : \mathbb{R}^d \longrightarrow \mathbb{R}$ *be the function defined by,* $h_i(y) = -\frac{1}{e} 1\!\!1_{|y| \leq \frac{1}{\varepsilon}} + \mid y \mid \log \mid y \mid 1\!\!1_{|y| \geq \frac{1}{\varepsilon}}$ *and define the function* g *by* $g(t, x, y) := (h_1(y) + \mid z \mid, ..., h_d(y) + \mid z \mid)$. *It is not difficult to check that* g *satisfies the assumptions of Corollary 2.23 and hence if* $\mathbb{E}(\mid \xi \mid^4) < \infty$ *then our BSDE has a unique solution.*

Chapter 3

Multidimensional Backward Stochastic Differential Equations with non-Lipschitz Coefficients

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The chapter is organized as follows. In Section 3.1, we study the existence and uniqueness of RBSDE with monotone generator. The existence and uniqueness of one solution to RBSDE with locally monotone coefficient is proved in Section 3.2. In Section 3.3, We prove existence, uniqueness and stability of the solution for multidimensional backward stochastic differential equation whose coefficient is neither locally Lipschitz in the variable Y nor in the variable Z. This is done with super-linear growth coefficient and a square integrable terminal condition.

3.1 RBSDE with Monotone Coefficient and polynomial growth

In many examples of semi-linear PDEs , the nonlinearity is not of linear growth but instead, it is of polynomial growth, see e.g. the linear heat equation analyzed by Escobedo et *al.* [29] or the Allen-Cahn equation (see Barles et *al.* [10]). If one attempts to study those equations by means of the formula (0.9) one has to deal with BSDEs whose generators with nonlinear (though polynomial) growth. The goal of this section is to study the reflected backward stochastic differential equation (1)-(5) (see below) under monotone and polynomial growth generator, via penalization technique in multidimensional case. Precisely, we prove that if the generator is monotone and has a polynomial growth, the RBSDE (1)-(5) below has one and only one solution.

3.1.1 Formulation of the problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(W_t, t \in [0,1])$ be a n-dimensional Wiener process defined on it. Let $(\mathcal{F}_t, t \in [0,1])$ denote the natural filtration of (W_t) augmented with the $\mathbb{P}$-null sets of $\mathcal{F}$. We define the following three objects:
(A.1) A process f defined on $\Omega \times [0,1] \times I\!R^d \times I\!R^{d\times n}$ with value in $\mathbb{R}^d$ and satisfies the

[0] A part of this work is accepted for publication in Stochastic Analysis and Applications.

following assumptions:
There exist constants $\gamma \geq 0$, $\mu \in \mathbb{R}$, $C \geq 0$ and $p \geq 1$ such that $\mathbb{P}-a.s.$, we have

$(i)\,\forall (y,z) \in I\!R^d \times I\!R^{d\times n}:\ \ (\omega,t) \longrightarrow f\,(\omega,t,y,z)$ is $\mathcal{F}_t-$progressively measurable
$(ii)\,\forall t,\forall y,\forall (z,z'),\quad \mid f(t,y,z)-f(t,y,z') \mid\leq \gamma \mid z-z' \mid$
$(iii)\,\forall t,\forall z,\forall (y,y'),\quad (y-y')(f(t,y,z)-f(t,y',z)) \leq \mu \mid y-y' \mid^2$
$(iv)\,\forall t,\forall y,\forall z,\quad \mid f(t,y,z) \mid\leq\mid f(t,0,z) \mid +K(1+\mid y \mid^p)$
$(v)\,\forall t,\forall z,\quad y \longrightarrow f(t,y,z)$ is continuous.

(A.2) A terminal value ξ which is F_1-measurable such that

$$\mathbb{E} \mid \xi \mid^{2p} + \mathbb{E}\left(\int_0^1 \mid f(s,0,0) \mid^2 ds\right)^p < +\infty.$$

(A.3) A proper lower semicontinuous convex function $\phi : \mathbb{R}^d \to]-\infty,+\infty]$.
We also assume that $\xi \in \overline{Dom(\phi)}$ and $\mathbb{E}(\phi(\xi)) < +\infty$.
Before stating our result, we recall some properties of a Yosida approximation of subdifferential operator. We define

$$\begin{aligned}
Dom(\phi) &= \{u \in \mathbb{R}^d : \phi(u) < +\infty\} \\
\partial\phi(u) &= \{u^* \in \mathbb{R}^d :< u^*, v-u > +\phi(u) \leq \phi(v), \forall v \in \mathbb{R}^d\} \\
Dom(\partial(\phi)) &= \{u \in \mathbb{R}^d : \partial(\phi) \neq \emptyset\} \\
Gr(\partial\phi) &= \{(u,u^*) \in \mathbb{R}^d \times \mathbb{R}^d : u \in Dom(\partial(\phi)) \quad \text{and} \quad u^* \in \partial\phi(u)\}.
\end{aligned}$$

For every $x \in \mathbb{R}^d$, we put

$$\phi_n(x) = \min_y \big(\frac{n}{2} \mid x-y \mid^2 + \phi(y)\big).$$

Let $J_n(x)$ be the unique solution of the diffrential inclusion $x \in J_n(x) + \frac{1}{n}\partial\phi(J_n(x))$ (see Barbu, Precupanu [8]). The map J_n is called the resolvent of the monotone operator $A = \partial\phi$. Note that $\phi_n : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a convex and C^1 class function with $\bigtriangledown\phi_n = A_n$ is the Yosida approximation of the operator $\partial\phi$ defined by $A_n = n(x - J_n(x))$. We also have

$$\inf_{y\in\mathbb{R}^d} \phi(y) \leq \phi(J_n(x)) \leq \phi_n(x) \leq \phi(x).$$

Moreover, one can show that there exist $a \in interior(Dom(\phi))$ and positive numbers R, C such that for every $z \in \mathbb{R}^d$

$$(\nabla\phi_n(z)^*(z-a)) \geq R \mid A_n(z) \mid -C \mid z \mid -C \text{ for all } n \in \mathbb{N}^*, \tag{3.1}$$

more details can be found in Cépa [19].
Now, let us introduce our RBSDE. The solution is a triplet (Y_t, Z_t, K_t), $0 \leq t \leq 1$ of

progressively measurable processes taking values in $\mathbb{R}^d \times \mathbb{R}^{d\times n} \times \mathbb{R}^d$ and satisfying:

$$\begin{cases} (1) Z \text{ is adapted process and } \mathbb{E}\displaystyle\int_0^1 \|Z_t\|^2 dt < +\infty \\ (2)\, Y_t = \xi + \displaystyle\int_t^1 f(s, Y_s, Z_s) ds - \int_t^1 Z_s dW_s + K_1 - K_t, 0 \le t \le 1 \\ (3) \text{ the process } Y \text{ is continuous} \\ (4) K \text{ is absolutely continuous}, K_0 = 0, \text{ and for every progressively measurable} \\ \text{and continuous processes } (\alpha, \beta) \text{ such that } (\alpha_t, \beta_t) \in Gr(\partial\phi), \text{ we have} \\ \qquad \displaystyle\int_0^{\cdot} (Y_t - \alpha_t)(dK_t + \beta_t dt) \le 0 \\ (5)\, Y_t \in \overline{Dom(\phi)},\ 0 \le t \le 1 \text{ a.s.} \end{cases}$$

Our goal in this section is to study the RBSDE (1)-(5) when the generator f satisfies the above assumptions.
Consider the following sequence of backward stochastic differential equation

$$Y_t^n = \xi + \int_t^1 (f(s, Y_s^n, Z_s^n) - A_n(Y_s^n)) ds - \int_t^1 Z_s^n dW_s, \tag{3.2}$$

where ξ, f satisfy the assumptions stated above and $(A_n)_n$ is the Yosida approximation of the operator $A = \partial\phi$. It is known, since A_n is Lipschitz and f is monotone, that the equation (3.2) has one and only one solution. We set

$$K_t^n = -\int_0^t A_n(Y_s^n) ds \quad \text{for} \ \ t \in [0, 1].$$

3.1.2 Existence and uniqueness results

The main result in this section is the following

Theorem 3.1. *Under the assumptions* ***(A.1), (A.2), (A.3)*** *on ξ, f, ϕ, the RBSDE (1)-(5) has a unique solution $\{(Y_t, Z_t, K_t);\ 0 \le t \le 1\}$. Moreover,*

$$\lim_{n \longrightarrow +\infty} \mathbb{E} \sup_{0 \le t \le 1} |Y_t^n - Y_t|^2 = 0$$

$$\lim_{n \longrightarrow +\infty} \mathbb{E} \int_0^1 |Z_t^n - Z_t|^2 ds = 0$$

$$\lim_{n \longrightarrow +\infty} \mathbb{E} \sup_{0 \le t \le 1} |K_t^n - K_t|^2 = 0.$$

In order to prove Theorem 3.1 we need the following lemmas.

Lemma 3.2. *Let assumptions of Theorem 3.1 hold. Then*

$$\sup_{n \in \mathbb{N}^*} \mathbb{E} \left(\sup_{0 \le t \le 1} |\ Y_t^n\ |^2 + \int_0^1 |Z_s^n|^2 ds + \int_0^1 |A_n(Y_s^n)| ds \right) < +\infty. \tag{3.3}$$

Proof . By Itô's formula we get

$$\begin{aligned} |Y_t^n - a|^2 + \int_t^1 |Z_s^n|^2 ds &= |\xi - a|^2 + 2\int_t^1 (Y_s^n - a)^* f(s, Y_s^n, Z_s^n) ds \\ &\quad -2\int_t^1 (Y_s^n - a)^* Z_s^n dW_s - 2\int_t^1 (Y_s^n - a)^* A_n(Y_s^n) ds. \end{aligned} \tag{3.4}$$

We Take expectation and use (3.1) to obtain,

$$\mathbb{E}\,|Y_t^n - a|^2 + \mathbb{E}\int_t^1 |Z_s^n|^2\, ds \le \mathbb{E}\,|\xi - a|^2 + 2\mathbb{E}\int_t^1 (Y_s^n - a)^* \, f(s, Y_s^n, Z_s^n) ds$$
$$- 2R\mathbb{E}\int_t^1 \mid A_n\,(Y_s^n) \mid ds + 2C\int_t^1 \mid Y_s^n \mid ds + 2C,$$

this implies that

$$\mathbb{E}\,|Y_t^n - a|^2 + \mathbb{E}\int_t^1 |Z_s^n|^2\, ds + 2R\mathbb{E}\int_t^1 |A_n\,(Y_s^n)|\, ds$$
$$\le \mathbb{E} \mid \xi - a \mid^2 + 2C\mathbb{E}\int_t^1 \mid Y_s^n \mid ds + 2C$$
$$+ 2\mathbb{E}\int_t^1 (Y_s^n - a)^* \left(f(s, Y_s^n, Z_s^n) - f(s, a, Z_s^n)\right) ds + 2\int_t^1 (Y_s^n - a) f(s, a, Z_s^n) ds,$$

Using assumptions **(A.1)**$(i) - (iii)$, we deduce

$$\mathbb{E}\left(|Y_t^n - a|^2 + \int_t^1 |Z_s^n|^2\, ds + 2R\int_t^1 |A_n\,(Y_s^n)|\, ds\right)$$
$$\le \mathbb{E} \mid \xi - a \mid^2 + 2\mu\mathbb{E}\int_t^1 \mid Y_s^n - a \mid^2 ds + 2\mathbb{E}\int_t^1 \mid Y_s^n - a \mid (\gamma \mid Z_s^n \mid + K(1 + \mid a \mid^p)) ds$$
$$+ \mathbb{E}\int_t^1 \mid Y_s^n - a \mid^2 ds + \mathbb{E}\int_t^1 \mid f(s, 0, 0) \mid^2 ds + C,$$

where C is a constant which can change from line to line.
Since $2ab \le \beta^2 a^2 + \frac{1}{\beta^2} b^2$ for each $a, b \ge 0$, we get

$$\mathbb{E}\left(|Y_t^n - a|^2 + \int_t^1 |Z_s^n|^2\, ds + 2R\int_t^1 |A_n\,(Y_s^n)|\, ds\right)$$
$$\le \mathbb{E} \mid \xi - a \mid^2 + (2 \mid \mu \mid + \beta^2 + 1)\mathbb{E}\int_t^1 |Y_t^n - a|^2\, ds + \frac{2\gamma^2}{\beta^2}\mathbb{E}\int_t^1 \mid Z_s^n \mid^2 ds + C.$$

If we take $\frac{2\gamma^2}{\beta^2} = \frac{1}{2}$, we obtain

$$\mathbb{E}\,|Y_t^n - a|^2 + \tfrac{1}{2}\mathbb{E}\int_t^1 |Z_s^n|^2\, ds \le C\left(1 + \mathbb{E}\int_t^1 |Y_s^n - a|^2\, ds\right),$$

Hence by Gronwall's lemma we have,

$$\sup_{0 \le t \le 1} \mathbb{E}\,|Y_t^n - a|^2 \le C, \ \forall n.$$

So that

$$\sup_{0 \le t \le 1} \mathbb{E}\,|Y_t^n|^2 \le C, \ \forall n.$$

Now, it is not difficult to show that,

$$\sup_{n \in \mathbb{N}^*} \mathbb{E}\left(\int_0^1 |Z_s^n|^2\, ds + \int_0^1 |A_n\,(Y_s^n)|\, ds\right) < +\infty. \tag{3.5}$$

We use equation (3.4) and Bulkholder-Davis-Gundy inequality to get,

$$\sup_{n\in\mathbb{N}^*}\mathbb{E}\sup_{0\le t\le 1}|Y_t^n|^2\le C. \tag{3.6}$$

Lemma 3.2 is proved. ■

We state the following lemma which is essential for the convergence of the sequence $(Y^n,Z^n)_{n\in IN^*}$.

Lemma 3.3. *Let assumptions of Theorem 3.1 hold. Then*
a) $\sup_{0\le t\le 1}\mathbb{E}\mid Y_t^n\mid^{2p}<+\infty,\ \forall n.$
b) $\sup_{n\in\mathbb{N}^*}\mathbb{E}\int_0^1|A_n(Y_s^n)|^2\,ds<+\infty.$

Proof . a) Itô's formula gives

$$|Y_t^n-a|^2+\int_t^1|Z_s^n|^2\,ds=|\xi-a|^2+2\int_t^1(Y_s^n-a)^*f(s,Y_s^n,Z_s^n,U_s^n)ds$$
$$-2\int_t^1Y_s^nZ_s^ndW_s-2\int_t^1(Y_s^n-a)^*A_n(Y_s^n)\,ds,$$

By assumptions **(A.1)**$(i)-(iii)$, we have

$$|Y_t^n-a|^2+\int_t^1|Z_s^n|^2\,ds+2R\int_t^1|A_n(Y_s^n)|\,ds$$
$$\le\mid\xi-a\mid^2+2\mu\int_t^1\mid Y_s^n-a\mid^2ds+2\int_t^1\mid Y_s^n-a\mid(\gamma\mid Z_s^n\mid+K(1+\mid a\mid^p))ds$$
$$+\int_t^1\mid Y_s^n-a\mid^2ds+\int_t^1\mid f(s,0,0)\mid^2ds+C-\int_t^1Y_s^nZ_s^ndW_s,$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ of both sides, we get that

$$|Y_t^n-a|^2\le\mathbb{E}\big[\mid\xi-a\mid^2/\mathcal{F}_t\big]+(2\mid\mu\mid+4\gamma^2+1)\mathbb{E}\big[\int_t^1\mid Y_s^n-a\mid^2ds/\mathcal{F}_t\big]$$
$$+\mathbb{E}\big[\int_0^1\mid f(s,0,0)\mid^2ds/\mathcal{F}_t\big]+2C\mathbb{E}\int_0^1(1+\mid a\mid^p)ds+C.$$

Jensen's inequality shows that for every $p>1$,

$$\mathbb{E}\,|Y_t^n-a|^{2p}\le C_p\big[\mathbb{E}\big[\mid\xi-a\mid^{2p}\big]+(2\mid\mu\mid+4\gamma^2+1)^p\mathbb{E}\big[\int_t^1\mid Y_s^n-a\mid^{2p}ds\big]$$
$$+\mathbb{E}\big(\int_0^1\mid f(s,0,0)\mid^2ds\big)^p+1\big]$$
$$\le C_p(1+\mathbb{E}\int_t^1|Y_s^n-a|^{2p}\,ds).$$

Gronwall's lemma implies that

$$\sup_{0\le t\le 1}\mathbb{E}\mid Y_t^n\mid^{2p}<+\infty,\ \ \forall n. \tag{3.7}$$

Assertion a) is proved.

b) We assume without loss of generality that ϕ is positive and $\phi(0) = 0$. Let us note that ϕ_n is a convex C^1-function with a lipschitz derivative, and put $\psi_n = \dfrac{\phi_n}{n}$.

By convolution of ψ_n with a smooth function, the convexity of ψ_n and Itô's formula, one can show that,

$$\begin{aligned}\psi_n(Y_t^n) \leq \psi_n(\xi) + &\int_t^1 \nabla\psi_n(Y_r^n)(f(r, Y_r^n, Z_r^n) - A_n(Y_r^n))dr \\ &- \int_t^1 \nabla\psi_n(Y_r^n) Z_r^n dW_r,\end{aligned}$$

thus

$$\begin{aligned}\mathbb{E}\psi_n(Y_s^n) &\leq \mathbb{E}\psi_n(\xi) + \mathbb{E}\int_t^1 \nabla\psi_n(Y_r^n)(f(r, Y_r^n, Z_r^n) - A_n(Y_r^n))dr \\ &= \mathbb{E}\psi_n(\xi) + \mathbb{E}\int_t^1 \nabla\psi_n(Y_r^n) f(r, Y_r^n, Z_r^n)dr - \frac{1}{n}\mathbb{E}\int_t^1 \mid A_n(Y_r^n) \mid^2 dr.\end{aligned}$$

Hence, using the elementary inequality $2ab \leq na^2 + \frac{1}{n}b^2$ we deduce,

$$\begin{aligned}\mathbb{E}\psi_n(Y_s^n) + \frac{1}{n}\mathbb{E}\int_t^1 \mid A_n(Y_r^n) \mid^2 dr \leq \mathbb{E}\psi_n(\xi) &+ \frac{1}{2n}\mathbb{E}\int_t^1 \mid A_n(Y_r^n) \mid^2 dr \\ &+ \frac{1}{2n}\mathbb{E}\int_t^1 \mid f(s, Y_s^n, Z_s^n) \mid^2 ds.\end{aligned}$$

We use assumptions **(A.1)**$(iv), (ii)$, to get

$$\begin{aligned}&\mathbb{E}\psi_n(Y_s^n) + \frac{1}{n}\mathbb{E}\int_t^1 \mid A_n(Y_r^n) \mid^2 dr \\ &\quad\leq \mathbb{E}\psi_n(\xi) + \frac{1}{2n}\mathbb{E}\int_t^1 \mid A_n(Y_r^n) \mid^2 dr + \frac{2\gamma^2}{n}\mathbb{E}\int_t^1 \mid Z_s^n \mid^2 ds \\ &\quad+ \frac{2}{n}\mathbb{E}\int_t^1 \mid f(s,0,0) \mid^2 ds + \frac{2K^2}{n}\mathbb{E}\int_t^1 (1+ \mid Y_s^n \mid^{2p})ds.\end{aligned}$$

The relations (3.5), (3.6) and (3.7) allowed us to prove that

$$\mathbb{E}\psi_n(Y_s^n) + \frac{1}{n}\mathbb{E}\int_t^1 \mid A_n(Y_r^n) \mid^2 dr \leq \frac{C}{n},$$

which implies that

$$\sup_n \mathbb{E}\int_0^1 \mid A_n(Y_r^n) \mid^2 dr < +\infty. \tag{3.8}$$

Lemma 3.3 is proved ■

Lemma 3.4. *Let assumptions of Theorem 3.1 hold. Then*

$$\mathbb{E}\sup_{0\leq t\leq 1} \mid Y_t^n - Y_t^m \mid^2 + \mathbb{E}\int_t^1 \mid Z_s^n - Z_s^m \mid^2 ds \leq C(\frac{1}{n} + \frac{1}{m})$$

Proof . Using Itô's formula, we get

$$
\begin{aligned}
&|Y_t^n - Y_t^m|^2 + \int_t^1 |Z_s^n - Z_s^m|^2\, ds \\
&= 2\int_t^1 (Y_s^n - Y_s^m)^* \left[f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m)\right] ds \\
&+ 2\int_t^1 (Y_s^n - Y_s^m)^* (Z_s^n - Z_s^m)\, dW_s \\
&- 2\int_t^1 (Y_s^n - Y_s^m)^* A_n(Y_s^n)\, ds + 2\int_t^1 (Y_s^n - Y_s^m)^* A_m(Y_s^m)\, ds,
\end{aligned}
$$

and then,

$$
\begin{aligned}
&|Y_t^n - Y_t^m|^2 + \int_t^1 |Z_s^n - Z_s^m|^2\, ds \\
&= 2\int_t^1 (Y_s^n - Y_s^m)^* \left[f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^n)\right] ds \\
&+ 2\int_t^1 (Y_s^n - Y_s^m)^* \left[f(s, Y_s^m, Z_s^n) - f(s, Y_s^m, Z_s^m)\right] ds \\
&+ 2\int_t^1 (Y_s^n - Y_s^m)^* (Z_s^n - Z_s^m)\, dW_s \\
&- 2\int_t^1 (Y_s^n - Y_s^m)^* A_n(Y_s^n)\, ds + 2\int_t^1 (Y_s^n - Y_s^m)^* A_m(Y_s^m)\, ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\mathbb{E}\,|Y_t^n - Y_t^m|^2 + \mathbb{E}\int_t^1 |Z_s^n - Z_s^m|^2\, ds \\
&\leq 2\mu\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2\, ds + 2\gamma\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|\mid Z_s^n - Z_s^m \mid ds \\
&-2\mathbb{E}\int_t^1 (Y_s^n - Y_s^m)^* \left(A_n(Y_s^n) - A_m(Y_s^m)\right) ds.
\end{aligned}
$$

Since, $Id = J_n + \frac{1}{n}A_n = J_m + \frac{1}{m}A_m$, $(A_m(Y_s^m), A_n(Y_s^n)) \in A(J_m(Y_s^m)) \times A(J_n(Y_s^n))$ and $xy \leq \frac{1}{4}x^2 + y^2$,$\forall\ x \geq 0\ \forall\ y \geq 0$, we can show that

$$
-\langle Y_s^n - Y_s^m, A_n(Y_s^n) - A_m(Y_s^m)\rangle \leq \frac{1}{4m}|A_n(Y_s^n)|^2 + \frac{1}{4n}|A_m(Y_s^m)|^2,
$$

and then

$$
\begin{aligned}
&\mathbb{E}\,|Y_t^n - Y_t^m|^2 + \mathbb{E}\int_t^1 |Z_s^n - Z_s^m|^2\, ds \\
&\leq (2\mid \mu \mid + \beta^2)\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2\, ds + \frac{\gamma^2}{\beta^2}\mathbb{E}\int_t^1 |Z_s^n - Z_s^m|^2\, ds \\
&+\mathbb{E}\int_t^1 \left(\frac{1}{4m}|A_n(Y_s^n)|^2 + \frac{1}{4n}|A_m(Y_s^m)|^2\right) ds.
\end{aligned}
$$

If we choose β such that $\dfrac{\gamma^2}{\beta^2} < \dfrac{1}{2}$, we get

$$\sup_{0\leq t\leq 1} \mathbb{E}\left|Y_t^n - Y_t^m\right|^2 + \frac{1}{2}\mathbb{E}\int_t^1 \left|Z_s^n - Z_s^m\right|^2 ds \leq C\left(\frac{1}{n} + \frac{1}{m}\right).$$

Using Burkholder-Davis-Gundy inequality, we obtain

$$\mathbb{E}\sup_{0\leq t\leq 1} \left|Y_t^n - Y_t^m\right|^2 + \frac{1}{2}\mathbb{E}\int_0^1 \left|Z_s^n - Z_s^m\right|^2 ds \leq C\left(\frac{1}{n} + \frac{1}{m}\right).$$

Lemma 3.4 is proved. ■

Lemma 3.5. *(see Saisho [75]) Let* $(k^n)_{n\in\mathbb{N}}$ *be a sequence of continuous and bounded variation functions from* $[0,1]$ *to* $\mathbb{R}^d$*, such that :*
(i) $\sup_n Var\,(k^n) \leq C < +\infty$.
(ii) $\lim_{n\to\infty} k^n = k$ *uniformly on* $[0,1]$.
(iii) Let $(f^n)_{n\in\mathbb{N}}$ *be a sequence of càdlàg functions from-* $[0,1]$ *to* R^d*, such that* $\lim_{n\to\infty} f^n = f$ *uniformly on* $[0,1]$.
Then for every $t\in[0,1]$ *we have:*

$$\lim_{n\to\infty}\int_0^t \left\langle f^n(s), dk^n(s)\right\rangle = \int_0^t \left\langle f(s), dk(s)\right\rangle.$$

Proof of Theorem 3.1
Existence. By Lemma 3.4, the sequence $(Y^n, Z^n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in the Banach space of progressively measurable processes $\mathbb{L}$ defined by,

$$\mathbb{L} = \left\{(Y,Z)\ /\ \mathbb{E}(\sup_{0\leq t\leq 1}|Y_t|^2) + \frac{1}{2}\mathbb{E}\int_0^1 |Z_s|^2 ds) < \infty\right\}.$$

Let (Y,Z) be the limit of (Y^n, Z^n) in $\mathbb{L}$.
If we return to the equation satisfied by $(Y^n, Z^n)_{n\in\mathbb{N}}$, we can whow that $(K^n)_{n\in\mathbb{N}}$ converges uniformly in $L^2(\Omega)$ to the process $K_\cdot = \lim_{n\longrightarrow+\infty}\int_0^\cdot A_n(Y_s^n)\,ds$, that is

$$\mathbb{E}\sup_{0\leq t\leq 1}\left|K_t^n - K_t\right|^2 = 0.$$

The relation (3.8) can be written as

$$\sup_{n\in\mathbb{N}^*}\mathbb{E}\left\|K_n\right\|^2_{H^1\left(0,1;\mathbb{R}^d\right)} < +\infty,$$

where $H^1\left(0,1;\mathbb{R}^d\right)$ is the usual Sobolev space consisting of all absolutely continuous functions with derivative in $L^2(0,1)$. Hence the sequence (K^n) is bounded in the Hilbert space $L^2\left(\Omega; H^1\left(0,1;\mathbb{R}^d\right)\right)$, and there exists then a subsequence of (K^n) which converges weakly. The limiting process K belongs to $L^2\left(\Omega; H^1\left(0,1;\mathbb{R}^d\right)\right)$ and a.s. $K_\cdot(\omega) \in H^1\left(0,1;\mathbb{R}^d\right)$. Hence K is absolutely continuous and $\dfrac{dK_t}{dt} = V_t$, where $-V_t \in \partial\phi(Y_t)$.
We shall prove that (Y,Z,K) is the unique solution to our equation. Taking a subsequence, if necessary, we can suppose that:

$$\sup_{t\in[0,1]}\left|K_t^n - K_t\right| \longrightarrow 0,\ a.s.$$

$$\sup_{t\in[0,1]} |Y_t^n - Y_t| \longrightarrow 0,\ a.s.$$

It follows that K_t and Y_t are continuous. Let (α,β) be a continuous processes with values in $Gr(\partial\phi)$. It holds that

$$\langle J_n(Y_t^n) - \alpha\,(t)\,, dK_t^n + \beta_t dt\rangle \le 0.$$

Since $J_n(Y_t^n)$ converge to $\mathbf{pr}(Y_t)$, where $\mathbf{pr}$ denotes the projection on $\overline{Dom(\phi)}$, then we use Lemma 3.5 to show that $\langle \mathbf{pr}(Y_t) - \alpha\,(t)\,, dK\,(t) + \beta_t dt\rangle \le 0$.
Since the process $(Y_t, 0\le t\le 1)$ is continuous, the proof of existence will complete if we show that

$$\mathbb{P}\left\{Y_t \in \overline{Dom(\phi)}\right\} = 1 \qquad \forall\ t \ge 0.$$

Assume that there exist $0 < t_0 < \infty$ and $B_0 \in\mathcal{F}$ such that $\mathbb{P}\,(B_0) > 0$ and $Y_{t_0}\,(\omega) \notin \overline{Dom(\phi)}$ $\forall\ \omega \in B_0$. By the continuity, there exist $\delta > 0, B_1 \in\mathcal{F}$ such that $\mathbb{P}\,(B_1) > 0$, $Y_t\,(\omega) \notin \overline{Dom(\phi)}$ for every $(\omega, t) \in B_1 \times [t_0, t_0 + \delta]$. Using the fact that

$$\sup_{n\in\mathbb{N}^*} \mathbb{E}\int_0^1 |A_n\,(Y_s^n)|\, ds < +\infty,$$

and by Fatou's lemma, we obtain

$$\int_{B_1}\int_{t_0}^{t_0+\delta} \liminf_{n\longrightarrow+\infty} |A_n\,(Y_s^n)|\, ds\, d\mathbb{P} < +\infty,$$

which contradict the fact that $\liminf\limits_{n\longrightarrow+\infty} |A_n\,(Y_s^n)| = +\infty$ on the set $B_1 \times [t_0, t_0 + \delta]$. This complete the existence proof. ■

Uniqueness. Let $\{(Y_t, Z_t, K_t)\,;\ 0\le t\le 1\}$ and $\{(Y_t', Z_t', K_t')\,;\ 0\le t\le 1\}$ denote two solutions of our BSDE. Define

$$\{(\Delta Y_t, \Delta Z_t, \Delta K_t)\ \ ;0\le t\le 1\} = \left\{\left(Y_t - Y_t', Z_t - Z_t', K_t - K_t'\right)\ \ ;0\le t\le 1\right\}.$$

It follows from Itô's formula that,

$$\mathbb{E}\left[|\Delta Y_t|^2 + \int_t^1 |\Delta Z_s|^2\, ds\right] = 2\mathbb{E}\int_t^1 \left\langle \Delta Y_s, f(s, Y_s, Z_s) - f(s, Y_s', Z_s)\right\rangle ds$$

$$+2\mathbb{E}\int_t^1 \left\langle \Delta Y_s, f(s, Y_s', Z_s) - f(s, Y_s', Z_s')\right\rangle ds + 2\mathbb{E}\int_t^1 \langle \Delta Y_s, d\Delta K_s\rangle\,.$$

By assumptions **(A.1)**$(ii) - (iii)$, we get

$$\mathbb{E}\left[|\Delta Y_t|^2 + \int_t^1 |\Delta Z_s|^2\, ds\right]$$

$$= (2\mu + \beta^2)\mathbb{E}\int_t^1 \mid \Delta Y_s \mid^2 ds + \frac{\gamma^2}{\beta^2}\mathbb{E}\int_t^1 \mid \Delta Z_s \mid^2 ds + 2\mathbb{E}\int_t^1 \langle Y_s, d\Delta K_s\rangle\,.$$

Since $\partial\phi$ is a monotone operator and $\dfrac{-dK_t}{dt} \in \partial\phi(Y_t)$, $\dfrac{-dK_t'}{dt} \in \partial\phi(Y_t')$, then

$$\mathbb{E}\int_t^1 \langle \Delta Y_s, d\Delta K_s\rangle \le 0.$$

Hence, taking $\dfrac{\gamma^2}{\beta^2} = \dfrac{1}{2}$, we have

$$\mathbb{E}\left[|\Delta Y_t|^2 + \int_t^1 |\Delta Z_s|^2\, ds\right] \leq C\mathbb{E}\int_t^1 |\Delta Y_s|^2\, ds + \frac{1}{2}\mathbb{E}\int_t^1 |\Delta Z_s|^2\, ds.$$

The result follows from Gronwall's lemma. ■

3.2 Reflected Backward Stochastic Differential Equation with Locally monotone Coefficient

The aim of this section is to extend the previous results to the case where the generator f is locally monotone on the y-variable and locally lipschitz on the z-variable. Our existence and uniqueness has been proved in Pardoux [66] for BSDE (without reflection) in the case where the generator f is globally monotone w.r.t. the variable y and Lipschitz w.r.t. the variable z, and more recently in Bahlali et *al.* [30] for BSDE with reflection and jumps in the case where the generator is locally Lipschitz w.r.t. the variables y and z. Our result is, in particular, an extension of the two results.
Consider the following assumptions:
(i) f is continuous in (y, z) for almost all (t, ω),
(ii) There exist $M > 0$ and $0 \leq \alpha \leq 1$ such that $\mid f(t, \omega, y, z) \mid \leq M(1 + \mid y \mid^{\alpha} + \mid z \mid^{\alpha})$.
(iii) There exists μ_N such that:

$$\langle y - y', f(t, y, z) - f(t, y', z)\rangle \leq \mu_N \mid y - y' \mid^2;\ \mathbb{P} - a.s.,\ a.e.t \in [0, 1] \text{ and}$$
$$\forall y,\ z \text{ such that } \mid y \mid \leq N,\ \mid y' \mid \leq N,\ \mid z \mid \leq N.$$

(iv) For each $N > 0$, there exists L_N such that:

$$\mid f(t, y, z) - f(t, y, z') \mid \leq L_N \mid z - z' \mid;\mid z \mid, \mid z' \mid \leq N;\ \mathbb{P} - a.s.,\ a.e.t \in [0, 1] \text{ and}$$
$$\forall y,\ z,\ z' \text{ such that } \mid y \mid \leq N,\ \mid z \mid \leq N,\ \mid z' \mid \leq N.$$

When the assumptions $(i), (ii)$, are satisfied, we can define the family of semi norms $(\rho_n(f))_n$

$$\rho_n(f) = (\mathbb{E}\int_0^1 \sup_{|y|,|z|\leq n} \mid f(s, y, z) \mid^2 ds)^{\frac{1}{2}}.$$

The main result of this section is the following

Theorem 3.6. *Let (i)-(iv) hold and ξ be a square integrable random variable. Assume moreover that $\alpha < 1$ and*

$$\lim_{N \to +\infty} \frac{\exp(L_N^2 + 2\mu_N^+)}{(L_N^2 + 2\mu_N^+)N^{2(1-\alpha)}} = 0, \tag{3.9}$$

where $\mu_N^+ = \sup(\mu_N, 0)$. Then equation $(1) - (5)$ has a unique solution.
In particular, if $\alpha \leq 1$ and there exists a constant $L \geq 0$ such that

$$L_N^2 + 2\mu_N^+ \leq L + 2(1 - \alpha)logN,$$

then equation $(1) - (5)$ has also a unique solution.

Remark 3.7. *It should be noted that there is existence and uniqueness if we replace condition (ii) by*
(ii') There exists $M > 0$ *and* $0 \leq \alpha \leq 1$ *such that* $\mid f(t,\omega,y,z) \mid \leq M(1+ \mid y \mid + \mid z \mid^{\alpha})$.

To prove Theorem 3.6 we need the following lemmas.

Lemma 3.8. *Let* f *be a process which satisfies (i), (ii), (iii), (iv). Then there exists a sequence of processes* (f_n) *such that,*
-(*a*)- *For each* n, f_n *is globally* L_n^1*-Lipschitz in* (y,z) *a.e.* t *and* P*-a.s.*ω.
-(*b*)- *For each* n, f_n *is* $\mu_{(N+\frac{1}{n})}$ *-locally monotone in* y *a.e.* t, P*-a.s.*ω *and for each* z.
-(*c*)- $\sup_n |f_n(t,\omega,y,z)| \leq |f(t,\omega,y,z)| \leq M(1+|y|^{\alpha}+|z|^{\alpha}) \quad P$*-a.s., a.e.* $t \in [0,1]$.
-(*d*)- *For every* N, $\rho_N(f_n - f) \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Proof . Let $\rho_n : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ be a sequence of smooth functions with compact support which approximate the Dirac measure at 0 and which satisfy $\int \rho_n(u)du = 1$. Let $\varphi_n : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ be a sequence of smooth functions such that $0 \leq \varphi_n \leq 1$, $\varphi_n(u) = 1$ for $|u| \leq n$ and $\varphi_n(u) = 0$ for $|u| \geq n+1$. Likewise we define the sequence ψ_n from $\mathbb{R}^{d\times r}$ to $\mathbb{R}_+$. We put, $f_{q,n}(t,y,z) = \int f(t,y-u,z)\rho_q(u)du\varphi_n(y)\psi_n(z)$. For $n \in \mathbb{N}^*$, let $q(n)$ be an integer such that $q(n) \geq M[n+n^{\alpha}]$. It is not difficult to see that the sequence $f_n := f_{q(n),n}$ satisfies all the assertions (a)-(d). Lemma 3.8 is proved. ■

Consider, for fixed (t,ω) the sequence $f_n(t,\omega,y,z)$ associated to f by Lemma 3.8. We get from the previous section that there exists a unique triplet $\{(Y_t^n, Z_t^n, K_t^n; 0 \leq t \leq 1)\}$ of progressively measurable processes which satisfy:

$$\begin{cases} (1') Z^n \text{ is adapted process and } \mathbb{E}\displaystyle\int_0^1 |Z_t^n|^2 dt < +\infty, \\ (2')\, Y_t^n = \xi + \displaystyle\int_t^1 f_n(s,Y_s^n,Z_s^n)ds - \int_t^1 Z_s^n dW_s + K_1^n - K_t^n, 0 \leq t \leq 1, \\ (3') \text{ the process } Y^n \text{ is continuous} \\ (4')\, K^n \text{ is absolutely continuous}, K_0^n = 0, \text{ and for every progressively measurable} \\ \text{and continuous processes } (\alpha,\beta) \text{ such that } (\alpha_t,\beta_t) \in Gr(\partial\phi), \text{ we have} \\ \qquad\qquad \displaystyle\int_0^{\cdot} (Y_t^n - \alpha_t)(dK_t^n + \beta_t dt) \leq 0. \\ (5')\, Y_t^n \in \overline{Dom(\phi)},\ 0 \leq t \leq 1 \text{ a.s.} \end{cases}$$

Lemma 3.9. *There exists a constant* C *depending only in* M *and* $\mathbb{E} \mid \xi \mid^2$*, such that*

$$\mathbb{E}(\sup_{0\leq t\leq 1} |Y_t^n|^2 + \int_0^1 |Z_s^n|^2\, ds + \mid K_1^n \mid^2) \leq C,\ \forall\, n \in \mathbb{N}^*.$$

Proof : Since $|x|^{\alpha} \leq 1+|x|\ \forall\alpha \in [0,1]$, the proof follows by standard arguments for BSDE. ■

Lemma 3.10. *There exist (Y,Z,K) such that*

$$\lim_{n\to\infty} \mathbb{E}\{\sup_{0\leq t\leq 1} \mid Y_1^n - Y_t \mid^2 + \sup_{0\leq t\leq 1} \mid K_1^n - K_t \mid^2 + \int_0^1 |Z_s^n - Z_s|^2\, ds\} = 0.$$

Proof . For $\alpha = 1$, the result follows from [23]. We shall treat the case $\alpha < 1$. By Itô's formula we have,

$$\begin{aligned}
&\mathbb{E}(|Y_t^n - Y_t^m|^2) + E\int_t^1 |Z_s^n - Z_s^m|^2 ds \\
&\quad = 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)\rangle ds \\
&\qquad + 2\int_t^1 (Y_s^n - Y_s^m) dK_s^n \\
&\quad = I_0(n,m) + I_1(n,m) + I_2(n,m) + I_3(n,m) + 2\int_t^1 (Y_s^n - Y_s^m) d(K_s^n - K_s^m)
\end{aligned}$$

where

$$\begin{aligned}
I_0(n,m) &= 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)\rangle \mathbf{1}_{A_{n,m}^N} ds \\
I_1(n,m) &= 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n)\rangle \mathbf{1}_{\overline{A}_{n,m}^N} ds \\
I_2(n,m) &= 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m)\rangle \mathbf{1}_{\overline{A}_{n,m}^N} ds \\
I_3(n,m) &= 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f(s, Y_s^m, Z_s^m) - f_m(s, Y_s^m, Z_s^m)\rangle \mathbf{1}_{\overline{A}_{n,m}^N} ds.
\end{aligned}$$

Since K^n, K^m are absolutely continuous, $Y^n, Y^m \in \overline{Dom(\phi)}$ and the measures

$$\langle Y_t^n - \alpha_t, dK_t^n - \beta_t dt\rangle$$

$$\langle Y_t^m - \alpha_t, dK_t^m - \beta_t dt\rangle,$$

are negatives, we deduce from Lemma 4.1 in Cépa [19] that

$$\langle Y_s^n - Y_s^m, d(K_s^n - K_s^m)\rangle$$

is also negative.
We shall estimate $I_0(n,m)$, $I_1(n,m)$, $I_2(n,m)$, $I_3(n,m)$. Let β be a strictly positive number. For a given $N > 1$, we put $A_{n,m}^N := \{(s,\omega); \quad |Y_s^n|^2 + |Z_s^n|^2 + |Y_s^m|^2 + |Z_s^m|^2 \geq N^2\}$, $\overline{A}_{n,m}^N := \Omega \setminus A_{n,m}^N$ and denote by $\mathbf{1}_E$ the indicator function of the set E. By standard arguments of BSDE we have,

$$\begin{aligned}
I_0(n,m) &\leq \beta^2 \mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 \mathbf{1}_{A_{n,m}^N} ds \\
&\quad + \frac{1}{\beta^2}\mathbb{E}\int_t^1 |\ f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n)\ |^2\ \mathbf{1}_{A_{n,m}^N} ds
\end{aligned}$$

We use Hölder inequality (since $\alpha < 1$) and Chebychev inequality to get,

$$I_0(n,m) \leq \beta^2 \mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 \mathbf{1}_{A_{n,m}^N} ds + \frac{K_2(M,\xi)}{\beta^2 N^{2(1-\alpha)}}. \tag{3.10}$$

Now

$$I_1(n,m) \leq \mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 ds + \mathbb{E}\int_t^1 |\ f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n)\ |^2\ \mathbf{1}_{\overline{A}_{n,m}^N} ds,$$

and then

$$I_1(n,m) \leq \mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 ds + \rho_N^2(f_n - f). \tag{3.11}$$

Likewise we show that,

$$I_3(n,m) \leq \mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 ds + \rho_N^2(f_m - f). \tag{3.12}$$

We use assumptions (iii) and (iv) to prove that,

$$\begin{aligned}
I_2(n,m) &\leq 2E\int_t^1 < Y_s^n - Y_s^m, \quad f(s,Y_s^n,Z_s^n) - f(s,Y_s^m,Z_s^n) > \mathbf{1}_{\overline{A}_{n,m}^N} ds \\
&\quad + 2\mathbb{E}\int_t^1 |Y_s^n - Y_s^m| \ |f(s,Y_s^m,Z_s^n) - f(s,Y_s^m,Z_s^m)|\mathbf{1}_{\overline{A}_{n,m}^N} ds \\
&\leq (2\mu_N + \gamma^2)\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 \mathbf{1}_{\overline{A}_{n,m}^N} ds + \frac{L_N^2}{\gamma^2}\mathbb{E}\int_t^1 |Z_s^n - Z_s^m|^2 ds.
\end{aligned}$$

We choose β and γ such that $\beta^2 = L_N^2 + 2\mu_N^+$ and $\gamma^2 = L_N^2$ then we use this last inequality (3.10), (3.11) and (3.12) to show that,

$$\begin{aligned}
\mathbb{E}(|Y_t^n - Y_t^m|^2) + \mathbb{E}\int_t^1 |Z_s^n - Z_s^m|^2 ds &\leq (L_N^2 + 2\mu_N^+ + 2)\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 ds \\
&\quad + [\rho_N^2(f_n - f) + \rho_N^2(f_m - f)] + \frac{K_3(M,\xi)}{(L_N^2 + 2\mu_N^+)N^{2(1-\alpha)}}.
\end{aligned}$$

Hence Gronwall Lemma implies that,

$$\mathbb{E}(|Y_t^n - Y_t^m|^2) \leq \left[[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)] + \frac{K_4(M,\xi)}{(L_N^2 + 2\mu_N^+)N^{2(1-\alpha)}}\right] \exp(L_N^2 + 2\mu_N^+ + 2).$$

Using Burkholder-Davis-Gundy inequality, we show that a universal positive constant C exists such that,

$$\begin{aligned}
\mathbb{E}(\sup_{0\leq t\leq 1} | Y_t^n - Y_t^m |^2) \leq C\Big[&[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)] \\
&+ \frac{K_4(M,\xi)}{(L_N^2 + 2\mu_N^+)N^{2(1-\alpha)}}\Big] \exp(L_N^2 + 2\mu_N^+ + 2).
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\int_0^1 |Z_s^n - Z_s^m|^2 ds \leq C\Big[&[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)] \\
&+ \frac{K_4(M,\xi)}{(L_N^2 + 2\mu_N^+)N^{2(1-\alpha)}}\Big] \exp(L_N^2 + 2\mu_N^+ + 2).
\end{aligned}$$

Passing to the limit successively on n,m and on N, to show that (Y^n, Z^n) is a Cauchy sequence in the Banach space $\mathbb{L}$.

Now, if we return to the equation satisfied by (Y^n, Z^n), we obtain that

$$\begin{aligned}
\mathbb{E}\sup_{0\leq t\leq 1} | K_t^n - K_t^m |^2 \leq \mathbb{E}\sup_{0\leq t\leq 1} &| Y_t^n - Y_t^m |^2 \\
&+ C\mathbb{E}\int_0^1 | f_n(s,Y_s^n,Z_s^n) - f_m(s,Y_s^m,Z_s^m) |^2 ds \\
&+ \mathbb{E}\int_0^1 | Z_s^n - Z_s^m |^2 ds.
\end{aligned}$$

We need to show that the sequence of processes $f_n(., Y^n, Z^n)_n$ converges to $f(., Y, Z)$ in L^2. We have

$$\begin{aligned}
&\mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 ds \\
&= \mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 \mathbf{1}_{A_n^N} ds \\
&+ 2\mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n) \mid^2 \mathbf{1}_{\overline{A}_n^N} ds \\
&+ 2\mathbb{E}\int_0^1 \mid f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 \mathbf{1}_{\overline{A}_n^N} ds \\
&\leq \frac{K_1}{N^{2(1-\alpha)}} + 2\rho_N^2(f_n - f) + I(n),
\end{aligned}$$

where

$$I(n) = 2\mathbb{E}\int_0^1 \mid f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 \mathbf{1}_{\overline{A}_n^N} ds.$$

We get for almost all ω that

$$f(s, Y_s^n, Z_s^n) \longrightarrow f(s, Y_s, Z_s), \quad dt - a.e. \text{ as } n \text{ goes to } +\infty,$$

and for all $\varepsilon > 0$

$$\begin{aligned}
&\mathbb{E}\int_0^1 \mid f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^{2+\varepsilon} ds \\
&\leq \mathbb{E}\int_0^1 (2+ \mid Y_s \mid^\alpha + \mid Y_s^n \mid^\alpha + \mid Z_s \mid^\alpha + \mid Z_s^n \mid^\alpha)^{2+\varepsilon} ds,
\end{aligned}$$

Put $\varepsilon = \dfrac{2-2\alpha}{\alpha}$, we have

$$\begin{aligned}
&\mathbb{E}\int_0^1 \mid f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^{2+\varepsilon} ds \\
&\leq \mathbb{E}\int_0^1 (2+ \mid Y_s \mid^2 + \mid Y_s^n \mid^2 + \mid Z_s \mid^2 + \mid Z_s^n \mid^2) ds \\
&< +\infty.
\end{aligned}$$

Hence

$$\lim_{n\to+\infty} I(n) = 0.$$

Therefore

$$\lim_{n\to+\infty} \mathbb{E}\int_0^1 \mid f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 ds = 0.$$

The proof of the existence, for the first result, is complete by passing to the limit successively on m, n and N.

Let us prove the second result. If $\alpha = 1$, the result follows from [23]. Suppose that $\alpha < 1$, arguing as above and using the fact that $L_N^2 + 2\mu_N^+ \leq L + 2(1-\alpha)\log(N)$ to show that

$$\mathbb{E}(\sup_{0\leq t\leq 1} \mid Y_t^n - Y_t^m \mid^2) \leq C\left[[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)]N^{2(1-\alpha)} + \frac{K_5(M, \xi)}{L_N^2 + 2\mu_N^+}\right] e^{(2+L)}$$

$$\mathbb{E}\int_0^1 |Z_s^n - Z_s^m|^2 ds \leq C\left[[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)]N^{2(1-\alpha)} + \frac{K_5(M,\xi)}{L_N^2 + 2\mu_N^+}\right] e^{(2+L)}.$$

We can assume that L_N or μ_N goes to infinity (if not, see Remark 3.11), passing to the limit we get the desired result.

Uniqueness: Let $\{(Y_t, Z_t, K_t)\ 0 \leq t \leq 1\}$ and $\{(Y_t', Z_t', K_t')\ 0 \leq t \leq 1\}$ be two solutions of our BSDE, we put

$$\{(\Delta Y_t, \Delta Z_t, \Delta K_t)\ 0 \leq t \leq 1\} = \left\{\left(Y_t - Y_t', Z_t - Z_t', K_t - K_t'\right)\ 0 \leq t \leq 1\right\}$$

It follows from Itô's formula that

$$\mathbb{E}\left[|\Delta Y_t|^2 + \int_t^1 |\Delta Z_s|^2\, ds)\right] = 2\mathbb{E}\int_t^1 \left\langle \Delta Y_s, f(s, Y_s, Z_s) - f(s, Y_s', Z_s')\right\rangle ds$$
$$+ 2\mathbb{E}\int_t^1 \langle \Delta Y_s, d\Delta K_s\rangle$$

By Saisho [75] (Lemma 3.5), we get

$$\mathbb{E}\int_t^1 \langle \Delta Y_s, d\Delta K_s\rangle \leq 0.$$

For $N > 1$, let μ_N the monotony constant of f in the balls $B(0,N)$, $\mathbf{1}_{A^N} := \{(s,w); |\ Y_s\ |^2 + |\ Y_s'\ |^2 + |\ Z_s\ |^2 + |\ Z_s'\ |^2 \geq N\}$, $\overline{A}^N := \Omega \setminus A^N$.

$$\mathbb{E}\left[|\Delta Y_t|^2 + \int_t^1 |\Delta Z_s|^2\, ds\right] \leq I_1(N) + I_2(N),$$

where

$$I_1(N) = 2\mathbb{E}\int_t^1 \langle \Delta Y_s, f(s, Y_s, Z_s) - f(s, Y_s', Z_s)\rangle \mathbf{1}_{\overline{A}^N} ds$$
$$+ 2\mathbb{E}\int_t^1 \langle \Delta Y_s, f(s, Y_s', Z_s) - f(s, Y_s', Z_s'\rangle \mathbf{1}_{\overline{A}^N} ds,$$

and

$$I_2(N) = 2\mathbb{E}\int_t^1 \langle \Delta Y_s, f(s, Y_s, Z_s) - f(s, Y_s', Z_s'\rangle \mathbf{1}_{A^N} ds.$$

We shall estimate $I_1(N)$ and $I_2(N)$. As above we obtain

$$I_1(N) \leq (2\mu_N^+ + \gamma^2)\mathbb{E}\int_t^1 |\ \Delta Y_s\ |^2\ \mathbf{1}_{\overline{A}^N} ds + \frac{L_N^2}{\gamma^2}\mathbb{E}\int_t^1 |\ \Delta Z_s\ |^2\ ds,$$

and

$$I_2(N) \leq \beta^2\mathbb{E}\int_t^1 |\ \Delta Y_s\ |^2\ \mathbf{1}_{A^N} ds + \frac{C}{\beta^2 N^{2(1-\alpha)}}$$

Taking $\beta^2 = L_N^2 + 2\mu_N^+$ and $\gamma^2 = L_N^2$ and using the estimates for $I_1(N)$ and $I_2(N)$, we have

$$\mathbb{E}\,|\Delta Y_t|^2 \leq (L_N^2 + 2\mu_N^+)\mathbb{E}\int_t^1 |\Delta Y_s|^2\, ds + \frac{C}{(L_N^2 + 2\mu_N^+)N^{2(1-\alpha)}}.$$

Using Gronwall's and Burkholder-Davis-Gundy inequalities, we get

$$\mathbb{E}\sup_{0\le t\le 1}|\Delta Y_t|^2 \le \frac{C}{(L_N^2+2\mu_N^+)N^{2(1-\alpha)}}\exp(L_N^2+2\mu_N^+),$$

$$\mathbb{E}\int_0^1 |\Delta Z_s|^2\,ds \le \frac{C}{(L_N^2+2\mu_N^+)N^{2(1-\alpha)}}\exp(L_N^2+2\mu_N^+),$$

the uniqueness follows by passing to the limit on N. ∎

Suppose now that f is globally Lipschitz with respect to z, that is

$$|\, f(t,y,z)-f(t,y,z') \,|\le L\mid z-z'\mid . \tag{iv'}$$

Remark 3.11. *Theorem 3.6 remains true under assumptions (i), (ii), (iii), (iv') and* $2\mu_N^+\le L+2(1-\alpha)logN$, *for* $L>0$.
Indeed, if μ_N *is also bounded the result of Theorem 3.6 follows from Pardoux [66]. Else, arguing as in the proof of Theorem 3.6 we obtain*

$$\mathbb{E}(\sup_{0\le t\le 1}\mid Y_t^n-Y_t^m\mid^2)\le C\Big(\big[\rho_N^2(f_n-f)+\rho_N^2(f_m-f)\big]N^{2(1-\alpha)}+\frac{K_6(M,\xi)}{2\mu_N^+}\Big)e^L$$

and

$$\mathbb{E}\int_0^1 |Z_s^n-Z_s^m|^2ds\le C\Big(\big[\rho_N^2(f_n-f)+\rho_N^2(f_m-f)\big]N^{2(1-\alpha)}+\frac{K_6(M,\xi)}{2\mu_N^+}\Big)e^L.$$

Passing to the limit, we get the desired result.

Corollary 3.12. *Assume that (i), (ii), (iii) and (iv') hold. If* $\lim_N \dfrac{\exp^{2\mu_N^+}}{2\mu_N^+N^{2(1-\alpha)}}=0$, *then the RBSDE (1)-(5) has one and only one solution.*

Example 3.13. *For example if* $2\mu_N^+\le 2(1-\alpha)\log(N)$, *then (1)-(5) has one solution.*

Proof of corollary 3.12. Arguing as in the proof of Theorem 3.6, we show that

$$\mathbb{E}(\sup_{0\le t\le 1}\mid Y_t^n-Y_t^m\mid^2)\le C\Big(\big[\rho_N^2(f_n-f)+\rho_N^2(f_m-f)\big]+\frac{K_6(M,\xi)}{2\mu_N^+N^{2(1-\alpha)}}\Big)e^{2\mu_N^+}$$

and

$$\mathbb{E}\int_0^1 |Z_s^n-Z_s^m|^2ds\le C\Big(\big[\rho_N^2(f_n-f)+\rho_N^2(f_m-f)\big]+\frac{K_6(M,\xi)}{2\mu_N^+N^{2(1-\alpha)}}\Big)e^{2\mu_N^+}.$$

Passing to the limit on n, m, N and using the same arguments as in the proof of theorem 3.6, one has the desired result. ∎

3.3 Multidimensional BSDE with non locally Lipschitz coefficient

Let $(W_t)_{0\le t\le T}$ be a r-dimensional Wiener process defined on a complete probability space $(\Omega,\mathcal{F},P)$. Let $(\mathcal{F}_t)_{0\le t\le T}$ denote the natural filtration of (W_t) such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, and ξ be an $\mathcal{F}_T$-measurable d-dimensional square integrable random variable. Let f be an $\mathbb{R}^d$-valued process defined on $\mathbb{R}_+\times\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times r}$ such that for all $(y,z)\in\mathbb{R}^d\times\mathbb{R}^{d\times r}$, the map $(t,\omega) \longrightarrow f(t,\omega,y,z)$ is $\mathcal{F}_t$-progressively measurable. We consider the following BSDE,

$$(E^f) \qquad\qquad Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \qquad\qquad 0 \leq t \leq T.$$

In this section, we extend our previous results essentially in two directions. First, the coefficient grow "almost" in quadratic fashion in the two variables Y and z, i.e. $\mid f(t,\omega,y,z) \mid \leq \overline{\eta} + M(\mid y \mid^\alpha + \mid z \mid^\alpha)$ for some $\alpha < 2$. Second the coefficient may be no locally Lipschitz. For example, our coefficient can take the form: $|z|\sqrt{|\log|z||}$ or $|y||\log|y||$.

3.3.1 The main result.

We denote by $\mathbb{E}$ the set of $\mathbb{R}^d \times \mathbb{R}^{d\times r}$-valued processes (Y,Z) defined on $\mathbb{R}_+ \times \Omega$ which are $\mathcal{F}_t$-adapted and such that: $||(Y,Z)||^2 = \mathbb{E}\big(\sup_{0\leq t\leq T} \mid Y_t \mid^2 + \int_0^T |Z_s|^2 ds\big) < +\infty$. The couple $(\mathbb{E}, ||.||)$ is then a Banach space.

Definition 3.14. *A solution of equation (E^f) is a couple (Y,Z) which belongs to the space $(\mathbb{E}, ||.||)$ and satisfies (E^f).*

Consider the following assumptions:
(H.1) f is continuous in (y,z) for almost all (t,ω).
(H.2) There exist $M > 0, \gamma < \frac{1}{2}$ and $\eta \in \mathbb{L}^1\left(\Omega; \mathbb{L}^1([0,T])\right)$ such that,

$$\langle y, f(t,\omega,y,z)\rangle \leq \eta + M|y|^2 + \gamma|z|^2 \quad P-a.s., \; a.e.\, t \in [0,T].$$

(H.3) There exist $M_1 > 0$, $0 \leq \alpha < 2$, $\alpha' > 1$ and $\overline{\eta} \in \mathbb{L}^{\alpha'}([0,T]\times\Omega)$ such that:

$$\mid f(t,\omega,y,z) \mid \leq \overline{\eta} + M_1(\mid y \mid^\alpha + \mid z \mid^\alpha).$$

(H.4) There exists a real valued sequence $(A_N)_{N>1}$ and constants $M_2 > 1$, $r > 0$ such that:
i) $\forall N > 1, \quad 1 < A_N \leq N^r$.
ii) $\lim_{N\to\infty} A_N = \infty$.
iii) For every $N \in \mathbb{N}$, $\forall y,\ y'\ z,\ z'$ such that $\mid y \mid, \mid y' \mid, \mid z \mid, \mid z' \mid \leq N$, we have

$$\langle y-y', f(t,y,z) - f(t,y',z')\rangle \leq \; M_2 \mid y-y' \mid^2 \log A_N + M_2 \mid y-y' \mid\mid z-z' \mid \sqrt{\log A_N} + M_2 A_N^{-1}.$$

For a given f, the solutions of equation (E^f) will be denoted by (Y^f, Z^f). When the assumption **(H.3)** is satisfied, we can define a family of semi-norms $\left(\rho_n(f)\right)_{n\in\mathbb{N}}$ by,

$$\rho_n(f) = \mathbb{E}\int_0^T \sup_{|y|,|z|\leq n} |f(s,y,z)|ds.$$

The main result is the following

Theorem 3.15. *Let ξ be a 2-integrable random variable. Assume that **(H.1)**–**(H.4)** are satisfied. Then equation (E^f) has a unique solution.*

In the following, we give a stability result for the solution with respect to the data (f,ξ). Roughly speaking, if f_n converges to f in the metric defined by the family of semi-norms (ρ_N) and ξ_n converges to ξ in $L^2(\Omega)$ then (Y^n, Z^n) converges to (Y, Z) in some reflexive Banach space which we will precise below. Let (f_n) be a sequence of processes which are $\mathcal{F}_t$-progressively measurable for each n. Let (ξ_n) be a sequence of random variables which are $\mathcal{F}_T$-measurable for each n and such that $E(|\xi_n|^2) < \infty$. We will assume that for each n, the BSDE (E^{f_n,ξ_n}) corresponding to the data (f_n, ξ_n) has a (not necessarily unique) solution. Each solution of the equation (E^{f_n,ξ_n}) will be denoted by (Y^n, Z^n). We suppose also that the following assumptions **(H.5)**, **(H.6)**, **(H.7)** are fulfilled,

(H.5) $\quad For$ every N, $\rho_N(f_n - f) \longrightarrow 0$ as $n \to \infty$.

(H.6) $\quad E(|\xi_n - \xi|^2) \longrightarrow 0$ as $n \to \infty$.

(H.7) There exist $M > 0, \gamma < \frac{1}{2}$ and $\eta \in \mathbb{L}^1\left(\Omega; \mathbb{L}^1([0,T])\right)$ such that,

$$\sup_n \langle y, f_n(t,\omega,y,z)\rangle \leq \eta + M|y|^2 + \gamma|z|^2 \quad P-a.s., \, a.e.\, t \in [0,T].$$

(H.8) There exist $M_1 > 0$, $0 \leq \alpha < 2$, $\alpha' > 1$ and $\overline{\eta} \in \mathbb{L}^{\alpha'}([0,T] \times \Omega)$ such that:

$$\sup_n \mid f_n(t,\omega,y,z) \mid \leq \overline{\eta} + M_1(\mid y\mid^\alpha + \mid z\mid^\alpha).$$

Theorem 3.16. *Let f and ξ be as in Theorem 3.15. Assume that **(H.5)**, **(H.6)**, **(H.7)** and **(H.8)** are satisfied. Then, for all $q < 2$ we have*

$$\lim_{n\to+\infty} \left(\mathbb{E} \sup_{0\leq t\leq T} |Y_t^n - Y_t|^q + \mathbb{E}\int_0^T |Z_s^n - Z_s|^q ds \right) = 0.$$

3.3.2 Proofs

To prove Theorem 3.15 we need the following lemmas.

Lemma 3.17. *Let f be a process which satisfies **(H.1)**–**(H.3)**. Then there exists a sequence of processes (f_n) such that,*
(a) For each n, f_n is bounded and globally Lipschitz in (y,z) a.e. t and P-a.s.ω.
There exists $M' > 0$, such that:
(b) $\sup_n |f_n(t,\omega,y,z)| \leq \overline{\eta} + M' + M_1(\mid y\mid^\alpha + \mid z\mid^\alpha)$. P-a.s., a.e. $t \in [0,T]$.
(c)

$$\sup_n < y, f_n(t,\omega,y,z) > \leq \eta + M' + M|y|^2 + \gamma|z|^2$$

(d) For every N, $\rho_N(f_n - f) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof . Let $\rho_n : \mathbb{R}^d \times \mathbb{R}^{d\times r} \longrightarrow \mathbb{R}_+$ be a sequence of smooth functions with compact support which approximate the Dirac measure at 0 and which satisfy $\int \rho_n(u)du = 1$. Let $\varphi_n : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ be a sequence of smooth functions such that $0 \leq \varphi_n \leq 1$, $\varphi_n(u) = 1$ for $|u| \leq n$ and $\varphi_n(u) = 0$ for $|u| \geq n+1$. Likewise we define the sequence ψ_n from $\mathbb{R}^{d\times r}$ to $\mathbb{R}_+$. We put, $f_{q,n}(t,y,z) = \mathbb{1}_{\{\overline{\eta}\leq q\}} \int f(t,(y,z)-u)\rho_q(u)du\varphi_n(y)\psi_n(z)$. For $n \in \mathbb{N}^*$, let $q(n)$ be an integer such that $q(n) \geq n + n^\alpha$. It is not difficult to see that the sequence $f_n := f_{q(n),n}$ satisfies all the assertions (a)-(d). ■

Using standard arguments of BSDEs one can prove the following estimates

Lemma 3.18. *Let f and ξ be as in Theorem 3.15. Let (f_n) be the sequence of processes associated to f by Lemma 3.17 and denotes by (Y^{f_n}, Z^{f_n}) the solution of equation (E^{f_n}). Then, there exits a universal constant ℓ such that*

a) $\mathbb{E}\displaystyle\int_0^T e^{2Ms}|Z_s^{f_n}|^2 ds \leq \frac{1}{1-2\gamma}\left[e^{2MT}\mathbb{E}\mid\xi\mid^2 + 2\mathbb{E}\int_0^T e^{2Ms}(\eta + M')ds\right] = K_1$

b) $\mathbb{E}\displaystyle\sup_{0\leq t\leq T}(e^{2Mt}\mid Y_t^{f_n}\mid^2) \leq \ell K_1 = K_2$

c) $\mathbb{E}\displaystyle\int_0^T e^{2Ms}|f_n(s, Y_s^{f_n}, Z_s^{f_n})|^{\overline{\alpha}} ds \leq 4^{\overline{\alpha}-1}\left[\mathbb{E}\int_0^T e^{2Ms}((\overline{\eta} + M')^{\overline{\alpha}} + 4)ds + M_1^{\overline{\alpha}} K_1 + TM_1^{\overline{\alpha}} K_2\right]$
$= K_3$

d) $\mathbb{E}\displaystyle\int_0^T e^{2Ms}|f(s, Y_s^{f_n}, Z_s^{f_n})|^{\overline{\alpha}} ds \leq K_3,$

where $\overline{\alpha} = \min(\alpha', \dfrac{2}{\alpha})$.

Proof . Using Itô's formula and Lemma 3.17 (c), we show that for all $t \leq T$

$$e^{2Mt}\mid Y_t^{f_n}\mid^2 + (1-2\gamma)\int_t^T e^{2Ms}\mid Z_s^{f_n}\mid^2 ds \leq e^{2MT}\mid\xi\mid^2 + 2\int_t^T e^{2Ms}(\eta_s + M')ds$$
$$-2\int_t^T e^{2Ms}\langle Y_s^{f_n}, \quad Z_s^{f_n} dW_s\rangle.$$

Taking expectation we get assertion a). Assertion b) is a direct of the Burkholder Davis Gundy inequality and assertion a). Finally, assertions c) and d) follow from Lemma 3.17 (b) and and assumption (H.3). Lemma 3.18 is proved. ■

After extracting a subsequence, if necessary, we have

Corollary 3.19. *There are $Y \in \mathbb{L}^2(\Omega, L^\infty[0,T])$, $Z \in \mathbb{L}^2(\Omega\times[0,T])$, $\Gamma \in \mathbb{L}^{\overline{\alpha}}(\Omega\times[0,T])$ such that*

$$Y^{f_n} \rightharpoonup Y, \textit{ weakly star in } \mathbb{L}^2(\Omega, L^\infty[0,T])$$
$$Z^{f_n} \rightharpoonup Z, \textit{ weakly in } \mathbb{L}^2(\Omega\times[0,T])$$
$$f_n(., Y^{f_n}, Z^{f_n}) \rightharpoonup \Gamma_. \textit{ weakly in } \mathbb{L}^{\overline{\alpha}}(\Omega\times[0,T]),$$

and moreover

$$Y_t = \xi + \int_t^T \Gamma_s ds - \int_t^T Z_s dW_s, \ \forall t \in [0,T].$$

The following lemma which we will need below is a direct consequence of Hölder's and Schwarz's inequalities .

Lemma 3.20. *For every $\beta \in]1,2]$, $A > 0$, $(y)_{i=1..d} \subset \mathbb{R}$, $(z)_{i=1..d, j=1..r} \subset \mathbb{R}$ we have,*

$$A\ [\sum_{i=1}^d y_i^2]^{\frac{1}{2}}\ [\sum_{i=1}^d\ \sum_{j=1}^r z_{ij}^2]^{\frac{1}{2}} - \frac{1}{2}\sum_{i=1}^d\ \sum_{j=1}^r z_{ij}^2 + \frac{2-\beta}{2}\ [\sum_{i=1}^d y_i^2]^{-1}\sum_{j=1}^r[\sum_{i=1}^d y_i\ \ z_{ij}]^2$$
$$\leq \frac{1}{\beta-1}\ A^2\ \sum_{i=1}^d y_i^2\ - \frac{\beta-1}{4}\ \sum_{i=1}^d\ \sum_{j=1}^r z_{ij}^2.$$

Proof . Using the inequality $ab \leq \dfrac{\alpha^2}{2}a^2 + \dfrac{1}{2\alpha^2}b^2$ we have

$$A\ [\sum_{i=1}^{d} y_i^2]^{\frac{1}{2}}\ [\sum_{i=1}^{d}\ \sum_{j=1}^{r} z_{ij}^2]^{\frac{1}{2}} - \frac{1}{2}\sum_{i=1}^{d}\ \sum_{j=1}^{r} z_{ij}^2 + \frac{2-\beta}{2}\ [\sum_{i=1}^{d} y_i^2]^{-1}\sum_{j=1}^{r}[\sum_{i=1}^{d} y_i\ \ z_{ij}]^2$$
$$\leq \frac{\alpha^2}{2}A^2\sum_{i=1}^{d} y_i^2 + \frac{1}{2\alpha^2}\sum_{i=1}^{d}\ \sum_{j=1}^{r} z_{ij}^2 - \frac{1}{2}\sum_{i=1}^{d}\ \sum_{j=1}^{r} z_{ij}^2 + \frac{2-\beta}{2}\ [\sum_{i=1}^{d} y_i^2]^{-1}\sum_{j=1}^{r}[\sum_{i=1}^{d} y_i\ \ z_{ij}]^2.$$

By Hölder inequality we have $\sum_{i=1}^{d} y_i z_{ij} \leq (\sum_{i=1}^{d} y_i^2)^{\frac{1}{2}}(\sum_{i=1}^{d} z_{ij}^2)^{\frac{1}{2}}$. Hence

$$A\ [\sum_{i=1}^{d} y_i^2]^{\frac{1}{2}}\ [\sum_{i=1}^{d}\ \sum_{j=1}^{r} z_{ij}^2]^{\frac{1}{2}} - \frac{1}{2}\sum_{i=1}^{d}\ \sum_{j=1}^{r} z_{ij}^2 + \frac{2-\beta}{2}\ [\sum_{i=1}^{d} y_i^2]^{-1}\sum_{j=1}^{r}[\sum_{i=1}^{d} y_i\ \ z_{ij}]^2$$
$$\leq \frac{\alpha^2}{2}A^2\sum_{i=1}^{d} y_i^2 + \frac{1}{2\alpha^2}\sum_{i=1}^{d}\ \sum_{j=1}^{r} z_{ij}^2 - \frac{1}{2}\sum_{i=1}^{d}\ \sum_{j=1}^{r} z_{ij}^2 + \frac{2-\beta}{2}\ \sum_{j=1}^{r}\sum_{i=1}^{d}\ \ z_{ij}^2.$$

The proof is finished by choosing $\alpha^2 = \dfrac{2}{(\beta-1)}$. Lemma 3.20 is proved. ■

The key estimate is given by,

Lemma 3.21. *For every $R \in \mathbb{N}$, $\beta \in]1, \min\left(3 - \frac{2}{\bar{\alpha}}, 2\right)[$, $\delta' < (\beta-1)\min\left(\frac{1}{4M_2^2}, \frac{3-\frac{2}{\bar{\alpha}}-\beta}{2rM_2^2\beta}\right)$ and $\varepsilon > 0$, there exists $N_0 > R$ such that for all $N > N_0$ and $T' \leq T$:*

$$\limsup_{n,m\to+\infty} E \sup_{(T'-\delta')^+\leq t\leq T'} |Y_t^{f_n} - Y_t^{f_m}|^\beta + E\int_{(T'-\delta')^+}^{T'} \frac{\left|Z_s^{f_n} - Z_s^{f_m}\right|^2}{\left(|Y_s^{f_n} - Y_s^{f_m}|^2 + \nu_R\right)^{\frac{2-\beta}{2}}}ds$$
$$\leq \varepsilon + \frac{\ell}{\beta-1}e^{C_N\delta'} \limsup_{n,m\to+\infty} E|Y_{T'}^{f_n} - Y_{T'}^{f_m}|^\beta.$$

where $\nu_R = \sup\left\{(A_N\log A_N)^{-1}, N \geq R\right\}$, $C_N = \frac{2M_2^2\beta}{(\beta-1)}\log A_N$ and ℓ is a universal positive constant.

Proof . Let $0 < T' \leq T$. It follows from Itô's formula that for all $t \leq T'$,

$$\left|Y_t^{f_n} - Y_t^{f_m}\right|^2 + \int_t^{T'}\left|Z_s^{f_n} - Z_s^{f_m}\right|^2 ds$$
$$= \left|Y_{T'}^{f_n} - Y_{T'}^{f_m}\right|^2 + 2\int_t^{T'}\langle Y_s^{f_n} - Y_s^{f_m},\quad f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f_m(s, Y_s^{f_m}, Z_s^{f_m})\rangle ds$$
$$- 2\int_t^{T'}\langle Y_s^{f_n} - Y_s^{f_m},\quad \left(Z_s^{f_n} - Z_s^{f_m}\right)dW_s\rangle.$$

For $N \in \mathbb{N}^\star$ we set, $\Delta_t := \left|Y_t^{f_n} - Y_t^{f_m}\right|^2 + (A_N\log A_N)^{-1}$.

Let $C > 0$ and $1 < \beta < \min\{(3 - \frac{2}{\overline{\alpha}}), 2\}$. Itô's formula shows that,

$$
\begin{aligned}
& e^{Ct}\Delta_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}} ds \\
&= e^{CT'}\Delta_{T'}^{\frac{\beta}{2}} + \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^{f_n} - Y_s^{f_m}, \quad f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f_m(s, Y_s^{f_m}, Z_s^{f_m})\rangle ds \\
&- \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left|Z_s^{f_n} - Z_s^{f_m}\right|^2 ds - \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^{f_n} - Y_s^{f_m}, \quad \left(Z_s^{f_n} - Z_s^{f_m}\right) dW_s\rangle \\
&- \beta(\frac{\beta}{2} - 1)\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-2}\sum_{j=1}^{r}\left(\sum_{i=1}^{d}(Y_{i,s}^{f_n} - Y_{i,s}^{f_m})(Z_{i,j,s}^{f_n} - Z_{i,j,s}^{f_m})\right)^2 ds.
\end{aligned}
$$

Put $\Phi(s) = |Y_s^{f_n}| + |Y_s^{f_m}| + |Z_s^{f_n}| + |Z_s^{f_m}|$. Then

$$
\begin{aligned}
& e^{Ct}\Delta_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}} ds \\
&= e^{CT'}\Delta_{T'}^{\frac{\beta}{2}} - \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^{f_n} - Y_s^{f_m}, \quad \left(Z_s^{f_n} - Z_s^{f_m}\right) dW_s\rangle \\
&- \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\left|Z_s^{f_n} - Z_s^{f_m}\right|^2 ds \\
&+ \beta\frac{(2-\beta)}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-2}\sum_{j=1}^{r}\left(\sum_{i=1}^{d}(Y_{i,s}^{f_n} - Y_{i,s}^{f_m})(Z_{i,j,s}^{f_n} - Z_{i,j,s}^{f_m})\right)^2 ds \\
&+ J_1 + J_2 + J_3 + J_4,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &:= \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^{f_n} - Y_s^{f_m},\ f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f_m(s, Y_s^{f_m}, Z_s^{f_m})\rangle 1\!\!1_{\{\Phi(s)>N\}} ds. \\
J_2 &:= \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^{f_n} - Y_s^{f_m},\ f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f(s, Y_s^{f_n}, Z_s^{f_n})\rangle 1\!\!1_{\{\Phi(s)\le N\}} ds. \\
J_3 &:= \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^{f_n} - Y_s^{f_m},\ f(s, Y_s^{f_n}, Z_s^{f_n}) - f(s, Y_s^{f_m}, Z_s^{f_m})\rangle 1\!\!1_{\{\Phi(s)\le N\}} ds. \\
J_4 &:= \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^{f_n} - Y_s^{f_m},\ f(s, Y_s^{f_m}, Z_s^{f_m}) - f_m(s, Y_s^{f_m}, Z_s^{f_m})\rangle 1\!\!1_{\{\Phi(s)\le N\}} ds.
\end{aligned}
$$

We shall estimate J_1, J_2, J_3, J_4. Let $\kappa = 3 - \frac{2}{\overline{\alpha}} - \beta$. Since $\frac{(\beta-1)}{2} + \frac{\kappa}{2} + \frac{1}{\overline{\alpha}} = 1$, we use Hölder inequality we obtain

$$
\begin{aligned}
J_1 &\le \beta e^{CT'}\frac{1}{N^\kappa}\int_t^{T'} \Delta_s^{\frac{\beta-1}{2}}\Phi^\kappa(s)|f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f_m(s, Y_s^{f_m}, Z_s^{f_m}|ds \\
&\le \beta e^{CT'}\frac{1}{N^\kappa}\left[\int_t^{T'} \Delta_s ds\right]^{\frac{\beta-1}{2}}\left[\int_t^{T'} \Phi(s)^2 ds\right]^{\frac{\kappa}{2}} \\
&\times \left[\int_t^{T'} |f_n(s, Y_s^{f_n}, Z_s^{f_n}) - f_m(s, Y_s^{f_m}, Z_s^{f_m}|^{\overline{\alpha}} ds\right]^{\frac{1}{\overline{\alpha}}}.
\end{aligned}
$$

Since $|Y_s^{f_n} - Y_s^{f_m}| \leq \Delta_s^{\frac{1}{2}}$, it easy to see that

$$J_2 + J_4 \leq 2\beta e^{CT'}[2N^2 + \nu_1]^{\frac{\beta-1}{2}}\bigg[\int_t^{T'} \sup_{|y|,|z|\leq N} |f_n(s,y,z) - f(s,y,z)|ds + \int_t^{T'} \sup_{|y|,|z|\leq N} |f_m(s,y,z) - f(s,y,z)|ds\bigg].$$

Using assumption **(H.4)**, we get

$$\begin{aligned} J_3 &\leq \beta M_2 \int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\Big[|Y_s^{f_n} - Y_s^{f_m}|^2 \log A_N \\ &+ A_N^{-1} + |Y_s^{f_n} - Y_s^{f_m}||Z_s^{f_n} - Z_s^{f_m}|\sqrt{\log A_N}\Big]\mathbb{1}_{\{\Phi(s)<N\}}ds \\ &\leq \beta M_2 \int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\Big[\Delta_s \log A_N + |Y_s^{f_n} - Y_s^{f_m}||Z_s^{f_n} - Z_s^{f_m}|\sqrt{\log A_N}\Big]\mathbb{1}_{\{\Phi(s)\leq N\}}ds. \end{aligned}$$

We choose $C = C_N = \dfrac{2M_2^2\beta}{\beta-1}\log A_N$, then we use Lemma 3.20 to show that

$$\begin{aligned} & e^{C_N t}\Delta_t^{\frac{\beta}{2}} + \frac{\beta(\beta-1)}{4}\int_t^{T'} e^{C_N s}\Delta_s^{\frac{\beta}{2}-1}\left|Z_s^{f_n} - Z_s^{f_m}\right|^2 ds \\ & \leq e^{C_N T'}\Delta_{T'}^{\frac{\beta}{2}} - \beta\int_t^{T'} e^{C_N s}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^{f_n} - Y_s^{f_m}, \ \left(Z_s^{f_n} - Z_s^{f_m}\right)dW_s\rangle \\ & +\beta e^{C_N T'}\frac{1}{N^\kappa}\left[\int_t^{T'}\Delta_s ds\right]^{\frac{\beta-1}{2}} \times \left[\int_t^{T'}\Phi(s)^2 ds\right]^{\frac{\kappa}{2}} \\ & \times \left[\int_t^{T'}|f_n(s,Y_s^{f_n},Z_s^{f_n}) - f_m(s,Y_s^{f_m},Z_s^{f_m}|^{\overline{\alpha}}\mathbb{1}_{\{\Phi(s)>N\}}ds\right]^{\frac{1}{\overline{\alpha}}} \\ & +\beta e^{C_N T'}[2N^2+\nu_1]^{\frac{\beta-1}{2}}\bigg[\int_t^{T'}\sup_{|y|,|z|\leq N}|f_n(s,y,z) - f(s,y,z)|ds \\ & + \int_t^{T'}\sup_{|y|,|z|\leq N}|f_m(s,y,z) - f(s,y,z)|ds\bigg]. \end{aligned}$$

Burkholder's inequality and Hölder's inequality (since $\frac{(\beta-1)}{2} + \frac{\kappa}{2} + \frac{1}{\overline{\alpha}} = 1$) allow us to show that there exists a universal constant $\ell > 0$ such that $\forall\delta' > 0$,

$$\begin{aligned} & \mathbb{E}\sup_{(T'-\delta')^+\leq t\leq T'}\left[e^{C_N t}\Delta_t^{\frac{\beta}{2}}\right] + \mathbb{E}\int_{(T'-\delta')^+}^{T'} e^{C_N s}\Delta_s^{\frac{\beta}{2}-1}\left|Z_s^{f_n} - Z_s^{f_m}\right|^2 ds \\ & \leq \frac{\ell}{\beta-1}e^{C_N T'}\bigg\{\mathbb{E}\left[\Delta_{T'}^{\frac{\beta}{2}}\right] + \frac{\beta}{N^\kappa}\left[\mathbb{E}\int_0^T \Delta_s ds\right]^{\frac{\beta-1}{2}}\left[\mathbb{E}\int_0^T \Phi(s)^2 ds\right]^{\frac{\kappa}{2}} \\ & \times \left[\mathbb{E}\int_0^T |f_n(s,Y_s^{f_n},Z_s^{f_n}) - f_m(s,Y_s^{f_m},Z_s^{f_m}|^{\overline{\alpha}}ds\right]^{\frac{1}{\overline{\alpha}}} \\ & + \beta[2N^2+\nu_1]^{\frac{\beta-1}{2}}\mathbb{E}\bigg[\int_0^T \sup_{|y|,|z|\leq N}|f_n(s,y,z) - f(s,y,z)|ds \\ & + \int_0^T \sup_{|y|,|z|\leq N}|f_m(s,y,z) - f(s,y,z)|ds\bigg]\bigg\}. \end{aligned}$$

We use assumption **(H.4)**-i) and Lemma 3.2 to obtain,

$\forall N > R$,

$$\mathbb{E}\sup_{(T'-\delta')^+\leq t\leq T'}|Y_t^{f_n}-Y_t^{f_m}|^\beta+\mathbb{E}\int_{(T'-\delta')^+}^{T'}\frac{\left|Z_s^{f_n}-Z_s^{f_m}\right|^2}{\left(|Y_s^{f_n}-Y_s^{f_m}|^2+\nu_R\right)^{\frac{2-\beta}{2}}}ds$$

$$\leq \frac{\ell}{\beta-1}e^{C_N\delta'}\Big\{(A_N\log A_N)^{\frac{-\beta}{2}}+\beta\frac{2K_3^{\frac{1}{\alpha}}}{N^\kappa}(4TK_2+T\ell)^{\frac{\beta-1}{2}}(8TK_2+8K_1)^{\frac{\kappa}{2}}$$

$$+\mathbb{E}|Y_{T'}^{f_n}-Y_{T'}^{f_m}|^\beta+\beta[2N^2+\nu_1]^{\frac{\beta-1}{2}}\big[\rho_N(f_n-f)+\rho_N(f_m-f)\big]\Big\}$$

$$\leq \frac{\ell}{\beta-1}e^{C_N\delta'}\mathbb{E}|Y_{T'}^{f_n}-Y_{T'}^{f_m}|^\beta+\frac{\ell}{\beta-1}\frac{A_N^{\frac{2M_2^2\delta'\beta}{\beta-1}}}{(A_N\log A_N)^{\frac{\beta}{2}}}$$

$$+\frac{2\ell}{\beta-1}\beta K_3^{\frac{1}{\alpha}}(4TK_2+T\ell)^{\frac{\beta-1}{2}}(8TK_2+8K_1)^{\frac{\kappa}{2}}\frac{A_N^{\frac{2M_2^2\delta'\beta}{\beta-1}}}{(A_N)^{\frac{\kappa}{r}}}$$

$$+\frac{2\ell}{\beta-1}e^{C_N\delta'}\beta[2N^2+\nu_1]^{\frac{\beta-1}{2}}\big[\rho_N(f_n-f)+\rho_N(f_m-f)\big].$$

Hence for $\delta' < (\beta-1)\min\left(\frac{1}{4M_2^2},\frac{\kappa}{2rM_2^2\beta}\right)$ we derive

$$\frac{A_N^{\frac{2M_2^2\delta'\beta}{\beta-1}}}{(A_N\log A_N)^{\frac{\beta}{2}}}\longrightarrow_{N\to\infty}0$$

and

$$\frac{A_N^{\frac{2M_2^2\delta'\beta}{\beta-1}}}{(A_N)^{\frac{\kappa}{r}}}\longrightarrow_{N\to\infty}0.$$

We conclude the proof of Lemma 3.21 by using assertion (d) of lemma 3.17. ■

Proof of Theorem 3.15. Taking successively $T'=T$, $T'=(T-\delta')^+$, $T'=(T-2\delta')^+$... in Lemma 3.21, we obtain, for every $\beta\in]1,\quad\min\left(3-\dfrac{2}{\alpha},2\right)[$

$$\lim_{n,m\to+\infty}\left(\mathbb{E}\sup_{0\leq t\leq T}|Y_t^{f_n}-Y_t^{f_m}|^\beta+\mathbb{E}\int_0^T\frac{\left|Z_s^{f_n}-Z_s^{f_m}\right|^2}{\left(|Y_s^{f_n}-Y_s^{f_m}|^2+\nu_R\right)^{\frac{2-\beta}{2}}}ds\right)=0.$$

But by Schwartz inequality we have

$$\mathbb{E}\int_0^T|Z_s^{f_n}-Z_s^{f_m}|ds\leq\left(\mathbb{E}\int_0^T\frac{\left|Z_s^{f_n}-Z_s^{f_m}\right|^2}{\left(|Y_s^{f_n}-Y_s^{f_m}|^2+\nu_R\right)^{\frac{2-\beta}{2}}}ds\right)^{\frac12}\left(\mathbb{E}\int_0^T\left(|Y_s^{f_n}-Y_s^{f_m}|^2+\nu_R\right)^{\frac{2-\beta}{2}}ds\right)^{\frac12}.$$

Hence

$$\lim_{n\to+\infty}\left(\mathbb{E}\sup_{0\leq t\leq T}|Y_t^{f_n}-Y_t|^\beta+\mathbb{E}\int_0^T|Z_s^{f_n}-Z_s|ds\right)=0.$$

In particular, there exists a subsequence, which we still denote (Y^{f_n}, Z^{f_n}), such that

$$\lim_{n\to+\infty}\Big(|Y_t^{f_n}-Y_t|+|Z_t^{f_n}-Z_t|\Big)=0 \quad a.e.\ (t,\omega).$$

On the other hand

$$\begin{aligned}
&\mathbb{E}\int_0^T|f_n(s,Y_s^{f_n},Z_s^{f_n})-f(s,Y_s^{f_n},Z_s^{f_n})|ds\\
&\le \mathbb{E}\int_0^T|f_n(s,Y_s^{f_n},Z_s^{f_n})-f(s,Y_s^{f_n},Z_s^{f_n})|1\!\!1_{\{|Y_s^{f_n}|+|Z_s^{f_n}|\le N\}}ds\\
&+\mathbb{E}\int_0^T|f_n(s,Y_s^{f_n},Z_s^{f_n})-f(s,Y_s^{f_n},Z_s^{f_n})|\frac{(|Y_s^{f_n}|+|Z_s^{f_n}|)^{(2-\frac{2}{\alpha})}}{N^{(2-\frac{2}{\alpha})}}1\!\!1_{\{|Y_s^{f_n}|+|Z_s^{f_n}|\ge N\}}ds\\
&\le \rho_N(f_n-f)+\frac{2K_3^{\frac{1}{\alpha}}\left[TK_2+K_1\right]^{1-\frac{1}{\alpha}}}{N^{(2-\frac{2}{\alpha})}}.
\end{aligned}$$

Passing to the limit first on n and next on N we obtain

$$\lim_n E\int_0^T|f_n(s,Y_s^{f_n},Z_s^{f_n})-f(s,Y_s^{f_n},Z_s^{f_n})|ds=0.$$

Finally, we use **(H.1)**, Lemma 3.17 and Lemma 3.18 to show that,

$$\lim_n E\int_0^T|f_n(s,Y_s^{f_n},Z_s^{f_n})-f(s,Y_s,Z_s)|ds=0.$$

The existence is proved.

Uniqueness. Let (Y,Z) and (Y',Z') be two solutions of equation (E^f). Arguing as previously one can show that:
for every $R>2$, $\beta\in]1,\min\left(3-\dfrac{2}{\alpha},2\right)[$, $\delta'<(\beta-1)\min\left(\frac{1}{4M_2^2},\frac{3-\frac{2}{\alpha}-\beta}{2rM_2^2\beta}\right)$ and $\varepsilon>0$
there exists $N_0>R$ such that for all $N>N_0$, $\forall T'\le T$

$$\begin{aligned}
\mathbb{E}\sup_{(T'-\delta')^+\le t\le T'}|Y_t-Y_t'|^\beta+\mathbb{E}\int_{(T'-\delta')^+}^{T'}\frac{\left|Z_s-Z_s'\right|^2}{(|Y_s-Y_s'|^2+\nu_R)^{\frac{2-\beta}{2}}}ds\\
\le \varepsilon+\frac{\ell}{\beta-1}e^{C_N\delta'}\mathbb{E}|Y_{T'}-Y_{T'}'|^\beta.
\end{aligned}$$

Again, taking successively $T'=T$, $T'=(T-\delta')^+$, $T'=(T-2\delta')^+$..., we establish the uniqueness of solution. Theorem 3.15 is proved. ■

Proof of Theorem 3.16. Also as in the proof of Theorem 3.15, we show that,
For every $R>2$, $\beta\in]1,\min\left(3-\dfrac{2}{\alpha},2\right)[$, $\delta'<(\beta-1)\min\left(\frac{1}{4M_2^2},\frac{3-\frac{2}{\alpha}-\beta}{2rM_2^2\beta}\right)$ and $\varepsilon>0$, there exists $N_0>R$ such that for all $N>N_0$, for all $T'\le T$:

$$\begin{aligned}
\limsup_{n\to+\infty}\mathbb{E}\sup_{(T'-\delta')^+\le t\le T'}|Y_t^n-Y_t|^\beta+\mathbb{E}\int_{(T'-\delta')^+}^{T'}\frac{|Z_s^n-Z_s|^2}{(|Y_s^n-Y_s|^2+\nu_R)^{\frac{2-\beta}{2}}}ds\\
\le \varepsilon+\frac{\ell}{\beta-1}e^{C_N\delta'}\limsup_{n\to+\infty}\mathbb{E}|Y_{T'}^n-Y_{T'}|^\beta.
\end{aligned}$$

Again as in the proof of Theorem 3.15, taking successively $T' = T$, $T' = (T-\delta')^+$, $T' = (T-2\delta')^+$..., we establish the convergence in the whole interval $[0,T]$. In particular, we have for every $q<2$, $\lim_{n\to+\infty}(|Y^n - Y|^q) = 0$ and $\lim_{n\to+\infty}(|Z^n - Z|^q) = 0$ *in measure* $P\times dt$. Since (Y^n) and (Z^n) are square integrable, the proof is finished by using an uniform integrability argument. Theorem 3.16 is proved. ■

To illustrate our results, let us consider the following example:

Example 3.22. *Let* $\varepsilon > 0$ *and*
$f_1(t,\omega,y,z) = g(t,\omega,y)\big[|z|\sqrt{|\log|z||}1_{|z|<\varepsilon} + h(z)1_{\varepsilon\le|z|\le 1+\varepsilon} + |z|\sqrt{\log|z|}1_{|z|>1+\varepsilon}\big]$
where g *is a bounded function which is continuous in* y *such that* $g(t,\omega,0)=0$ *and* $\langle y-y', g(t,y)-g(t,y')\rangle \le 0$. h *is a lipchitz and positive function which is choosing such that* f_1 *is continuous.*
Let $f_2(t,\omega,y,z)$ *be a continuous function in* (y,z) *such that:*
i) There exist $M>0$, *and* $\eta\in\mathbb{L}^1([0,T]\times\Omega)$: $\langle y, f_2(t,\omega,y,z)\rangle \le \eta + M|y|^2 + M\mid z\mid^2$
ii) There exist $M>0$, $1<\alpha<2$ *and* $\overline{\eta}\in\mathbb{L}^{\frac{2}{\alpha}}([0,T]\times\Omega)$: $\mid f_2(t,\omega,y,0)\mid \le \overline{\eta} + M\mid y\mid^\alpha$.
iii) There exists a constant $C>0$:

$$\langle y-y', f_2(t,y,z) - f_2(t,y',z')\rangle$$
$$\le C\mid y-y'\mid^2\big[1+|\log\mid y-y'\mid|\big] + C\mid y-y'\mid\mid z-z'\mid\big[1+\sqrt{|\log\mid z-z'\mid|}\big].$$

Our work shows that equation $(E^{f_1+f_2})$ *has a unique solution.*

Chapter 4

Homogenization of Multivalued Partial Differential Equations via Reflected Backward Stochastic Differential Equations

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The chapter is organized as follows. In Section 4.1, we introduce some notations and assumptions to be used in the sequel. Section 4.2 is devoted to the proof of weak convergence of RBSDE. In Section 4.3, we apply our result to the homogenization of a class of multivalued PDE's.

4.1 Problem formulation

Let $\{X_t^\varepsilon; t \geq 0\}$ be a diffusion process with values in $\mathbb{R}^d$, such that $X^\varepsilon \Longrightarrow X$ in $\mathbb{C}([0,t], \mathbb{R}^d)$ equipped with the topology of convergence on compact subsets of $\mathbb{R}_+$, where X itself is a diffusion with generator L. We suppose that the martingale problem associated to X is well posed, and there exist $p, q \geq 0$ such that

$$\sup_{\varepsilon} \mathbb{E}(\mid X_t^\varepsilon \mid^{2p} + \int_0^t \mid X_s^\varepsilon \mid^{2q} ds) < \infty. \tag{4.1}$$

Moreover, we assume that $g : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ and $f : \mathbb{R}^d \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ are continuous, and that

$$\mid g(x) \mid \leq C(1 + \mid x \mid^p) \tag{4.2}$$

$$\mid f(x,y) \mid \leq C(1 + \mid x \mid^q + \mid y \mid^2) \tag{4.3}$$

$$\mid f(x,y) \mid \leq C(1 + \mid x \mid^q + \mid y \mid^\alpha) \tag{4.4}$$

$$\mid f(x,y) - f(x,y') \mid \leq K \mid y - y' \mid, \tag{4.5}$$

for some $C > 0$, $K > 0$, $0 \leq \alpha < 1$ and for all $x \in \mathbb{R}^d$, $y, y' \in \mathbb{R}^k$.
For each $N > 0$, there exists μ_N such that:

$$\langle f(x,y) - f(x,y'), y - y' \rangle \leq \mu_N \mid y - y' \mid^2, \quad \mid y \mid, \mid y' \mid \leq N. \tag{4.6}$$

[0]This work is accepted for publication in Stochastic Analysis and Applications.

Let ϕ be a lower semi-continuous, proper and convex function. We assume that

$$\phi(g(x)) \leq C(1+ \mid x \mid^p), \forall x \in \mathbb{R}^d. \tag{4.7}$$

Now, let us recall some properties of a Yosida approximate of subdifferential operator. We put

$$\begin{array}{lcl} Dom(\phi) & = & \{u \in \mathbb{R}^k : \phi(u) < +\infty\} \\ \partial\phi(u) & = & \{u^* \in \mathbb{R}^k :< u^*, v - u > +\phi(u) \leq \phi(v), \forall v \in \mathbb{R}^k\} \\ Dom(\partial(\phi)) & = & \{u \in \mathbb{R}^k : \partial(\phi) \neq \emptyset\} \\ Gr(\partial\phi) & = & \{(u, u^*) \in \mathbb{R}^k \times \mathbb{R}^k : u \in Dom(\partial(\phi)) \quad \text{and} \quad u^* \in \partial\phi(u)\}. \end{array}$$

For every $x \in \mathbb{R}^d$

$$\phi_n(x) = \min_y \big(\frac{n}{2} \mid x - y \mid^2 + \phi(y)\big),$$

and $J_n(x)$ is the unique solution of the inclusion $x \in J_n(x) + \frac{1}{n}\partial\phi(J_n(x))$ (see Barbu, Precupanu [8]). The map J_n is called the resolvent of the monotone operator $A = \partial\phi$. Let us note that $\phi_n : \mathbb{R}^k \longrightarrow \mathbb{R}$ is a convex function of class C^1 with $\bigtriangledown\phi_n = A_n$ is the yosida approximations of the operator $\partial\phi$, defined by $A_n = n(x - J_n(x))$. We also have

$$\inf_{y \in \mathbb{R}^k} \phi(y) \leq \phi(J_n(x)) \leq \phi_n(x) \leq \phi(x).$$

Let $\{(Y_s^\varepsilon, Z_s^\varepsilon, K_s^\varepsilon); 0 \leq s \leq t)\}$ be the unique solution of the reflected BSDE

$$\begin{cases} Y_s^\varepsilon = g(X_t^\varepsilon) + \displaystyle\int_s^t f(X_r^\varepsilon, Y_r^\varepsilon)dr - \int_s^t Z_r^\varepsilon dB_r + K_t^\varepsilon - K_s^\varepsilon \\ K_t^\varepsilon = - \displaystyle\int_0^t U_s^\varepsilon ds, \quad (Y^\varepsilon, U^\varepsilon) \in \mathbf{Gr}(\partial\phi), \end{cases} \tag{4.8}$$

where $\{B_s, 0 \leq s \leq t\}$ is a Brownian motion. Next, we shall prove that the family of processes $(X^\varepsilon; Y^\varepsilon; Z^\varepsilon; K^\varepsilon)$ converges on law to the unique solution (X, Y, Z, K) of the RBSDE

$$\begin{cases} Y_s = g(X_t) + \displaystyle\int_s^t f(X_r, Y_r)dr - \int_s^t Z_r dB_r + K_t - K_s \\ K_t = - \displaystyle\int_0^t U_s ds, \quad (Y, U) \in \mathbf{Gr}(\partial\phi), \end{cases}$$

and then we shall apply this result to the homogenization of a class of multivalued PDE's.

Theorem 4.1. *(See Meyer-Zheng [56] or Kurtz [47]).*
The sequence of quasi-martingale $\{V_s^n; 0 \leq s \leq t\}$ defined on the filtred probability space $\{\Omega; \mathcal{F}_s, 0 \leq s \leq t; \mathbb{P}\}$ is tight whenever

$$\sup_n \big(\sup_{0 \leq s \leq t} \mathbb{E} \mid V_s^n \mid + CV_t(V^n)\big) < +\infty,$$

where $CV_t(V^n)$, denotes the "conditional variation of V^n on $[0, t]$" defined by

$$CV_t(V^n) = \sup \mathbb{E}(\sum_i \mid \mathbb{E}\big(V_{t_{i+1}}^n - V_{t_i}^n / \mathcal{F}_{t_i}\big) \mid \big),$$

with "sup" meaning that the supremum is taken over all partitions of the interval $[0, t]$.

We put

$$M_t^{\varepsilon} = -\int_0^t Z_s^{\varepsilon} dB_r.$$

In the rest of thesis we denote by:
• $\mathbb{C}([0,t],\mathbb{R}^d)$ the space of functions of $[0,t]$ with values in $\mathbb{R}^d$ equipped with the topology of uniform convergence.
• $\mathbb{D}([0,t],\mathbb{R}^k)$ the space of càdlàg functions of $[0,t]$ with values in $\mathbb{R}^k$ equipped with the topology of Meyer-Zheng.

4.2 The main results.

The main results are the following

Theorem 4.2. *Let assumptions (4.1)-(4.3), (4.5), (4.7) hold. Then, the family of processes* $(X^{\varepsilon}, Y^{\varepsilon}, M^{\varepsilon}, K^{\varepsilon})$ *converges in law to* (X, Y, M, K) *on* $\mathbb{C}([0,t],\mathbb{R}^d) \times \mathbb{D}([0,t],\mathbb{R}^{2k}) \times \mathbb{C}([0,t],\mathbb{R}^k)$.

Proposition 4.3. *Let assumptions (4.1)-(4.2), (4.4), (4.6)-(4.7) hold. If* $\lim_{N\to\infty} \dfrac{e^{\mu_N^+}}{N^{2(1-\alpha)}} = 0$, *then the family of processes* $(X^{\varepsilon}, Y^{\varepsilon}, M^{\varepsilon}, K^{\varepsilon})$ *converges in law to* (X, Y, M, K) *on* $\mathbb{C}([0,t],\mathbb{R}^d) \times \mathbb{D}([0,t],\mathbb{R}^{2k}) \times \mathbb{C}([0,t],\mathbb{R}^k)$.

To do the proofs of Theorem 4.2 and Proposition 4.3, we need the following lemmas

Lemma 4.4. *Let* U^{ε} *be a family of random variables defined on the same probability spaces. For each* $\varepsilon \geq 0$, *we assume the existence of a family of random variables* $(U^{\varepsilon,n})_n$, *such that*
• $U^{\varepsilon,n} \overset{\text{dist}}{\Longrightarrow} U^{0,n}$ *as* ε *goes to zero*
• $U^{\varepsilon,n} \Longrightarrow U^{\varepsilon}$ *as* $n \to +\infty$, *uniformly in* ε
• $U^{0,n} \Longrightarrow U^0$ *as* $n \to +\infty$.
Then, U^{ε} *converges in distribution to* U^0.

Proof : This lemma is a simplified version of Theorem 4.2 in [Billingsley [13], p.25]. ■

Consider the backward stochastic differential equation

$$Y_s^{\varepsilon,n} = g(X_s^{\varepsilon}) + \int_s^t f(X_s^{\varepsilon}, Y_r^{\varepsilon,n})dr - \int_s^t Z_r^{\varepsilon,n} dB_r - \int_s^t A_n(Y_r^{\varepsilon,n})dr, \tag{4.9}$$

where $A_n(y)$ is defined as above.
Let (Y^n, Z^n) be the unique solution of the backward stochastic differential equation

$$Y_s^n = g(X_s) + \int_s^t f(X_s, Y_r^n)dr - \int_s^t Z_r^n dB_r - \int_s^t A_n(Y_r^n)dr.$$

We set

$$M_t^{\varepsilon,n} = -\int_0^t Z_r^{\varepsilon,n} dB_r \quad \text{and} \quad M_t^n = -\int_0^t Z_r^n dB_r.$$

Lemma 4.5. *Let assumptions of Theorem 4.2 hold. Then, the family of processes* $(Y^{\varepsilon,n}, M^{\varepsilon,n})$ *converges in law to the the family of processes* (Y^n, M^n) *on* $\mathbb{D}([0,t],\mathbb{R}^{2k})$.

Before proving this lemma, let us recall that there exist $a \in \mathbb{R}^k$ and (μ, γ) a pair of positive numbers such that for any $x \in \mathbb{R}^k$ and any $n \in \mathbb{N}$

$$\langle A_n(x), x-a\rangle \geq \gamma \mid A_n(x) \mid -\mu \mid x-a \mid -\gamma\mu; \tag{4.10}$$

for more details, see Cépa thesis [19].
Proof . Step1. A priori estimates for $(Y^{\varepsilon,n}, M^{\varepsilon,n})$.
Fix n and let $a \in \mathbb{R}^k$ satisfying (4.10). By Itô's formula, one has

$$\begin{aligned}
&\mid Y_s^{\varepsilon,n} - a \mid^2 + \int_s^t \mid Z_r^{\varepsilon,n} \mid^2 dr \\
&= \mid g(X_t^\varepsilon) - a \mid^2 + 2\int_s^t (Y_r^{\varepsilon,n} - a) f(X_r^\varepsilon, Y_r^{\varepsilon,n}) dr \\
&- 2\int_s^t (Y_r^{\varepsilon,n} - a) Z_s^{\varepsilon,n} dB_r - 2\int_s^t < A_n(Y_r^{\varepsilon,n}), Y_r^{\varepsilon,n} - a > dr.
\end{aligned}$$

It follows from Pardoux [66] that the expectation of the above stochastic integral is zero. Moreover, from (4.5) and (4.10), we deduce

$$\begin{aligned}
&\mathbb{E} \mid Y_s^{\varepsilon,n} - a \mid^2 + \mathbb{E}\int_s^t \mid Z_r^{\varepsilon,n} \mid^2 dr + 2\gamma \int_s^t \mid A_n(Y_r^{\varepsilon,n}) \mid dr \\
&\leq \mathbb{E} \mid g(X_t^\varepsilon) - a \mid^2 + 2\mathbb{E}\int_s^t (Y_r^{\varepsilon,n} - a) f(X_r^\varepsilon, Y_r^{\varepsilon,n}) dr + 2\mu\mathbb{E}\int_s^t \mid Y_r^{\varepsilon,n} - a \mid^2 dr + C \\
&\leq \mathbb{E} \mid g(X_t^\varepsilon) - a \mid^2 + (2K + 1 + 2\mu)\mathbb{E}\int_s^t \mid Y_r^{\varepsilon,n} - a \mid^2 dr + \mathbb{E}\int_0^t \mid f(X_r^\varepsilon, a) \mid^2 dr + C.
\end{aligned}$$

Hence from Gronwall's lemma

$$\begin{aligned}
&\sup_{0\leq s\leq t} \mathbb{E}\big(\mid Y_s^{\varepsilon,n} - a \mid^2 + \int_s^t \mid Z_r^{\varepsilon,n} \mid^2 dr + 2\gamma \int_s^t \mid A_n(Y_r^{\varepsilon,n}) \mid dr \big) \\
&\leq C(\mathbb{E} \mid g(X_t^\varepsilon) - a \mid^2 + \mathbb{E}\int_0^t \mid f(X_r^\varepsilon, a) \mid^2 dr) + C.
\end{aligned}$$

Finally, from this last inequality, assumptions (4.1), (4.3) and Burkholder-Davis-Gundy inequality, we get

$$\sup_\varepsilon \mathbb{E}\big(\sup_{0\leq s\leq t} \mid Y_s^{\varepsilon,n} - a \mid^2 + \int_0^t \mid Z_r^{\varepsilon,n} \mid^2 dr + 2\gamma \int_s^t \mid A_n(Y_r^{\varepsilon,n}) \mid dr \big) < +\infty. \tag{4.11}$$

Step2.Tightness.
Clearly, we have

$$CV_t(Y^{\varepsilon,n}) \leq \int_0^t \mid f(X_r^\varepsilon, Y_r^{\varepsilon,n}) \mid dr + \int_0^t \mid A_n(Y_r^{\varepsilon,n}) \mid dr,$$

and it follows from step 1 and assumptions (4.1), (4.3) and (4.5) that

$$\sup_\varepsilon (CV_t\big((Y^{\varepsilon,n}) + \mathbb{E} \sup_{0\leq s\leq t} \mid Y_s^{\varepsilon,n} - a \mid^2 + \int_0^t \mid Z_r^{\varepsilon,n} \mid^2 dr)\big) < +\infty, \tag{4.12}$$

hence the sequence $\{(Y_s^{\varepsilon,n}, M_s^{\varepsilon,n}); 0 \leq s \leq t\}$ satisfies Meyer-Zheng's tightness criterion for quasi-martingales under $\mathbb{P}$.

Step.3. Convergence in law.
By step.2 there exists a subsequence (which we still denote $(Y^{\varepsilon,n}, M^{\varepsilon,n})$) such that

$$(Y^{\varepsilon,n}, M^{\varepsilon,n}) \Longrightarrow (Y^n, M^n),$$

on $(\mathbb{D}([0,t],\mathbb{R}^k))^2$, where the first factor is equipped with the topology of convergence in ds measure, and the second with the topology of uniform convergence.
Clearly, for each $0 \leq s \leq t$, $(x,y) \longrightarrow \int_s^t f(x(r), y(r))dr$ is continuous for $\mathbb{C}([0,t],\mathbb{R}^d) \times \mathbb{D}([0,t],\mathbb{R}^k)$ equipped with the same topology as above, and $y \longrightarrow \int_s^t A_n(y(r))dr$ is continuous in $\mathbb{C}([0,t],\mathbb{R}^k)$ as ε goes to 0. We can now take the limit in (4.9), yielding

$$Y_s^n = g(X_s) + \int_s^t f(X_r, Y_r^n)dr + M_t^n - M_s^n - \int_s^t A_n(Y_r^n)dr.$$

Moreover, for any $0 \leq s_1 < s_2 \leq t$, $\phi \in \mathbb{C}_b^\infty$ and ψ_s a function of $X_r^\varepsilon, Y_r^{\varepsilon,n}$, $0 \leq r \leq t$, bounded and continuous in $\mathbb{C}([0,t],\mathbb{R}^d) \times \mathbb{D}([0,t],\mathbb{R}^k)$, we have

$$\mathbb{E}\big(\psi_{s_1}(X^\varepsilon, Y^{\varepsilon,n})(\phi(X_{s_2}^\varepsilon) - \phi(X_{s_1}^\varepsilon) - \int_{s_1}^{s_2} L\phi(X_r^\varepsilon)dr)\big) \longrightarrow 0 \quad \text{as} \quad n \to +\infty,$$

and for each $n \in \mathbb{N}$,

$$\mathbb{E}(\psi_{s_1}(X^\varepsilon, Y^{\varepsilon,n}) \int_0^\alpha (M_{s_2+r}^{\varepsilon,n} - M_{s_1+r}^{\varepsilon,n})dr) = 0.$$

From the weak convergence and the fact that $\mathbb{E}(\sup_{0\leq s\leq t} \mid M_s^{\varepsilon,n} \mid^2) < +\infty$, by dividing the second identity by α and letting α go to zero, we have

$$\mathbb{E}(\psi_{s_1}(X, Y^n)(\phi(X_{s_2}) - \phi(X_{s_1}) - \int_{s_1}^{s_2} L\phi(X_r)dr)\big) \longrightarrow 0,$$

$$\mathbb{E}(\psi_{s_1}(X, Y^n)(M_{s_2}^n - M_{s_1}^n) = 0.$$

Therefore, both M^n and M^X -the martingale part of X- are $\mathcal{F}_t^{X,Y^n}$ martingales.
Step.4. Identification of the limit.
Let $(\overline{Y}^n, \overline{U}^n)$ denotes the unique solution of the BSDE

$$\overline{Y}_s^n = g(X_t) + \int_s^t f(X_r, \overline{Y}_r^n)dr - \int_s^t \overline{U}_r^n dM_r^X - \int_s^t A_n(\overline{Y}_r^n)dr,$$

which satisfies $\mathbb{E}Tr \int_s^t \overline{U}_r^n < M^X >_r \overline{U}_r^n < +\infty$. Set also $\widetilde{M}_s^n = \int_0^s \overline{U}_r^n dM_r^X$. Since $\overline{Y}^n$ and $\overline{U}^n$ are $\mathcal{F}_t^X$ adapted, and M^X is $\mathcal{F}_t^{X,Y^n}$ martingale, hence so is $\widetilde{M}^n$.
From Itô's formula, it follows that

$$\begin{aligned}
&\mathbb{E} \mid \overline{Y}_s^n - Y_s^n \mid^2 + \mathbb{E}[M^n - \widetilde{M}^n]_t - \mathbb{E}[M^n - \widetilde{M}^n]_s \\
&= 2\int_s^t < f(X_r, Y_r^n) - f(X_r, \overline{Y}_r^n), \overline{Y}_r^n - Y_r^n > dr \\
&- 2\int_s^t < A_n(Y_r^n) - A_n(\overline{Y}_r^n), \overline{Y}_r^n - Y_r^n > dr \\
&\leq C_n \mathbb{E} \int_s^t \mid \overline{Y}_r^n - Y_r^n \mid^2 dr.
\end{aligned}$$

(We use the fact that the operator is n-lipschitz).
We conclude from Gronwall's lemma that $\overline{Y}_r^n = Y_r^n, 0 \leq s \leq t$, and $M^n = \widetilde{M}^n$. ∎

Lemma 4.6. *Under the assumptions of Lemma 4.5 the family of processes* $(Y^{\varepsilon,n}, M^{\varepsilon,n}, K^{\varepsilon,n})_n$ *converges uniformly of* $\varepsilon \in]0,1]$ *in probability to the family of processes* $(Y^{\varepsilon}, M^{\varepsilon}, K^{\varepsilon})$ *as* n *goes to* $+\infty$.

Proof . By the same proof as in step.1 of Lemma 4.5, we have

$$\sup_{\varepsilon}\sup_{n}\mathbb{E}\sup_{0\leq s\leq t}\big(\mid Y_s^{\varepsilon,n}-a\mid^2+\int_0^t\mid Z_r^{\varepsilon,n}\mid^2 dr+2\gamma\int_s^t\mid A_n(Y_r^{\varepsilon,n})\mid dr\big)<+\infty. \qquad (4.13)$$

Now, we will prove that

$$\sup_{n}\sup_{\varepsilon}\int_0^t\mid A_n(Y_r^{\varepsilon,n})\mid^2 dr<+\infty,$$

which is essential for the convergence of the sequence $(Y^{\varepsilon,n}, Z^{\varepsilon,n})_n$. Without loss of generality we may suppose that ϕ is positive and $\phi(0)=0$. Let us note that ϕ_n is a convex C^1-function with a lipschitz derivative, and put $\psi_n=\frac{\phi_n}{n}:=\frac{1}{n}\min_y(\frac{n}{2}\mid x-y\mid^2+\phi(y))$.
By convolution of ψ_n with a smooth function, the convexity of ψ_n and Itô's formula, one has

$$\begin{aligned}\psi_n(Y_s^{\varepsilon,n})&\leq\psi_n(g(X_t^{\varepsilon}))+\int_s^t\nabla\psi_n(Y_r^{\varepsilon,n})(f(X_r^{\varepsilon},Y_r^{\varepsilon,n})-A_n(Y_r^{\varepsilon,n}))dr\\&-\int_s^t\nabla\psi_n(Y_r^{\varepsilon,n})Z_r^{\varepsilon,n}dB_r,\end{aligned}$$

thus

$$\begin{aligned}\mathbb{E}\psi_n(Y_s^{\varepsilon,n})&\leq\mathbb{E}\psi_n(g(X_t^{\varepsilon}))+\mathbb{E}\int_s^t\nabla\psi_n(Y_r^{\varepsilon,n})(f(X_r^{\varepsilon},Y_s^{\varepsilon,n})-A_n(Y_r^{\varepsilon,n}))dr\\&=\mathbb{E}\psi_n(g(X_t^{\varepsilon}))+\mathbb{E}\int_s^t\nabla\psi_n(Y_r^{\varepsilon,n})f(X_r^{\varepsilon},Y_r^{\varepsilon,n})dr-\frac{2}{n}\mathbb{E}\int_s^t\mid A_n(Y_r^{\varepsilon,n})\mid^2 dr.\end{aligned}$$

Using the simple inequality $ab\leq\dfrac{2}{n}\,a^2+\dfrac{n}{2}\,b^2$, we get

$$\begin{aligned}\mathbb{E}\psi_n(Y_s^{\varepsilon,n})+\frac{1}{n}\mathbb{E}\int_s^t\mid A_n(Y_r^{\varepsilon,n})\mid^2 dr&\leq\mathbb{E}\psi_n(g(X_t^{\varepsilon}))+\frac{1}{2n}\mathbb{E}\int_s^t\mid A_n(Y_r^{\varepsilon,n})\mid^2 dr\\&+\frac{C}{n}\mathbb{E}\sup_{0\leq s\leq t}\mid Y_r^{\varepsilon,n}\mid^2 dr+\frac{1}{n}\mathbb{E}\int_s^t\mid X_r^{\varepsilon}\mid^{2p} dr+\frac{C}{n}.\end{aligned}$$

By relations (4.1), (4.7) and (4.11), we deduce that

$$\mathbb{E}\psi_n(Y_s^{\varepsilon,n})+\frac{1}{n}\mathbb{E}\int_s^t\mid A_n(Y_r^{\varepsilon,n})\mid^2 dr\leq\frac{C}{n},$$

and finally

$$\sup_{\varepsilon}\sup_{n}\mathbb{E}\int_0^t\mid A_n(Y_r^{\varepsilon,n})\mid^2 dr<+\infty. \qquad (4.14)$$

Now, let us prove the convergence of $(Y^{\varepsilon,n}, Z^{\varepsilon,n})_n$ for every $(n,m)\in\mathbb{N}^*$.
By Itô's formula, one has

$$\begin{aligned}&\mid Y_s^{\varepsilon,n}-Y_s^{\varepsilon,m}\mid^2+\int_s^t\mid Z_r^{\varepsilon,n}-Z_r^{\varepsilon,m}\mid^2 dr\\&=2\int_s^t(Y_r^{\varepsilon,n}-Y_r^{\varepsilon,m})(f(X_r^{\varepsilon},Y_r^{\varepsilon,n}-f(X_r^{\varepsilon},Y_r^{\varepsilon,m}))dr\\&+2\int_s^t(Y_r^{\varepsilon,n}-Y_r^{\varepsilon,m})(Z_r^{\varepsilon,n}-Z_r^{\varepsilon,m})dB_r\\&-2\int_s^t(Y_r^{\varepsilon,n}-Y_r^{\varepsilon,m})(A_n(Y_r^{\varepsilon,n})-A_m(Y_r^{\varepsilon,m}))dr,\end{aligned}$$

we deduce from (4.5) that

$$\begin{aligned}&\mathbb{E} \mid Y_s^{\varepsilon,n} - Y_s^{\varepsilon,m} \mid^2 + \mathbb{E} \int_s^t \mid Z_r^{\varepsilon,n} - Z_r^{\varepsilon,m} \mid^2 dr \\ &\leq 2K \int_s^t \mid Y_r^{\varepsilon,n} - Y_r^{\varepsilon,m} \mid^2 dr - 2\mathbb{E} \int_s^t (Y_r^{\varepsilon,n} - Y_r^{\varepsilon,m})(A_n(Y_r^{\varepsilon,n}) - A_m(Y_r^{\varepsilon,m})dr.\end{aligned}$$

We set

$$A(x) := \partial\phi(x) = \frac{1}{2} grad(\min_y \{\mid x - y \mid^2 + \phi(y)\}.$$

By the relation

$$I = J_n + \frac{1}{n} A_n = J_m + \frac{1}{m} A_m, A_m(Y_r^{\varepsilon,m}) \in A(J_m(Y_r^{\varepsilon,m})), A_n(Y_r^{\varepsilon,n}) \in A(J_n(Y_r^{\varepsilon,n})),$$

where $J_n(x)$ is the unique solution of the inclusion $x \in J_n(x) + \frac{1}{n} A(J_n(x))$, we have

$$\begin{aligned}&-(Y_r^{\varepsilon,n} - Y_r^{\varepsilon,m})(A_n(Y_r^{\varepsilon,n}) - A_m(Y_r^{\varepsilon,m})) \\ &= - < A_n(Y_r^{\varepsilon,n}) - A_m(Y_r^{\varepsilon,m}), J_n(Y_r^{\varepsilon,n}) - J_m(Y_r^{\varepsilon,m}) > \\ &\leq - < A_n(Y_r^{\varepsilon,n}) - A_m(Y_r^{\varepsilon,m}), \frac{1}{n} A_n(Y_r^{\varepsilon,n}) - \frac{1}{m} A_m(Y_r^{\varepsilon,m}) > \\ &\leq (\frac{1}{n} + \frac{1}{m}) < A_n(Y_r^{\varepsilon,n}), A_m(Y_r^{\varepsilon,m}) > -\frac{1}{n} \mid A_n(Y_r^{\varepsilon,n} \mid^2 - \frac{1}{m} \mid A_m(Y_r^{\varepsilon,m}) \mid^2 \\ &\leq \frac{1}{4n} \mid A_m(Y_r^{\varepsilon,m}) \mid^2 + \frac{1}{4m} \mid A_n(Y_r^{\varepsilon,n}) \mid^2,\end{aligned}$$

Hence, from Gronwall's lemma, we deduce that

$$\sup_{0\leq s\leq t} \mathbb{E}(\mid Y_s^{\varepsilon,n} - Y_s^{\varepsilon,m} \mid^2 + \int_s^t \mid Z_r^{\varepsilon,n} - Z_r^{\varepsilon,m} \mid^2 dr) \leq C(\frac{1}{n} + \frac{1}{m}).$$

Using Bulkholder-Davis-Gundy inequality, we obtain

$$\sup_\varepsilon \mathbb{E}(\sup_{0\leq s\leq t} \mid Y_s^{\varepsilon,n} - Y_s^{\varepsilon,m} \mid^2 + \int_0^t \mid Z_r^{\varepsilon,n} - Z_r^{\varepsilon,m} \mid^2 dr) \leq C(\frac{1}{n} + \frac{1}{m}).$$

We set

$$\lim_{n\to+\infty} Y^{\varepsilon,n} = \overline{Y}^\varepsilon, \quad \lim_{n\to+\infty} Z^{\varepsilon,n} = \overline{Z}^\varepsilon.$$

If we return to the equation satisfied by the $(Y^{\varepsilon,n}, Z^{\varepsilon,n})$, we find that $(K^{\varepsilon,n})_n$ converges uniformly in $\mathbb{L}^2(\Omega)$ to the limit $\overline{K}^\varepsilon$ where

$$\overline{K}_t^\varepsilon = \lim_n \int_0^t A_n(Y_r^{\varepsilon,n}) dr.$$

Condition (4.14) can be written as follows

$$\sup_{\varepsilon,n} \mathbb{E} \| K^{\varepsilon,n} \|^2_{H^1([0,t],\mathbb{R}^d)} < \infty,$$

where $H^1([0,t],\mathbb{R}^d)$ is the Sobolev space. In this way, the sequence $(K^{\varepsilon,n})_n$ is bounded independently of ε in $\mathbb{L}^2(\Omega; H^1([0,t],\mathbb{R}^d))$ and there exists a subsequence of $(K^{\varepsilon,n})_n$ which converges weakly. The limiting process $\overline{K}^\varepsilon$ belong to $\mathbb{L}^2(\Omega; H^1([0,t],\mathbb{R}^d))$, hence K^ε is absolutely continuous. By uniqueness of solution of the reflected BSDE (4.8) (see [61]), we can find that $\overline{Y}^\varepsilon = Y^\varepsilon$, $\overline{Z}^\varepsilon = Z^\varepsilon$, $\overline{K}^\varepsilon = K^\varepsilon$. ■

Lemma 4.7. *Under the assumption of the above lemma, the family of processes* (Y^n, M^n, K^n) *converges in probability to* (Y, M, K) *as* n *goes to* $+\infty$.

Proof . The proof of this lemma is similar to the Lemma 4.6. ■

Now, we are ready for the proof of Theorem 4.2.

Proof of theorem 4.2

Combining the above lemmas, we find that $(X^\varepsilon, Y^\varepsilon, M^\varepsilon, K^\varepsilon)$ converges in law to (X, Y, M, K) in the sense defined as above, where

$$Y_s = g(X_t) + \int_s^t f(X_r, Y_r)dr - \int_s^t Z_r dB_r + K_t - K_s.$$

■

Corollary 4.8. *Under the assumptions of theorem,* $\{Y_0^\varepsilon\}$ *converges to* Y_0 *as* ε *goes to* 0.

Proof : Since Y_0^ε is deterministic, we have

$$Y_0^\varepsilon = \mathbb{E}(g(X_t^\varepsilon) + \int_0^t f(X_s^\varepsilon, Y_s^\varepsilon)ds - K_t^\varepsilon).$$

Put

$$A_\varepsilon = g(X_t^\varepsilon) + \int_0^t f(X_s^\varepsilon, Y_s^\varepsilon)ds - K_t^\varepsilon,$$

we have

$$\mathbb{E} \mid A_\varepsilon \mid^2 \leq C(1+ \mid X_t^\varepsilon \mid^{2p}) + \mathbb{E}\int_0^t \mid Y_s^\varepsilon \mid^2 ds + \mathbb{E}\int_0^t \mid X_s^\varepsilon \mid^{2q} ds + \mathbb{E} \mid K_t^\varepsilon \mid^2 .$$

According to Lemma 4.6 and by assumption (4.1), we have

$$\sup_\varepsilon \mathbb{E} \mid A_\varepsilon \mid^2 < \infty.$$

Since Theorem 4.2 states that A_ε converges in law, as ε goes to 0, toward

$$g(X_t) + \int_0^t f(X_r, Y_r)dr + K_t,$$

the uniform integrability of A_ε implies that

$$\lim_{\varepsilon\to 0} \mathbb{E}(A_\varepsilon) = \mathbb{E}(\lim_{\varepsilon\to 0} A_\varepsilon).$$

This means that Y_0^ε converges to

$$Y_0 = g(X_t) + \int_0^t f(X_r, Y_r)dr + K_t.$$

■

Proof Proposition 4.3. Using the same arguments as in the the proof of theorem 4.2, one can prove the result. The only problem is to identify the limit:
Let β be a strictly positive number. For a given $N > 1$, we put $A_n^N := \{(s,\omega);\quad |Y_s^n|^2 + |\overline{Y}_s^n|^2 \geq N^2\}$, $\overline{A}_n^N := \Omega \setminus A_n^N$ and denote by $\mathbf{1}_E$ the indicator function of the set E. From Itô's formula, it follows that

$$\begin{aligned}
&\mathbb{E} \mid \overline{Y}_s^n - Y_s^n \mid^2 + \mathbb{E}[M^n - \widetilde{M}^n]_t - \mathbb{E}[M^n - \widetilde{M}^n]_s \\
&= 2\int_s^t \langle f(X_r, \overline{Y}_r^n) - f(X_r, Y_r^n), \overline{Y}_r^n - Y_r^n \rangle (\mathbf{1}_{\overline{A}_n^N} + \mathbf{1}_{A_n^N}) dr \\
&- 2\int_s^t \langle A_n(\overline{Y}_r^n) - A_n(Y_r^n), \overline{Y}_r^n - Y_r^n \rangle dr.
\end{aligned}$$

Since A_n is monotone, we have for every $x, z \in \mathbb{R}^d$

$$\langle A_n(x) - A_n(z), x - z \rangle \geq 0.$$

Thus

$$\mathbb{E} \mid \overline{Y}_s^n - Y_s^n \mid^2 + \mathbb{E}[M^n - \widetilde{M}^n]_t - \mathbb{E}[M^n - \widetilde{M}^n]_s \leq 2\mu_N^+ \mathbb{E} \int_s^t \mid \overline{Y}_r^n - Y_r^n \mid^2 dr + \frac{C}{N^{2(1-\alpha)}}.$$

We conclude from Gronwall's lemma that

$$\mathbb{E} \mid \overline{Y}_s^n - Y_s^n \mid^2 + \mathbb{E}[M^n - \widetilde{M}^n]_t - \mathbb{E}[M^n - \widetilde{M}^n]_s \leq \frac{C}{N^{2(1-\alpha)}} e^{2\mu_N^+ t}.$$

Passing to the limit on N we obtain, $\overline{Y}_r^n = Y_r^n, 0 \leq s \leq t$, and $M^n = \widetilde{M}^n$. ■

4.3 Application to a class of PDEs

Now, we apply our result to the proof of an homogenization result for PDEs.

4.3.1 Application to the viscosity solutions of multivalued PDEs

Let u^ε be the solution of the PDE

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial s}(s,x) - L_\varepsilon u^\varepsilon(s,x) - f(x, u^\varepsilon(s,x)) \in \partial\phi(u^\varepsilon(s,x)), for \quad s \in [0,t] \\ u^\varepsilon(0,x) = g(x), u^\varepsilon(t,x) \in \overline{Dom(\phi)}, x \in \mathbb{R}^d, \end{cases} \tag{4.15}$$

and u be the solution of the PDE

$$\begin{cases} \dfrac{\partial u}{\partial s}(s,x) - Lu(s,x) - f(x, u(s,x)) \in \partial\phi(u(s,x)), for \quad s \in [0,t] \\ u(0,x) = g(x), u(t,x) \in \overline{Dom(\phi)}, x \in \mathbb{R}^d. \end{cases} \tag{4.16}$$

Theorem 4.9. *Assume $k = 1$, under the conditions of Theorem 4.2 $u^\varepsilon(t,x)$ converges to $u(t,x)$ for all $(t,x) \in [0,t] \times \mathbb{R}^d$ as ε goes to 0.*

Proof . Let $x \in \mathbb{R}^d$ and $\{X_s^{x,\varepsilon}; 0 \leq s \leq t\}$ be the diffusion process defined as above, starting at x. For all $t \in \mathbb{R}^+$, we denote by $(\{Y_s^{t,x,\varepsilon}, Z_s^{t,x,\varepsilon}, K_s^{t,x,\varepsilon}\}; 0 \leq s \leq t)$ be the solution of the reflected BSDE

$$Y_s^{t,x,\varepsilon} = g(X_t^{x,\varepsilon}) + \int_s^t f(X_r^{x,\varepsilon}, Y_r^{t,x,\varepsilon}) dr - \int_s^t Z_r^{t,x,\varepsilon} dB_r + K_t^{t,x,\varepsilon} - K_s^{t,x,\varepsilon}.$$

By virtue of Pardoux, Rascanu [70] (see also Chapter 1), the function $u^\varepsilon : \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by $u^\varepsilon(t,x) = Y_0^{t,x}$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, is the unique viscosity solution of the PDE (4.15). Let $\{X_s^x; s \geq 0\}$ be the diffusion process with infenitesimal L, strating at $x \in \mathbb{R}$ and $(\{Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x}\}; 0 \leq s \leq t)$ be the unique solution of the RBSDE

$$Y_s^{t,x} = g(X_t^x) + \int_s^t f(X_r^x, Y_r^{t,x})dr - \int_s^t Z_r^{t,x}dB_r + K_t^{t,x} - K_s^{t,x}.$$

Again, in view of [70] (see also Chapter 1) the function $u : [0,t] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by $u(t,x) = Y_0^{t,x}$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, is the unique viscosity solution of the PDE (4.16). Therefore, the result follows from corollary 4.8. ■

4.3.2 Application to the solutions of PDEs in Sobolev spaces

In this subsection, we prove an homogenization result for solutions of semi-linear PDEs with obstacle $h = 0$ in Sobolev sense (see [9]), of the form

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial s}(s,x) - L_\varepsilon u^\varepsilon(s,x) - F(x, \alpha^\varepsilon(s,x), u^\varepsilon(s,x)) = 0, \\ u^\varepsilon(0,x) = g(x), \end{cases} \tag{4.17}$$

where $F(x,\alpha,y) = f(x,y) + \alpha 1_{\{y=0\}} f^-(x,y)$ with $f^- = \sup(-f, 0)$.
Now, let us introduce our assumptions.
Let X^ε be the diffusion defined as in Section 4.2 with smooth coefficients.
We assume that
- $g : \mathbb{R}^d \longrightarrow \mathbb{R}$ is continuous, positive and satisfies (4.2).
- $f : \mathbb{R}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, bounded and satisfies (4.5).

We consider the RBSDE

$$Y_s^\varepsilon = g(X_s^\varepsilon) + \int_s^t f(X_r^\varepsilon, Y_r^\varepsilon)dr - \int_s^t Z_r^\varepsilon dB_r + K_t^\varepsilon - K_s^\varepsilon,$$

where $K_t^\varepsilon = \displaystyle\int_0^t \alpha^\varepsilon(r, X_r^\varepsilon) 1_{\{Y_r^\varepsilon = 0\}} f^-(X_r^\varepsilon, Y_r^\varepsilon)dr$ and $Y^\varepsilon \geq 0$.
We will prove that the family of processes $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon, \alpha^\varepsilon)$ converges to the unique solution (see [7]) (X, Y, Z, α) of the RBSDE

$$Y_s = g(X_t) + \int_s^t f(X_r, Y_r)dr - \int_s^t Z_r dB_r + K_t - K_s,$$

where $K_t = \int_0^t \alpha(r, X_r) 1_{\{Y_r = 0\}} f^-(X_r, Y_r)dr$ and $Y \geq 0$.

Theorem 4.10. *Under the above conditions, the family of processes $(X^\varepsilon, Y^\varepsilon, M^\varepsilon, \alpha^\varepsilon)$ converges in law to (X, Y, M, α) on $\mathbb{C}([0,t], \mathbb{R}^d) \times \mathbb{D}([0,t], \mathbb{R}^2) \times (\mathbb{C}[0,t], \mathbb{R})$. Furthermore, $Y_0^\varepsilon \longrightarrow Y_0$ in $\mathbb{R}$.*

Proof . Approximate the reflection term by

$$K_t^{\varepsilon,n} := \int_0^t \alpha_s^{\varepsilon,n} f^-(X_s^\varepsilon, Y_s^{\varepsilon,n})ds, \ \alpha_s^{\varepsilon,n} = \phi_n(Y_s^{\varepsilon,n}),$$

where $\phi_n \in C^\infty$ such that $0 \leq \phi_n \leq 1$ and

$$\begin{aligned} \phi_n(y) \quad &= 1 \quad \text{if} \quad | y | \leq \tfrac{1}{2^n} \\ &= 0 \quad \text{if} \quad | y | \geq \tfrac{2}{2^n}. \end{aligned}$$

Using the same arguments as in Subsection 4.3.1, we get the result. Indeed, we need to prove only that the limit Y^n of $Y^{\varepsilon,n}$ converges in probability to Y. In fact, it is easy to check that

$$\sup_n \mathbb{E} \sup_{0\leq s\leq t} \mid Y_s^n \mid^2 < +\infty.$$

Since $f_n(y) := \phi_n(y)f^-(y)$ is decreasing, the comparison theorem yields that so is Y^n. Define $\lim_{n\to\infty} Y^n = Y$.
By Lebesgue theorem, we have

$$\mathbb{E}\int_0^t \mid Y_s^n - Y_s \mid^2 ds \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

■

Now, let us apply our result to the homogenization of solutions of semi-linear PDEs in Sobolev sense.
Let u be the weak solution (see Bally et *al.* [7]) of the reflected PDE

$$\begin{cases} \dfrac{\partial u}{\partial s}(s,x) - Lu(s,x) - F(x,\alpha(s,x),u(s,x)) = 0, \\ u(0,x) = g(x), \end{cases} \tag{4.18}$$

Theorem 4.11. *Under the conditions of theorem 4.10, $u^\varepsilon(t,x)$ the weak solution of PDE (4.17) converges to $u(t,x)$ the solution of (4.17) for all $(t,x)\in[0,t]\times\mathbb{R}^d$ as ε goes to* 0.

Proof . Following of Bally et *al.* [7] $u^\varepsilon(t,x) = Y_0^{t,x,\varepsilon}$, and using the same arguments as in Subsection 4.3.1 and Theorem 4.10, we get our assertion. ■

Chapter 5

Averaging of Backward Stochastic Differential Equations and Homogenization of Partial Differential Equations with Periodic Coefficients

The chapter is organized as follows. In Section 5.1, we introduce some notations and assumptions to be used in the sequel. Section 5.2 is devoted to the proof of averaging result of BSDE. In section 5.3, we apply our result to the homogenization of a class of semi-linear PDE's. We give another application to the homogenization of nonlinear Cauchy problem in Section 5.4.

5.1 Preliminary Results

Let us consider the singularity perturbed stochastic differential equation, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$

$$\begin{cases} dX_s^{1,\varepsilon} = F(X_s^{1,\varepsilon}, X_s^{2,\varepsilon})ds + \varepsilon^{-1}G(X_s^{1,\varepsilon}, X_s^{2,\varepsilon})ds + K(X_s^{1,\varepsilon}, X_s^{2,\varepsilon})dB_s^\varepsilon, X_0^{1,\varepsilon} = x_0^1 \\ dX_s^{2,\varepsilon} = \varepsilon^{-2}b(X_s^{2,\varepsilon})ds + \varepsilon^{-1}\sigma(X_s^{2,\varepsilon})dB_s^\varepsilon, X_0^{2,\varepsilon} = x_0^2, \end{cases} \tag{5.1}$$

where $X^{1,\varepsilon} \in \mathbb{R}^d$, $X^{2,\varepsilon} \in \mathbb{R}^l$, $\{B_s^\varepsilon, 0 \leq s \leq t\}$ is a d_1-Brownian motion, $d_1 \geq d$ and F, G, K, σ, b are a vector-functions.
We assume that F, G, K, σ, b are periodic of period one in each direction of the variable x_2, so that the process $\{X^{2,\varepsilon}\}$ can be considered as taking values in l-dimensional torus $\mathbb{T}^l$.
Moreover, we assume that σ, b are continuous functions.
We suppose in addition that

$$a := \sigma\sigma^* \geq \alpha I > 0, \tag{5.2}$$

there exist λ_-, λ_+ and γ the best constants such that for any $x \in \mathbb{R}^d/\{0\}$

$$0 < \lambda_- \leq (\sigma\sigma^* x/ \mid x \mid, x/ \mid x \mid) \leq \lambda_+, \quad Tr\sigma\sigma^*/l \leq \gamma, \tag{5.3}$$

$$(b(x), x/ \mid x \mid) \leq -r \mid x \mid^2, \quad \mid x \mid \geq M_0, \tag{5.4}$$

with $M_0 \geq 0$ and $r > 0$.

$$\begin{aligned} &\mid F(x_1,x_2)-F(x_1,x_2') \mid \quad + \mid G(x_1,x_2)-G(x_1,x_2') \mid + \|K(x_1,x_2)-K(x_1,x_2')\| \\ &\leq C(x_1) \mid x_2-x_2' \mid . \end{aligned} \tag{5.5}$$

Note that (5.2), (5.3), (5.4) and (5.5) insure that the system (5.1) of SDEs is well posed (see Pardoux, Veretennikov [72]).

We assume that for all $x_1 \in \mathbb{R}^d$, $G(.,x_2) \in C^2(\mathbb{R}^d,\mathbb{R}^d)$, uniformly with respect to x_2, $\dfrac{\partial^2 G}{\partial x_1^2} \in C(\mathbb{R}^{d+l},\mathbb{R}^{d^3})$ and the functions F, K and G satisfy the following conditions:

- F, G and K are continuous in x_1 uniformly with respect to x_2
- $\exists C_1 > 0$ such that

$$\begin{aligned} &\mid K(x_1,x_2) \mid + \mid G(x_1,x_2) \mid + \| \frac{\partial G}{\partial x_1}(x_1,x_2) \| + \| \frac{\partial^2 G}{\partial x_1^2}(x_1,x_2) \| \leq C_1, \\ &\forall (x_1,x_2) \in \mathbb{R}^d \times \mathbb{R}^l. \end{aligned} \tag{5.6}$$

- $\exists C_2 > 0$ such that

$$\mid F(x_1,x_2) \mid \leq C_2(1+ \mid x_1 \mid), \forall (x_1,x_2) \in \mathbb{R}^d \times \mathbb{R}^l. \tag{5.7}$$

We assume moreover that for all $x_2 \in \mathbb{R}^d$ and $j = 1,...,d$

$$\int_{\mathbb{T}^l} G_j(x_1,x_2)\mu(dx_2) = 0, \quad \forall x_1 \in \mathbb{R}^d, \tag{5.8}$$

where $\mu(dx)$ denotes the unique invariant measure of the diffusion process $\{X_s^{2,1}\}, 0 \leq s \leq t$ (see Pardoux, Veretennikov [72]). It then follows from [72] that the Poisson equation

$$L_2 \widehat{G}_j(x_1,x_2) = -G_j(x_1,x_2), \quad j = 1,..,d$$

has unique centred solution

$$\widehat{G}_j(x_1,x_2) = \int_0^{+\infty} \mathbb{E}_{x_2} G_j(x_1, X_t^{2,1})dt,$$

where L_2 is the generator of $X^{2,\varepsilon}$ when $\varepsilon = 1$.

It was shown in [72] that under the above conditions $X^{1,\varepsilon}$ converges in distribution to a d-dimensional diffusion process with generator

$$L = \frac{1}{2}\sum_{i,j} \overline{a}_{ij}(x_1)\frac{\partial^2}{\partial x_{1i}x_{1j}} + \sum_i \overline{b}_i(x_1)\frac{\partial}{\partial x_{1i}},$$

where

$$\begin{aligned} \overline{b}(x) = \overline{F}(x) &+ \sum_i \int G_i(x,y)\partial x_i \widehat{G}(x,y)\mu(dy) \\ &+ \sum_{i,k} \int (K\sigma^*)_{i,k}(x,y)\partial y_k \partial x_i \widehat{G}(x,y)\mu(dy), \end{aligned}$$

and

$$\overline{a}(y) = (\overline{H} + \overline{K} + \overline{g}),$$

with

$$\overline{F}(x) = \int F(x,y)\mu(dy)$$

$$\overline{H}(x) = \int H(x,y)\mu(dy)$$

$$\overline{g}(x) = \int (G(x,y)\widehat{G}^*(x,y) + \widehat{G}G^*(x,y))\mu(dy)$$

$$\overline{K}(x) = \int ((K\sigma^*)_{i,k}(x,y)\partial y_k \widehat{G}_j(x,y) + (K\sigma^*)_{j,k}(x,y)\partial y_k \widehat{G}_j(x,y))\mu(dy).$$

In the sequel, we assume that $d = 1$.
Now, we can state the lemma which we will use below

Lemma 5.1. $\forall p \in \mathbb{N}/\{0\}$

$$\sup_{\varepsilon} \sup_{0\le s\le t} \mathbb{E}(\mid X_s^{1,\varepsilon} \mid^{2p}) < +\infty.$$

Proof . It follows from the equation (5.1), Itô-Krylov formula (see Krylov [45] or more precisely, Pardoux-Veretennikov [72]) and the fact that $L_2\widehat{G} = -G$

$$\begin{aligned}
&X_s^{1,\varepsilon} + \varepsilon(\widehat{G}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}) - \widehat{G}(x_0^1, x_0^2)) \\
&= x_0^1 + \int_0^s (F + \frac{\partial \widehat{G}}{\partial x_1} G + \frac{\partial^2 \widehat{G}}{\partial x_1 \partial x_2}\sigma K^* + \varepsilon \frac{\partial^2 \widehat{G}}{\partial x_1^2} KK^* + \varepsilon \frac{\partial \widehat{G}}{\partial x_1} F)(X_r^{1,\varepsilon}, X_r^{2,\varepsilon})dr \\
&+ \int_0^s (K + \frac{\partial \widehat{G}}{\partial x_2}\sigma + \varepsilon \frac{\partial \widehat{G}}{\partial x_1} K)(X_r^{1,\varepsilon}, X_r^{2,\varepsilon})dB_r^\varepsilon.
\end{aligned}$$

Thanks to Itô's formula, assumptions (5.6), (5.7), the boundedness of the coefficients and the fact that for each $1 \le r \le q$, $\mid x \mid^r \le C(1 + \mid x \mid^q)$, we deduce that

$$\mathbb{E} \mid X_s^{1,\varepsilon} \mid^{2p} \le C(1 + \mathbb{E}\int_0^s \mid X_r^{1,\varepsilon} \mid^{2p} dr),$$

hence the result follows from Granwall's lemma. ∎

Now, for each $\varepsilon > 0$, let $\{(Y_s^\varepsilon, Z_s^\varepsilon); 0 \le s \le t\}$ be the $\mathbb{R} \times \mathbb{R}^{d_1}$ valued progressively measurable process solution of the BSDE

$$Y_s^\varepsilon = g(X_t^{1,\varepsilon}) + \frac{1}{\varepsilon}\int_s^t e(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr + \int_s^t f(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr - \int_s^t Z_r^\varepsilon dB_r^\varepsilon, \tag{5.9}$$

satisfying

$$\mathbb{E}(\sup_{0\le s\le t} \mid Y_s^\varepsilon \mid^2 + \int_0^t \mid Z_s^\varepsilon \mid^2 ds) < +\infty.$$

The problem under consideration is the averaging of the process Y^ε. A first application is the homogenization of the following parabolic semi-linear PDE

$$\begin{cases}
\dfrac{\partial u^\varepsilon}{\partial s}(s,x_1,x_2) = \varepsilon^{-2}L_2 u^\varepsilon(s,x_1,x_2) + (\varepsilon^{-1}F(x_1,x_2 + G(x_1,x_2))\dfrac{\partial u^\varepsilon}{\partial x_1}(s,x_1,x_2) \\
+\frac{1}{2}KK^*(x_1)\dfrac{\partial^2 u^\varepsilon}{\partial x_1^2}(s,x_1,x_2) + (\frac{1}{\varepsilon}e(x_1,x_2,u^\varepsilon(s,x_1,x_2)) + f(x_1,x_2,u^\varepsilon(s,x_1,x_2)) \\
\forall s \in [0,t], \quad (x_1,x_2) \in \mathbb{R} \times \mathbb{R}^l \\
u^\varepsilon(0,x_1,x_2) = g(x_1), \quad (x_1,x_2) \in \mathbb{R} \times \mathbb{R}^l.
\end{cases} \tag{5.10}$$

A second application is the averaging of non singular Cauchy problem, $u^{\varepsilon} \in C^{1,1,2}((0,t) \times \mathbb{R} \times \mathbb{R})$

$$\begin{cases} \dfrac{\partial u^{\varepsilon}}{\partial s}(s,x_1,x_2) = \varepsilon^{-2} L_2 u^{\varepsilon}(s,x_1,x_2) + \varepsilon^{-1}(F(x_1,x_2+G(x_1,x_2))\dfrac{\partial u^{\varepsilon}}{\partial x_1}(s,x_1,x_2) \\ +\frac{1}{2}KK^*(x_1)\dfrac{\partial^2 u^{\varepsilon}}{\partial x_1^2}(s,x_1,x_2) + (\frac{1}{\varepsilon}e(x_1,x_2,u^{\varepsilon}(s,x_1,x_2)) + f(x_1,x_2,u^{\varepsilon}(s,x_1,x_2)) \\ +\varepsilon^{-2}\sigma^2(x_2)h(u^{\varepsilon}(s,x_1,x_2))(\dfrac{\partial u^{\varepsilon}}{\partial x_2}(s,x_1,x_2))^2 \quad \forall s \in [0,t], \quad (x_1,x_2) \in \mathbb{R}\times\mathbb{R} \\ u^{\varepsilon}(0,x_1,x_2) = g(x_1), \quad (x_1,x_2) \in \mathbb{R}\times\mathbb{R} \\ u^{\varepsilon}(s,x) \leq C(1+\mid x \mid^{2\delta}), \quad x=(x_1,x_2) \in \mathbb{R}\times\mathbb{R}, \quad for some\ C>0, \quad \delta \geq 1. \end{cases} \tag{5.11}$$

We now formulate our assumptions on g, e and f .
g is continuous and there exists $p \in \mathbb{N}$ such that

$$g(x_1) \leq C(1+\mid x_1 \mid^p).$$

e and f are measurable from $\mathbb{R} \times \mathbb{R}^l \times \mathbb{R} \longrightarrow \mathbb{R}$, which are periodic of period one in each direction, in the second argument, continuous in x_1 (resp. y) uniformly with respect to x_2 and y (resp. x_1 and x_2) and that for all $x_1 \in \mathbb{R}$ and $y \in \mathbb{R}$

$$\int_{\mathbb{T}^l} e(x_1,x_2,y)\mu(dx_2) = 0, \tag{5.12}$$

and e is twice continuously differentiable in x_1 (resp. y) uniformly with respect to x_2 and y (resp. x_1 and x_2). Moreover, for some $\mu \in \mathbb{R}$, all $(x_1,x_2) \in \mathbb{R}\times\mathbb{R}^l$, $y, y' \in \mathbb{R}$

$$(f(x_1,x_2,y) - f(x_1,x_2,y'))(y-y') \leq \mu \mid y-y' \mid^2 \tag{5.13}$$

$$\mid f(x_1,x_2,y) \mid \leq C(1+\mid y \mid^2). \tag{5.14}$$

We finally assume that there exists K such that $\forall (x_1,x_2) \in \mathbb{R}\times\mathbb{T}^l, y \in \mathbb{R}$

$$\begin{aligned} \mid e(x_1,x_2,y) \mid + \mid \frac{\partial e}{\partial y}(x_1,x_2,y) \mid + \mid \frac{\partial e}{\partial x_1}(x_1,x_2,y) \mid + \mid \frac{\partial^2 e}{\partial y^2}(x_1,x_2,y) \mid \\ + \mid \frac{\partial^2 e}{\partial x_1^2}(x_1,x_2,y) \mid \leq K. \end{aligned}$$

Remark 5.2. *Assumptions 5.8 and 5.12 are standard in homogenization theory. The study of the problem when those assumptions are not satisfied is largely open.*

5.2 Statement of the result

Denote

$$A(x_1,y) = \int_{\mathbb{T}^l} (F + \frac{\partial \widehat{G}}{\partial x_1}G + \frac{\partial^2 \widehat{G}}{\partial x_1 \partial x_2}\sigma K^* + \frac{\partial^2 \widehat{e}}{\partial x_2 \partial y}\sigma K^* + \frac{\partial^2 \widehat{e}}{\partial x_2 \partial y}\sigma\sigma^* \frac{\partial \widehat{G}}{\partial x_2}^*)(x_1,.,y))(x_2)\mu(dx_2)$$

$$D(x_1) = \int_{\mathbb{T}^l} \{(KK^* + \frac{\partial \widehat{G}}{\partial x_2}\sigma\sigma^* \frac{\partial \widehat{G}}{\partial x_2}^* + \frac{\partial \widehat{G}}{\partial x_2}\sigma K^* + K\sigma^* \frac{\partial \widehat{G}}{\partial x_2}^* \}(x_1,.))(x_2)\mu(dx_2)$$

$$C(x_1,y) = \int_{\mathbb{T}^l} (f - \frac{\partial \widehat{e}}{\partial x_1}e + \frac{\partial e^*}{\partial x_1}G + \frac{\partial^2 \widehat{e}}{\partial x_2 \partial y}a\frac{\partial e^*}{\partial x_1} + \frac{\partial^2 \widehat{e}}{\partial x_1 \partial x_2}\sigma K^*)(x_1,.,y))(x_2)\mu(dx_2).$$

Let $\{(Y_s, Z_s), 0 \le s \le t\}$ be the unique solution of the BSDE

$$Y_s = g(X_t^1) + \int_s^t C(X_r^1, Y_r)dr - \int_s^t Z_r dB_r,$$

where X^1 is the solution of SDE

$$X_s^1 = x_0^1 + \int_0^s A(X_r^1, Y_r)dr + \int_0^s \lambda(X_r^1)dB_r,$$

with B is a Brownian motion and $\lambda\lambda^* = D$.
Let $u : [0,t] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be the viscosity solution of the system of parabolic semi-linear PDE's

$$\begin{cases} \dfrac{\partial u}{\partial s}(s, x_1) = \frac{1}{2}D(x_1)\dfrac{\partial^2 u}{\partial x_1^2}(s, x_1) + A(x_1, u(s, x_1))\dfrac{\partial u}{\partial x_1}(s, x_1) \\ \qquad\qquad +C(x_1, , u(s, x_1)) \quad \forall s \in [0,t], \quad x_1 \in \mathbb{R} \\ u(0, x_1) = g(x_1), \quad x_1 \in \mathbb{R}. \end{cases} \tag{5.15}$$

Recall that

$$Y_s^\varepsilon = g(X_t^{1,\varepsilon}) + \frac{1}{\varepsilon}\int_s^t e(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr + \int_s^t f(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr - \int_s^t Z_r^\varepsilon dB_r^\varepsilon,$$

and

$$\begin{cases} dX_s^{1,\varepsilon} = F(X_s^{1,\varepsilon}, X_s^{2,\varepsilon})ds + \varepsilon^{-1}G(X_s^{1,\varepsilon}, X_s^{2,\varepsilon})ds + K(X_s^{1,\varepsilon}, X_s^{2,\varepsilon})dB_s^\varepsilon, X_0^{1,\varepsilon} = x_0^1 \\ dX_s^{2,\varepsilon} = \varepsilon^{-2}b(X_s^{2,\varepsilon})ds + \varepsilon^{-1}\sigma(X_s^{2,\varepsilon})dB_s^\varepsilon, X_0^{2,\varepsilon} = x_0^2. \end{cases}$$

We put

$$M_t^\varepsilon = -\int_0^t Z_s^\varepsilon dB_s^\varepsilon.$$

The main result is the following.

Theorem 5.3. *Under the above conditions, there exists a d_1-dimensional Brownian motion B such that the family of processes $(Y^\varepsilon, M^\varepsilon)$ converges in law to $(Y, M := -\int_0^s Z_s dB_s)$ on $(D([0,t], \mathbb{R}))^2$ equipped with the same topology as above. Moreover, $Y_0^\varepsilon \longrightarrow Y_0$, as $\varepsilon \to 0$.*

The proof will be divided into a several steps.
Step1. Transformation of systems (5.1) and (5.9).
It follows from Itô-Krylov's formula (see Krylov [45] or Pardoux, Veretennikov [72])

$$\begin{aligned} &X_s^{1,\varepsilon} + \varepsilon(\widehat{G}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}) - \widehat{G}(x_0^1, x_0^2)) \\ &= x_0^1 + \int_0^s (F + \frac{\partial \widehat{G}}{\partial x_1}G + \frac{\partial^2 \widehat{G}}{\partial x_1 \partial x_2}\sigma K^* + \varepsilon\frac{\partial^2 \widehat{G}}{\partial x_1^2}KK^* + \varepsilon\frac{\partial \widehat{G}}{\partial x_1}F)(X_r^{1,\varepsilon}, X_r^{2,\varepsilon})dr \\ &+ \int_0^s (K + \frac{\partial \widehat{G}}{\partial x_2}\sigma + \varepsilon\frac{\partial \widehat{G}}{\partial x_1}K)(X_r^{1,\varepsilon}, X_r^{2,\varepsilon})dB_r^\varepsilon. \end{aligned} \tag{5.16}$$

$$
\begin{aligned}
&Y_s^\varepsilon+\varepsilon(\widehat{e}(X_t^{1,\varepsilon},X_t^{2,\varepsilon},Y_t^\varepsilon)-\widehat{e}(X_s^{1,\varepsilon},X_s^{2,\varepsilon},Y_s^\varepsilon))\\
&=g(X_t^{1,\varepsilon})+\int_s^t(f-\frac{\partial\widehat{e}}{\partial y}e-\varepsilon\frac{\partial\widehat{e}}{\partial y}f)(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)dr\\
&+\varepsilon\int_s^t\frac{\partial^2\widehat{e}}{\partial x_1\partial y}K(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)Z_r^\varepsilon dr+\int_s^t(\nabla_{x_2}\widehat{e}(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)\sigma(X_r^{2,\varepsilon})\\
&+\varepsilon\frac{\partial\widehat{e}}{\partial x_1}(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)K(X_r^{1,\varepsilon}X_r^{2,\varepsilon})-Z_r^\varepsilon)dB_r^\varepsilon+\varepsilon\int_s^t\frac{\partial\widehat{e}}{\partial y}(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)Z_r^\varepsilon dB_r^\varepsilon\\
&+\tfrac{\varepsilon}{2}\int_s^t\frac{\partial^2\widehat{e}}{\partial y^2}(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)\mid Z_r^\varepsilon\mid^2dr+\varepsilon\int_s^t\frac{\partial^2\widehat{e}}{\partial x_1^2}KK^*(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)dr\\
&+\int_s^t(\frac{\partial^2\widehat{e}}{\partial x_1\partial x_2}\sigma K^*)(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)dr+\varepsilon\int_s^t\frac{\partial\widehat{e}}{\partial x_1}F(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)dr\\
&+\int_s^t\frac{\partial\widehat{e}}{\partial x_1}G(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)dr+\int_s^t(\frac{\partial^2\widehat{e}}{\partial x_2\partial y}\sigma)(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)Z_r^\varepsilon dr.
\end{aligned}
\tag{5.17}
$$

Now define

$$
\widetilde{Z}_s^\varepsilon=Z_s^\varepsilon-\nabla_{x_2}\widehat{e}(X_s^{1,\varepsilon},X_s^{2,\varepsilon},Y_s^\varepsilon)\sigma(X_s^{2,\varepsilon})-\varepsilon\frac{\partial\widehat{e}}{\partial x_1}K(X_s^{1,\varepsilon},X_s^{2,\varepsilon},Y_s^\varepsilon),\quad 0\le s\le t,
$$

note that the difference between Z^ε and $\widetilde{Z}^\varepsilon$ is uniformly bounded process.
Hence, from (5.17) one has

$$
\begin{aligned}
&Y_s^\varepsilon+\varepsilon(\widehat{e}(X_t^{1,\varepsilon},X_t^{2,\varepsilon},Y_t^\varepsilon)-\widehat{e}(X_s^{1,\varepsilon},X_s^{2,\varepsilon},Y_s^\varepsilon))\\
&=g(X_t^{1,\varepsilon})+\int_s^t(f-\frac{\partial\widehat{e}}{\partial y}e-\varepsilon\frac{\partial\widehat{e}}{\partial y}f+\varepsilon\frac{\partial\widehat{e}}{\partial x_1}F+\frac{\partial\widehat{e}}{\partial x_1}G)(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)dr\\
&+\varepsilon\int_s^t\frac{\partial^2\widehat{e}}{\partial x_1\partial y}(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)K(X_r^{1,\varepsilon},X_r^{2,\varepsilon})Z_r^\varepsilon dr+\varepsilon\int_s^t\frac{\partial\widehat{e}}{\partial y}(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)Z_r^\varepsilon dB_r^\varepsilon\\
&+\frac{\varepsilon}{2}\int_s^t\frac{\partial^2\widehat{e}}{\partial^2 y}(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)\mid Z_r^\varepsilon\mid^2dr+\varepsilon\int_s^t\frac{\partial^2\widehat{e}}{\partial x_1^2}KK^*(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)dr\\
&+\int_s^t\frac{\partial^2\widehat{e}}{\partial x_1\partial x_2}\sigma K^*(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)dr+\int_s^t\frac{\partial^2\widehat{e}}{\partial x_2\partial y}\sigma(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)Z_r^\varepsilon dr-\int_s^t\widetilde{Z}_r^\varepsilon dB_r^\varepsilon.
\end{aligned}
$$

We put

$$
\widetilde{B}_s^\varepsilon=B_s^\varepsilon-\int_0^s\{\frac{\partial^2\widehat{e}}{\partial x_2\partial y}\sigma+\varepsilon\frac{\partial^2\widehat{e}}{\partial x_1\partial y}K\}(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)dr.
$$

It follows from Girsanov's theorem that there exists a new probability measure $\widetilde{P}$, under which $\{\widetilde{B}_s^\varepsilon;0\le s\le t\}$ is a Brownian motion. We have that

$$
\begin{aligned}
&X_s^{1,\varepsilon}+\varepsilon(\widehat{G}(X_s^{1,\varepsilon},X_s^{2,\varepsilon})-\widehat{G}(x_0^1,x_0^2))\\
&=x_0^1+\int_0^s\{F+\frac{\partial\widehat{G}}{\partial x_1}G+\frac{\partial^2\widehat{G}}{\partial x_1\partial x_2}\sigma K^*+\varepsilon\frac{\partial^2\widehat{G}}{\partial x_1^2}KK^*+\varepsilon\frac{\partial\widehat{G}}{\partial x_1}F+\frac{\partial^2\widehat{e}}{\partial x_2\partial y}\sigma K^*\\
&+\frac{\partial^2\widehat{e}}{\partial x_2\partial y}\sigma\sigma^*\frac{\partial\widehat{G}}{\partial x_2}^*+\varepsilon\frac{\partial^2\widehat{e}}{\partial x_1\partial y}KK^*+\varepsilon\frac{\partial^2\widehat{e}}{\partial x_1\partial y}K\sigma^*\frac{\partial\widehat{G}}{\partial x_2}\\
&+\varepsilon^2\frac{\partial^2\widehat{e}}{\partial x_1\partial y}KK^*\frac{\partial\widehat{G}}{\partial x_1}\}(X_r^{1,\varepsilon},X_r^{2,\varepsilon},Y_r^\varepsilon)dr\\
&+\int_0^s(K+\frac{\partial\widehat{G}}{\partial x_2}\sigma+\varepsilon\frac{\partial\widehat{G}}{\partial x_1}K)(X_r^{1,\varepsilon},X_r^{2,\varepsilon})d\widetilde{B}_r^\varepsilon.
\end{aligned}
\tag{5.18}
$$

$$
\begin{aligned}
&Y_s^\varepsilon + \varepsilon(\widehat{e}(X_t^{1,\varepsilon}, X_t^{2,\varepsilon}, Y_t^\varepsilon) - \widehat{e}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)) \\
&= g(X_t^{1,\varepsilon}) + \int_s^t \{f - \frac{\partial \widehat{e}}{\partial y} e - \varepsilon \frac{\partial \widehat{e}}{\partial y} f + \varepsilon \frac{\partial \widehat{e}}{\partial x_1} F + \frac{\partial \widehat{e}}{\partial x_1} G + \frac{\partial^2 \widehat{e}}{\partial x_2 \partial y} a \nabla_{x_2} \widehat{e}^* \\
&+\varepsilon \frac{\partial^2 \widehat{e}}{\partial x_1 \partial y} K \sigma^* \nabla_{x_2} \widehat{e}^* + \varepsilon \frac{\partial^2 \widehat{e}}{\partial x_2 \partial y} \sigma K^* \frac{\partial \widehat{e}}{\partial x_1} + \varepsilon^2 \frac{\partial^2 \widehat{e}}{\partial x_1 \partial y} K K^* \frac{\partial \widehat{e}}{\partial x_1}\}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) dr \\
&+\varepsilon \int_s^t \frac{\partial \widehat{e}}{\partial y}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) Z_r^\varepsilon (d\widetilde{B}_r^\varepsilon + (\frac{\partial^2 \widehat{e}}{\partial x_2 \partial y}\sigma + \varepsilon \frac{\partial^2 \widehat{e}}{\partial x_1 \partial y} K)(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) dr \\
&+\frac{\varepsilon}{2} \int_s^t \frac{\partial^2 \widehat{e}}{\partial y^2}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) \mid Z_r^\varepsilon \mid^2 dr + \varepsilon \int_s^t \frac{\partial^2 \widehat{e}}{\partial x_1^2} K K^* (X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) dr \\
&+ \int_s^t \frac{\partial^2 \widehat{e}}{\partial x_1 \partial x_2} \sigma K^* (X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) dr - \int_s^t \widetilde{Z}_r^\varepsilon d\widetilde{B}_r^\varepsilon.
\end{aligned}
\tag{5.19}
$$

Since $\frac{\partial^2 \widehat{e}}{\partial x_2 \partial y}$ and $\frac{\partial^2 \widehat{e}}{\partial x_1 \partial y}$ are bounded, the Radon-Nikodym derivative $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}$ belong to all space $L^q(\mathbb{P})$, hence from Lemma 5.1, for any $p \in \mathbb{N}/\{0\}$, $\sup_\varepsilon \widetilde{\mathbb{E}} \mid X_t^{1,\varepsilon} \mid^{2p} < +\infty$, consequently, $\sup_\varepsilon \widetilde{\mathbb{E}} \mid g(X_t^{1,\varepsilon}) \mid^2 < +\infty$.

Step2. A priori estimates for $(Y^\varepsilon, Z^\varepsilon)$

$$
\begin{aligned}
Y_s^\varepsilon = g(X_t^{1,\varepsilon}) &+ \int_s^t \frac{1}{\varepsilon}(e(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) - (Z_r^\varepsilon(\frac{\partial^2 \widehat{e}}{\partial x_2 \partial y}\sigma + \varepsilon \frac{\partial^2 \widehat{e}}{\partial x_1 \partial y} K))(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)))dr \\
&+ \int_s^t f(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) dr - \int_s^t Z_r^\varepsilon d\widetilde{B}_r^\varepsilon.
\end{aligned}
$$

It follows from the same argument as in Pardoux [65] that

$$
\varepsilon \widetilde{E} \int_s^t \mid Y_r^\varepsilon \mid\mid Z_r^\varepsilon \mid^2 dr \le C(\varepsilon + \widetilde{E} \int_s^t \mid Y_r^\varepsilon \mid^2 dr). \tag{5.20}
$$

We return to equation (5.19), let $\widehat{Y}_s^\varepsilon = Y_s^\varepsilon - \varepsilon \widehat{e}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)$, from Itô's formula, we have

$$
\begin{aligned}
&\mid \widehat{Y}_s^\varepsilon \mid^2 + \int_s^t \mid \widetilde{Z}_r^\varepsilon - \varepsilon \frac{\partial \widehat{e}}{\partial y}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) Z_r^\varepsilon \mid^2 dr \\
&= \mid g(X_t^{1,\varepsilon}) - \varepsilon \widehat{e}(X_t^{1,\varepsilon}, X_t^{2,\varepsilon}, Y_t^\varepsilon) \mid^2 + 2 \int_s^t \widehat{Y}_r^\varepsilon \{f - \frac{\partial \widehat{e}}{\partial y} e - \varepsilon \frac{\partial \widehat{e}}{\partial y} f + \varepsilon \frac{\partial \widehat{e}}{\partial x_1} F \\
&+ \frac{\partial \widehat{e}}{\partial x_1} G + \frac{\partial^2 \widehat{e}}{\partial x_2 \partial y} a \nabla_{x_2} \widehat{e}^* + \varepsilon \frac{\partial^2 \widehat{e}}{\partial x_1 \partial y} K \sigma^* \nabla_{x_2} \widehat{e}^* + \varepsilon \frac{\partial^2 \widehat{e}}{\partial x_2 \partial y} \sigma K^* \frac{\partial \widehat{e}}{\partial x_1} \\
&+ \varepsilon^2 \frac{\partial^2 \widehat{e}}{\partial x_1 \partial y} K K^* \frac{\partial \widehat{e}}{\partial x_1}\}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) dr - 2 \int_s^t \widehat{Y}_r^\varepsilon \widetilde{Z}_r^\varepsilon d\widetilde{B}_r^\varepsilon \\
&+ 2\varepsilon \int_s^t \widehat{Y}_r^\varepsilon \frac{\partial \widehat{e}}{\partial y}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) Z_r^\varepsilon (d\widetilde{B}_r^\varepsilon + (\frac{\partial^2 \widehat{e}}{\partial x_2 \partial y}\sigma + \varepsilon \frac{\partial^2 \widehat{e}}{\partial x_1 \partial y} K)(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) dr \\
&+ \frac{\varepsilon}{2} \int_s^t \widehat{Y}_r^\varepsilon \frac{\partial^2 \widehat{e}}{\partial y^2}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) \mid Z_r^\varepsilon \mid^2 dr + \varepsilon \int_s^t \widehat{Y}_r^\varepsilon \frac{\partial^2 \widehat{e}}{\partial x_1^2} K K^* (X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) dr \\
&+ \int_s^t \widehat{Y}_r^\varepsilon \frac{\partial^2 \widehat{e}}{\partial x_1 \partial x_2} \sigma K^* (X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) dr.
\end{aligned}
$$

From assumptions (5.7), (5.13), (5.20) and the fact that $1 - \varepsilon \dfrac{\partial \widehat{e}}{\partial y} \ge \frac{1}{2}$ for ε small enough and standard inequality, we deduce that

$$
\widetilde{E} \mid Y_s^\varepsilon \mid^2 + \frac{1}{2} \int_s^t \mid \widetilde{Z}_r^\varepsilon \mid^2 dr \le C(1 + \widetilde{E} \int_s^t \mid Y_r^\varepsilon \mid^2 dr).
$$

Hence from Burkholder-Davis-Gundy inequality

$$\sup_{\varepsilon>0} \widetilde{E}(\sup_{0\le s\le t} \mid Y_s^\varepsilon \mid^2 + \int_0^t \mid \widetilde{Z}_r^\varepsilon \mid^2 dr) < +\infty.$$

Step3. Tightness
We write our BSDE in the form

$$Y_s^\varepsilon = g(X_t^{1,\varepsilon}) + V_t^\varepsilon - V_s^\varepsilon + M_t^\varepsilon - M_s^\varepsilon + N_t^\varepsilon - N_s^\varepsilon$$

where

$$\begin{aligned}
V_s^\varepsilon &= \int_0^s \{f - \frac{\partial \widehat{e}}{\partial y} e - \frac{\partial \widehat{e}}{\partial x_1} G + \frac{\partial^2 \widehat{e}}{\partial x_2 \partial y} a \nabla_{x_2} \widehat{e}^* + \frac{\partial^2 \widehat{e}}{\partial x_1 \partial x_2} K\sigma^*\}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) dr \\
M_s^\varepsilon &= -\int_0^s \widetilde{Z}_r^\varepsilon d\widetilde{B}_r^\varepsilon \\
N_s^\varepsilon &= \varepsilon(\widehat{e}(X_t^{1,\varepsilon}, X_t^{2,\varepsilon}, Y_t^\varepsilon) - \widehat{e}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)) \\
&+ \int_s^t \{-\varepsilon \frac{\partial \widehat{e}}{\partial y} f + \varepsilon \frac{\partial \widehat{e}}{\partial x_1} F \\
&+ \varepsilon \frac{\partial^2 \widehat{e}}{\partial x_1 \partial y} K\sigma^* \nabla_{x_2} \widehat{e}^* + \varepsilon \frac{\partial^2 \widehat{e}}{\partial x_2 \partial y} \sigma K^* \frac{\partial \widehat{e}}{\partial x_1} + \varepsilon^2 \frac{\partial^2 \widehat{e}}{\partial x_1 \partial y} KK^* \frac{\partial \widehat{e}}{\partial x_1}\}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) dr \\
&+ \varepsilon \int_s^t \frac{\partial \widehat{e}}{\partial y}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) Z_r^\varepsilon (d\widetilde{B}_r^\varepsilon + (\frac{\partial^2 \widehat{e}}{\partial x_2 \partial y} \sigma + \varepsilon \frac{\partial^2 \widehat{e}}{\partial x_1 \partial y} K)(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) dr) \\
&+ \frac{\varepsilon}{2} \int_s^t \frac{\partial^2 \widehat{e}}{\partial y^2}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) \mid Z_r^\varepsilon \mid^2 dr + \varepsilon \int_s^t \frac{\partial^2 \widehat{e}}{\partial x_1^2} KK^*(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) dr
\end{aligned}$$

We can check that

$$\widetilde{E}(\sup_{0\le s\le t} \mid N_s^\varepsilon \mid) \longrightarrow 0,$$

hence $\sup_{0\le s\le t} \mid N_s^\varepsilon \mid \longrightarrow 0$ tend to zero in $\widetilde{P}$ probability, or equivalently in law.
Clearly (see Theorem 4.1 of Chapter 4)

$$CV_t(V^\varepsilon) \le \widetilde{E}(\int_0^t \mid f(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) \mid dr),$$

and it follows from step2 that

$$\sup_\varepsilon (CV_t(V^\varepsilon) + \sup_{0\le s\le t} \widetilde{E} \mid Y_s^\varepsilon \mid + \sup_{0\le s\le t} \widetilde{E} \mid \int_0^s \widetilde{Z}_s^\varepsilon dB_r^\varepsilon \mid < +\infty,$$

hence the sequence $\{(Y_s^\varepsilon, \int_0^s \widetilde{Z}_s^\varepsilon dB_r^\varepsilon); 0 \le s \le t\}$ satisfies Meyer-Zheng's tightness criterion under $\widetilde{\mathbb{P}}$.

Step4. Passage to the limit
After extraction of a suitable subsequence, which we still denote $(X^\varepsilon, Y^\varepsilon, M^\varepsilon)$, we have

$$(X^\varepsilon, Y^\varepsilon, M^\varepsilon) \Longrightarrow (X, \overline{Y}, \overline{M})$$

weakly on $C([0,t], \mathbb{R}^d) \times (D([0,t]))^2$ equipped with the same topology as above.
Let us admits for a moment the following

Lemma 5.4. *Let $h : \mathbb{R}\times\mathbb{R}^l\times\mathbb{R} \longrightarrow \mathbb{R}$ be measurable, periodic of period one in each direction with respect to it's second argument, continuous with respect to the first (resp. third) argument uniformly with respect to the second and the third one (resp. first and second). Then*

$$\sup_{0\le s\le t} \mid \int_0^s h(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^{\varepsilon})dr - \int_0^s \overline{h}(X_r^1, Y_r)dr \mid \longrightarrow 0$$

in $\widetilde{\mathbb{P}}$ probability as $\varepsilon \longrightarrow 0$, where

$$\overline{h}(x_1, y) = \int_{\mathbb{T}^l} h(x_1, x_2, y)\mu(dx_2).$$

We can now passe to the limit in (5.18) and (5.19), we obtain that

$$X_s^1 = x_0^1 + \int_0^s A(X_r^1, \overline{Y}_r)dr + M_r^X$$

$$\overline{Y}_s = g(X_t^1) + \int_s^t C(X_r^1, \overline{Y}_r)dr + \overline{M}_s - \overline{M}_t,$$

where A and C are defined in Section 5.2, $\{\overline{M}_s, 0 \le s \le t\}$ is a martingale and M_r^X is the martingale part of X^1.

Step5. Identification of the limit

We need to show that $\overline{Y}_t = Y_t$, $\overline{M}_t = -\int_0^t Z_s dB_s$. Using the same argument as in Pardoux, Veretennikov [72] (see also Pardoux [65]), one can prove that there exists a $\mathcal{F}_t^{X^1,\overline{Y}}$−Brownian motion $\{B_t\}$ such that

$$X_s^1 = x_0^1 + \int_0^s A(X_r^1, \overline{Y}_r)dr + \int_0^s \lambda(X_r^1)dB_r,$$

where $\lambda\lambda^* = D$.

Let (Y, Z) the unique solution of the BSDE

$$Y_s = g(X_t^1) + \int_s^t C(X_r^1, Y_r)dr - \int_s^t Z_r dB_r.$$

Since $(\overline{Y}, \overline{M})$ is $\mathcal{F}_t^{X^1,\overline{Y}}$-adapted, $\overline{M}$ is a square integrable $\mathcal{F}_t^{X^1,\overline{Y}}$ martingale, and

$$\overline{Y}_s = g(X_t^1) + \int_s^t C(X_r^1, \overline{Y}_r)dr + \overline{M}_s - \overline{M}_t,$$

if we denote $M_t := -\int_0^t Z_r dB_r$, we deduce from Itô's formula that

$$\begin{aligned}\mid Y_s - \overline{Y}_s \mid^2 + [M - \overline{M}]_t - [M - \overline{M}]_s = 2\int_s^t (C(X_r^1, Y_r) - C(X_r^1, \overline{Y}_r), Y_r - \overline{Y}_r)dr \\ + 2\int_s^t < Y_r - \overline{Y}_r, dM_r - d\overline{M}_r > .\end{aligned}$$

Taking the expectation and using Gronwall's lemma, one obtain that $\overline{Y} = Y$ and $M = \overline{M}$. ■

In order to prove Lemma 5.4, we need the following lemmas

Lemma 5.5. *For any $\delta > 0$, there exist $N \in \mathbb{N}$ and $\mathbb{R}$-valued step functions $x^1, ..., x^N$ such that*

$$\mathbb{P}(\bigcap_{l=1}^{N}\{\sup_{0\leq s\leq t} \mid X_s^{1,\varepsilon} - x_s^l \mid > \delta\}) < \delta, \forall\varepsilon > 0.$$

Proof . The result follows from the tightness of the sequence $X^{1,\varepsilon}$ and the separability of $C[0,t],\mathbb{R}^d)$ and the fact that for any continuous function we can associate a step function which is arbitrarily close the former in sup norm (see Pardoux, Veretennikov [72]). ■

Lemma 5.6. *(see Pardoux, Veretennikov [72]) For any $\delta > 0$, there exist $N_2 \in \mathbb{N}$ and $y^1, ..., y^{N_2} \in D([0,t],\mathbb{R}^l)$ such that*

$$\mathbb{P}(\bigcap_{k=1}^{N_2}\{\lambda\{0 \leq s \leq t; \mid Y_s^{1,\varepsilon} - y_s^k > \delta\} \mid > \delta\}) < \delta, \forall\varepsilon > 0,$$

where λ denotes the Lebesgue measure.

Now, let us prove Lemma 5.4.

Proof of Lemma 5.4. We can assume that h vanishes outside $[-K,K] \times \mathbb{R}^d \times [-M,M]$, for some $M > 0$, and $K > 0$. We put $\widetilde{h} = h(x_1,x_2,y) - \overline{h}(x_1,y)$, we want to prove that

$$\int_0^s \widetilde{h}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^{\varepsilon})dr \longrightarrow 0, \text{ in } \widetilde{\mathbb{P}} \text{ probability}.$$

From Lemma 5.5 and Lemma 5.6, $\forall\delta > 0$, $\exists N_2, y^1, ..., y^{N_2}, N, x^1, ..., x^N$ step functions such that if $A_\varepsilon := \bigcap_{k=1}^{N_2}\{\lambda\{0 \leq s \leq t; \mid Y_s^{1,\varepsilon} - y_s^k \mid > \delta\} > \delta\}$, $\widetilde{\mathbb{P}}(A_\varepsilon) < \delta$, and if $A_1 = \bigcap_{l=1}^{N}\{\sup_{0\leq s\leq t} \mid X_s^{1,\varepsilon} - x_s^l \mid > \delta\}$, $\widetilde{\mathbb{P}}(A_1) < \delta$. Now
$A_\varepsilon^c = \bigcup_{k=1}^{N} B_k^\varepsilon$ where $B_k^\varepsilon = \{\lambda\{0 \leq s \leq t; \mid Y_s^{1,\varepsilon} - y_s^k \mid > \delta\} \leq \delta\}$,
and
$A_1^c = \bigcup_{l=1}^{N}\{\sup_{0\leq s\leq t} \mid X_s^{1,\varepsilon} - x_s^l \mid \leq \delta\} = \bigcup_{k=1}^{N} B_1^k$,
we can rewrite $A_1^c = \bigcup_{k=1}^{N} \overline{B}_1^k$ where $\overline{B}_1^k \subset B_1^k \forall k$, and the $\overline{B}_1^k$ are disjoint.
Then we have

$$\begin{aligned}
&\mid \int_0^s \widetilde{h}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^{\varepsilon})dr \mid \\
&\leq \mid \int_0^s \widetilde{h}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^{\varepsilon})dr \mid 1_{A_\varepsilon} + \sum_{k=1}^{N_2} \mid \int_0^s \widetilde{h}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^{\varepsilon})dr \mid 1_{B_k^\varepsilon} \\
&\leq sC_{M,K}1_{A_\varepsilon} + \sum_{k=1}^{N_2} \mid \int_0^s \widetilde{h}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^{\varepsilon})dr \mid 1_{B_k^\varepsilon}.
\end{aligned}$$

Indeed

$$\sum_{k=1}^{N_2} \mid \int_0^s \mid \widetilde{h}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr \mid 1_{B_k^\varepsilon}$$
$$\leq \sum_{l=1}^{N}\sum_{k=1}^{N_2} \mid \int_0^s \widetilde{h}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr - \int_s^t \widetilde{h}(x_r^l, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr \mid 1_{B_k^\varepsilon \bigcap \overline{B}_1^l}$$
$$+ \sum_{l=1}^{N}\sum_{k=1}^{N_2} \mid \int_0^s \widetilde{h}(X_r^l, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr - \int_s^t \widetilde{h}(x_r^l, X_r^{2,\varepsilon}, y_r^k)dr \mid 1_{B_k^\varepsilon \bigcap \overline{B}_1^l}$$
$$+ \sum_{l=1}^{N}\sum_{k=1}^{N_2} \mid \int_0^s \widetilde{h}(x_r^l, X_r^{2,\varepsilon}, y_r^k)dr \mid 1_{B_k^\varepsilon \bigcap \overline{B}_1^l} + \sum_{k=1}^{N} \mid \int_0^s \widetilde{h}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon) \mid 1_{B_k^\varepsilon \bigcap A_1}$$
$$\leq sW_K(\delta) + W_M(\delta)s + 2\delta C_{K,M} + \sum_{l=1}^{N}\sum_{k=1}^{N_2} \int_0^s \mid \widetilde{h}(x_r^1, X_r^{2,\varepsilon}, y_r^k)dr \mid + C_{K,M} 1_{\overline{B}_k^\varepsilon \bigcap B_1^l}.$$

where $C_{M,K} = \sup_{[-K,K]\times\mathbb{R}^d\times[-M,M]} \mid \widetilde{h}(x_1, x_2, y) \mid$ and W_K (resp W_M) is the modulus for continuity of $\widetilde{h}$ in it's first (resp third) argument.
We have that

$$\mid \int_0^s \widetilde{h}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr \mid \leq C_{M,K} 1_{A_\varepsilon} + W_M(\delta)s + 2\delta C_{K,M} + sW_K(\delta) + sC_{M,K} 1_{B_k^\varepsilon \bigcap B_1^l}$$
$$+ \sum_{k=1}^{N} \mid \int_0^s \widetilde{h}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr \mid .$$

Now, ergodicity implies that for each k, l

$$\int_0^s \mid \widetilde{h}(x_r^l, X_r^{2,\varepsilon}, y_r^k)dr \longrightarrow 0,$$

in $\mathbb{P}$-probability, also in $\widetilde{\mathbb{P}}$-probability. Finally, for any $\delta > 0$ such that

$$C_{M,K} 1_{A_\varepsilon} + W_M(\delta)s + 2\delta C_{K,M} + sW_K(\delta) + sC_{K,M} 1_{B_k^\varepsilon \bigcap B_1^l} < \frac{\delta}{2}.$$

Then

$$\widetilde{\mathbb{P}}(\mid \int_0^s \widetilde{h}(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr \mid > \delta) \leq \sum_{l=1}^{N_2}\sum_{k=1}^{N} \widetilde{\mathbb{P}}(\mid \int_0^s \widetilde{h}(x_r^l, X_r^{2,\varepsilon}, y_r^k)dr \mid > \frac{\delta}{2NN_2}),$$

and this tend to zero as $\varepsilon \longrightarrow 0$. ■

5.3 Homogenization of parabolic PDE with periodic coefficients

Now, let us apply Theorem 5.3 to the averaging of the parabolic semi-linear PDE.

Theorem 5.7. *Under the assumptions of Theorem 5.3 , $u^\varepsilon(t, x_1, x_2)$ solution of (5.10) converges to $u(t, x_1)$ solution of (5.15) for all $(s, x_1, x_2) \in [0, t] \times \mathbb{R} \times \mathbb{R}^l$ as ε goes to zero.*

Proof . Let $x = (x_1, x_2) \in \mathbb{R}\times\mathbb{R}^l$ and $\{X_s^{x,\varepsilon} = (X_s^{1,x,\varepsilon}, X_s^{2,x,\varepsilon}); 0 \le s \le t\}$ denote the solution of the SDE (5.1), starting at x. For all $t \in \mathbb{R}^+$, we denote by $(\{Y_s^{t,x,\varepsilon}, Z_s^{t,x,\varepsilon}\}; 0 \le s \le t)$ the solution of the BSDE

$$Y_s^{t,x,\varepsilon} = g(X_t^{1,x,\varepsilon}) + \frac{1}{\varepsilon}\int_s^t e(X_r^{1,x,\varepsilon}, X_r^{2,x,\varepsilon}, Y_r^{t,x,\varepsilon}) + \int_s^t f(X_r^{1,x,\varepsilon}, X_r^{2,x,\varepsilon}, Y_r^{t,x,\varepsilon})dr - \int_s^t Z_r^{t,x,\varepsilon} dB_r^{\varepsilon}.$$

By virtue of Pardoux, Peng [67] (see also [65]), the function $u^\varepsilon : \mathbb{R}^+ \times \mathbb{R}^{1+l} \longrightarrow \mathbb{R}$ defined by $u^\varepsilon(t,x) = Y_0^{t,x}$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^{1+l}$, is the unique viscosity solution of the systems of PDE (5.10). Similarly, Let $\{X_s^{1,x_1}; s \ge 0\}$ denote the solution of the SDE associated to the limiting process X^1 starting at $x_1 \in \mathbb{R}$ and $(\{Y_s^{t,x_1}, Z_s^{t,x_1}\}; 0 \le s \le t)$ be the unique solution to the BSDE

$$Y_s^{t,x_1} = g(X_t^{1,x_1}) + \int_s^t C(X_r^{1,x_1}, Y_r^{t,x_1})dr - \int_s^t Z_r^{t,x_1} dB_r.$$

Again, in view of [67] the function $u : [0,t] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by $u(t,x_1) = Y_0^{t,x_1}$ for $(t,x_1) \in \mathbb{R}_+ \times \mathbb{R}$, is the unique viscosity solution of the PDE (5.15). Therefore, the result follows from Theorem 5.3. ■

5.4 Application to the nonlinear Cauchy problem

Let $(Y^\varepsilon, Z^\varepsilon)$ be the solution of the BSDE

$$\begin{aligned} Y_s^\varepsilon &= g(X_t^{1,\varepsilon}) + \tfrac{1}{\varepsilon}\int_s^t e(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr + \int_s^t f(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr - \int_s^t Z_r^\varepsilon dB_r^\varepsilon \\ &+ \int_{\mathbb{R}} (L_t^a(Y^\varepsilon) - L_s^a(Y^\varepsilon))\nu(da), \end{aligned} \tag{5.21}$$

where $L_t^a(Y^\varepsilon)$ stands for the local time of the semimartingale Y^ε at level a. Such equations are introduced by Dermoune et *al.* [24].
In this section, we consider only the case of absolutely continuous measure ν, i.e. $\nu(da) = h(a)da$, $\int_{\mathbb{R}} h(x)dx = 1, h \ge 0$.
From the equality $d\langle Y^\varepsilon, Y^\varepsilon\rangle_t = \mid Z_t^\varepsilon \mid^2 dt$ and occupation time formula, we have

$$\int_0^t h(Y_s^\varepsilon) \mid Z_t^\varepsilon \mid^2 ds = \int_{\mathbb{R}} L_t^a(Y^\varepsilon) h(a)da.$$

The equation (5.21), becomes

$$\begin{aligned} Y_s^\varepsilon &= g(X_t^{1,\varepsilon}) + \tfrac{1}{\varepsilon}\int_s^t e(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr + \int_s^t f(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, Y_r^\varepsilon)dr - \int_s^t Z_r^\varepsilon dB_r^\varepsilon \\ &+ \int_s^t h(Y_r^\varepsilon) \mid Z_r^\varepsilon \mid^2 dr. \end{aligned} \tag{5.22}$$

Now let (Y, Z) be the unique solution of the BSDE

$$Y_s = g(X_t^1) + \int_s^t C(X_r^1, Y_r)dr - \int_s^t Z_r dB_r + \int_s^t h(Y_r) \mid Z_r \mid^2 dr.$$

We assume that

$$h \text{ and its first derivative are bounded.} \tag{5.23}$$

Theorem 5.8. *Under the conditions of Theorem 5.3 and assumption (5.23), there exists a d-dimensional Brownian motion B such that the family of processes* $(Y^\varepsilon, M^\varepsilon)$ *converges in law to* $(Y, M := -\int_0^s Z_s dB_s)$ *on* $(D([0,t],\mathbb{R}))^2$ *equipped with the same topology as above. Moreover* $Y_0^\varepsilon \longrightarrow Y_0$ *in* $\mathbb{R}$.

Proof . We put

$$(\widehat{Y}^\varepsilon, \widehat{Z}^\varepsilon) = (F(Y^\varepsilon), Z^\varepsilon k(Y^\varepsilon)),$$

where $k(x) = \exp(2\int_{-\infty}^x h(y)dy)$ and $F(y) = \int_0^y k(x)dx$.
By virtue of Dermoune et *al.* [24], $(\widehat{Y}^\varepsilon, \widehat{Z}^\varepsilon)$ is the unique solution of the BSDE

$$\begin{aligned}\widehat{Y}_s^\varepsilon &= \widehat{g}(X_t^{1,\varepsilon}) + \frac{1}{\varepsilon}\int_s^t e(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, F^{-1}(\widehat{Y}_r^\varepsilon))R(\widehat{Y}_r^\varepsilon)dr \\ &+ \int_s^t f(X_r^{1,\varepsilon}, X_r^{2,\varepsilon}, F^{-1}(\widehat{Y}_r^\varepsilon))R(\widehat{Y}_r^\varepsilon)dr - \int_s^t \widehat{Z}_r^\varepsilon dB_r^\varepsilon,\end{aligned}$$

with $\widehat{g}(x) = g(F^{-1}(x))$ and $R(y) = k(F^{-1}(y))$. By condition (5.12), we have

$$\int_{\mathbb{T}^l} e(x_1, x_2, F^{-1}(y))R(y)\mu(dx_2) = 0,$$

then the Poisson equation $L_2\widehat{N}(x_1,x_2,y) + N(x_1,x_2,y) = 0$ has one solution given by

$$\widehat{N}(x_1,x_2,y) = \int_0^{+\infty} \mathbb{E}_{x_2} N(x_1, X_t^{2,1}, y)dt,$$

where $N(x_1,x_2,y) = e(x_1,x_2,F^{-1}(y))R(y)$. Thanks to assumption (5.23), N satisfy the same conditions as e in Section 5.2. Hence, by Theorem 5.3 $(\widehat{Y}_r^\varepsilon, \widehat{Z}_r^\varepsilon)$ converges in law to $(\widehat{Y}, \widehat{Z})$ on $(D([0,t]))^2$, where $(\widehat{Y}, \widehat{Z})$ is the unique solution of the BSDE

$$\widehat{Y}_s = \widehat{g}(X_t^1) + \int_s^t \overline{C}(X_r^1, \widehat{Y}_r)dr - \int_s^t \widehat{Z}_r dB_r,$$

where $\overline{C}(x,y)$ is the same as $C(x,y)$ in Section 5.2, but we replace $\widehat{e}$ by $\widehat{N}$ and e by N. Now, again by Dermoune et *al.* [24] and the continuity of F^{-1}

$$(Y^\varepsilon = F^{-1}(\widehat{Y}^\varepsilon), Z^\varepsilon = \frac{\widehat{Z}^\varepsilon}{k(\widehat{Y}^\varepsilon)}) \Longrightarrow (F^{-1}(\widehat{Y}) = Y, \frac{\widehat{Z}}{k(\widehat{Y})} = Z)$$

where (Y,Z) satisfies the following BSDE

$$Y_s = g(X_t^1) + \int_s^t C(X_1^r, Y_r)dr - \int_s^t Z_r dB_r + \int_s^t h(Y_r) \mid Z_r \mid^2 dr.$$

■

Now, let us apply Theorem 5.8 to the homogenization of nonlinear Cauchy problem. Consider the following PDE: $u \in C^{1,2}(]0,t[\times\mathbb{R})$

$$\begin{cases} \dfrac{\partial u}{\partial s}(s,x_1) = \frac{1}{2}D(x_1)\dfrac{\partial^2 u}{\partial x_1^2}(s,x_1) + A(x_1, u(s,x_1,x_2))\dfrac{\partial u}{\partial x_1}(s,x_1) \\ +C(x_1,,u(s,x_1)) + \lambda^2(x_1)h(u(s,x_1))(\dfrac{\partial u}{\partial x_1}(s,x_1))^2 \;\forall s\in[0,t], \quad x_1\in\mathbb{R} \\ u(0,x_1) = g(x_1), \quad x_1 \in \mathbb{R} \\ \mid u(s,x_1) \mid \leq C(1+\mid x_1 \mid^{2\delta}), x_1\in\mathbb{R}, \quad for \quad some \quad C>0, \delta\geq 1. \end{cases} \qquad (6.4)$$

Theorem 5.9. *Under the assumptions of Theorem 5.8, if* $u^\varepsilon(t,x_1,x_2)$ *is the unique solution of (5.11) and* $u(t,x_1)$ *solution of (6.4), then* $u^\varepsilon(t,x_1,x_2)$ *converges towards* $u(t,x_1)$ *as* ε *goes to zero for all* $(x_1,x_2)\in\mathbb{R}\times\mathbb{R}$.

Proof . Let $x=(x_1,x_2)\in\mathbb{R}\times\mathbb{R}$ and $\{X_s^{x,\varepsilon}=(X_s^{1,x,\varepsilon},X_s^{2,x,\varepsilon});0\le s\le t\}$ denote the solution of the SDE (5.1), starting at x. For all $t\in\mathbb{R}^+$, we denote by $(\{Y_s^{t,x,\varepsilon},Z_s^{t,x,\varepsilon}\};0\le s\le t)$ the solution of the BSDE

$$
\begin{aligned}
Y_s^{t,x,\varepsilon}=g(X_t^{1,x,\varepsilon})+\frac{1}{\varepsilon}\int_s^t e(X_r^{1,x,\varepsilon},X_r^{2,x,\varepsilon},Y_r^{t,x,\varepsilon})+\int_s^t f(X_r^{1,x,\varepsilon},X_r^{2,x,\varepsilon},Y_r^{t,x,\varepsilon})dr\\
+\int_s^t h(Y_r^{t,x,\varepsilon})\mid Z_r^{t,x,\varepsilon}\mid^2 dr-\int_s^t Z_r^{t,x,\varepsilon}dB_r^\varepsilon.
\end{aligned}
$$

By virtue of Dermoune et *al.* [24], the function $u^\varepsilon:\mathbb{R}^+\times\mathbb{R}\longrightarrow\mathbb{R}$ defined by $u^\varepsilon(t,x)=Y_0^{t,x}$, $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, is the solution of the systems of PDEs (5.11). Let $\{X_s^{1,x_1};s\ge 0\}$ denote the solution of the SDE associated to the limiting process X^1 starting at $x_1\in\mathbb{R}$ and $(\{Y_s^{t,x_1},Z_s^{t,x_1}\};0\le s\le t)$ be the unique solution to the BSDE

$$
Y_s^{t,x_1}=g(X_t^{1,x_1})+\int_s^t C(X_r^{1,x_1},Y_r^{t,x_1})dr+\int_s^t h(Y_r^{t,x_1})\mid Z_r^{t,x_1}\mid^2 dr-\int_s^t Z_r^{t,x_1}dB_r.
$$

Again, in view of [24] the function $u:[0,t]\times\mathbb{R}\longrightarrow\mathbb{R}$ defined by $u(t,x_1)=Y_0^{t,x_1}$ for $(t,x_1)\in\mathbb{R}_+\times\mathbb{R}$, is the solution of the PDE (6.4). Theorem 5.8 implies that $u^\varepsilon(t,x_1,x_2)$ converges to $u(t,x_1)$ as ε goes to 0. ■

Chapter 6

Weak Convergence of Reflected BSDE's and Homogenization of Semi-linear Variational Inequalities

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The chapter is organized as follows. In Section 6.2, we give our standing assumptions and some notations to be used in the sequel. Section 6.3 is devoted to the proof of weak convergence of reflected BSDE. In Section 6.4, we apply our result to the homogenization of a class of semi-linear variational inequalities with periodic coefficients.

6.1 Introduction

In this chapter, we shall study the stability properties of BSDE's and their applications to the homogenization of systems of semi-linear variational inequalities involving a second order differential operator of parabolic type with periodic coefficients and highly oscillating term. The approach is based upon the nonlinear Feynman-Kac formula, which gives the probabilistic interpretation of the solutions of systems of semi-linear parabolic PDE's, and the weak convergence of an associated reflected backward stochastic differential equation in the sense of Meyer and Zheng topology [56].

6.2 Preliminaries

Consider the following stochastic differential equation (SDE): for $\varepsilon > 0$, $x \in R^d$,

$$X_s^\varepsilon = x + \int_0^s \frac{1}{\varepsilon} b(\frac{X_r^\varepsilon}{\varepsilon})dr + \int_0^s c(\frac{X_r^\varepsilon}{\varepsilon})dr + \int_0^s \sigma(\frac{X_r^\varepsilon}{\varepsilon})dB_r, \tag{6.1}$$

where $\{B_t;\ t \geq 0\}$ is a d-dimensional Brownian motion , $c : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ are measurable, bounded and periodic of period one in each direction. We assume that $a(x) := \sigma\sigma^*(x)$ is continuous and satisfies

$$a(x) \geq \alpha I > 0, \quad \forall x \in R^d, \tag{6.2}$$

[0]This work is published in Bull. Sci. math. 126, 413-431, (2002).

$$\sum_{i=1}^{d} \frac{\partial a_{ij}}{\partial x_i} \in \mathbb{L}^\infty(\mathbb{R}^d), \quad j = 1, ..., d. \tag{6.3}$$

Note that the above conditions insure that the SDE (6.1) has a unique solution $\{X_t^\varepsilon;\ t \geq 0\}$ (see Stroock-Varadhan [76]).

For all $x \in \mathbb{R}^d$, $\varepsilon > 0$, $t > 0$, let $\{(Y_s^\varepsilon, Z_s^\varepsilon, U_s^\varepsilon);\ 0 \leq s \leq t\}$ be the $\mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathbb{R}^k$-valued progressively measurable process solution of the BSDE's, depending on parameters $\{\varepsilon > 0\}$:

$$\begin{cases} Y_s^\varepsilon = g(X_t^\varepsilon) + \dfrac{1}{\varepsilon}\displaystyle\int_s^t e(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon)dr + \int_s^t f(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon)dr - \int_s^t Z_r^\varepsilon dB_r - \int_s^t U_r^\varepsilon dr \\ \\ (Y^\varepsilon, U^\varepsilon) \in Gr(\partial\phi)\ \ d\mathbb{P} \times dt \text{ on } \Omega \times [0,t], \end{cases} \tag{6.4}$$

where g is continuous with polynomial growth at infinity and takes values in a bounded, open and convex Θ of $\mathbb{R}^k$ and

$$\phi(x) := \partial I_{\overline{\Theta}}(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbf{cl}(\Theta) \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathbf{cl}(\Theta)$ denotes the closure of Θ and

$$\partial\phi(u) = \{u^* \in \mathbb{R}^k : \langle u^*, v - u\rangle + \phi(u) \leq \phi(v), \forall v \in \mathbb{R}^k\}$$

$$Dom(\partial(\phi)) = \{u \in \mathbb{R}^k : \partial\phi(u) \neq \emptyset\}$$

$$Gr(\partial\phi) = \{(u, u^*) \in \mathbb{R}^k \times \mathbb{R}^k : u \in Dom(\partial(\phi)) \quad \text{and} \quad u^* \in \partial\phi(u)\}.$$

We can verify that ϕ is convex, lower semi-continuous and proper with $Dom(\phi) = \mathbf{cl}(\Theta)$ and

$$\partial\phi(x) = \left\{\ y \in \mathbb{R}^k; \langle y, x - z\rangle \geq 0, \forall z \in \mathbf{cl}(\Theta),\ \text{for } x \in \mathbf{cl}(\Theta)\ \right\}.$$

We put $\mathbb{T}^d = \frac{\mathbb{R}^d}{\mathbb{Z}^d}$ and assume that:

b satisfies the centering condition

$$\int_{\mathbb{T}^d} b_i(x)\mu(dx) = 0 \quad i = 1, ..., d, \tag{6.5}$$

and a satisfies conditions (6.2) and (6.3).

e and f are measurable and $\mathbb{R}^d \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$-valued periodic functions with period one in each direction and continuous in y, uniformly with respect to x and for all $y \in \mathbb{R}^k$

$$\int_{\mathbb{T}^d} e(x, y)\mu(dx) = 0, \tag{6.6}$$

and e is twice continuously differentiable in y, uniformly with respect to x.

Moreover, for some $\mu \in \mathbb{R}$, $\forall x \in \mathbb{R}^d$, $y, y' \in \mathbb{R}^k$,

$$\langle f(x, y) - f(x, y'), y - y'\rangle \leq \mu \mid y - y' \mid^2 . \tag{6.7}$$

In addition, we assume that there exists a constant $K' > 0$ such that

$$| e(x,y) | + | \frac{\partial e}{\partial y}(x,y) | + | \frac{\partial^2 e}{\partial y^2}(x,y) | \leq K', \forall x \in \mathbb{T}^d, y \in \mathbb{R}^k, \tag{6.8}$$

and

$$| f(x,y) | \leq K, \tag{6.9}$$

and that

$$0 \in \Theta. \tag{6.10}$$

Let $\{\widetilde{X}_t; t \geq 0\}$ be a $\mathbb{T}^d$-valued process with generator

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

and $\mu(dx) = p(x)dx$ its invariant probability (see Section 2 in Pardoux [65]).
From (6.5) and (6.6), we deduce that for all $y \in \mathbb{R}^k$, the Poisson equations

$$\begin{cases} L\widehat{b}_i(x) + b_i(x) = 0, \quad i = 1, ..., d \\ L\widehat{e}(x,y) + e(x,y) = 0, \quad x \in \mathbb{T}^d, \quad y \in \mathbb{R}^k, \end{cases}$$

have respectively a solution given by

$$\widehat{b}_i(x) = \int_0^\infty \mathbb{E}_x b(\widetilde{X}_t)dt, \quad x \in \mathbb{T}^d, \quad i = 1, ..., d;$$

respectively

$$\widehat{e}(x,y) = \int_0^\infty \mathbb{E}_x e(\widetilde{X}_s, y)ds,$$

and $\widehat{e}$ satisfies: $\widehat{e} \in \mathcal{C}^{0,2}(\mathbb{T}^d \times \mathbb{R}^k)$, $\widehat{e}(.,y), \frac{\partial \widehat{e}}{\partial y}(.,y), \frac{\partial^2 \widehat{e}}{\partial y^2}(.,y) \in W^{2,p}(\mathbb{T}^d)$, $\forall p \geq 1$, $y \in \mathbb{R}^k$, and for some $K" > 0$,

$$\|\widehat{e}(.,y)\|_{W^{2,p}(\mathbb{T}^d)} + \|\frac{\partial \widehat{e}}{\partial y}(.,y)\|_{W^{2,p}(\mathbb{T}^d)} + \|\frac{\partial^2 \widehat{e}}{\partial y^2}(.,y)\|_{W^{2,p}(\mathbb{T}^d)} \leq K",$$

(see Pardoux-Veretennikov [72]). Put

$$A = \int_{\mathbb{T}^d} (I + \nabla\widehat{b})a(I + \nabla\widehat{b})^*(x)\mu(dx).$$

$$C(y) = \int_{\mathbb{T}^d} (I + \nabla\widehat{b})(c + a\frac{\partial^2 \widehat{e}^*}{\partial x \partial y}(.,y))(x)\mu(dx).$$

$$D(y) = \int_{\mathbb{T}^d} [\langle \frac{\partial \widehat{e}}{\partial x}, c\rangle(.,y) - \frac{\partial \widehat{e}}{\partial y}(.,y)e(.,y) + \frac{\partial^2 \widehat{e}}{\partial x \partial y}(.,y)a\frac{\partial \widehat{e}^*}{\partial x}(.,y) + f(.,y)](x)\mu(dx).$$

and

$$M_t^\varepsilon = -\int_0^t Z_r^\varepsilon dB_r; \qquad K_t^\varepsilon = -\int_0^t U_r^\varepsilon ds.$$

Our objective is to prove that the family of processes $(X^\varepsilon, Y^\varepsilon, M^\varepsilon, K^\varepsilon)$ converges in law to (X,Y,M,K), where

$$X_s = x + \int_0^s C(Y_r)dr + M_s^x \quad \text{with} \quad \{M_s^x; s \geq 0\} \tag{6.11}$$

is a non standard Brownian motion satisfying $<< M^x >>_s = As$, and

$$Y_s = g(X_t) + \int_s^t D(Y_r)dr + M_t - M_s + K_t - K_s.$$

From the stability of $(X^\varepsilon, Y^\varepsilon, M^\varepsilon, K^\varepsilon)$ one can deduce some homogenization results of the following semi-linear variational inequality

$$\begin{cases} \forall s \in [0,t],\ x \in \mathbb{R}^d \\ \dfrac{\partial u^\varepsilon}{\partial s}(s,x) - \frac{1}{2}\sum_{i,j=1}^d a_{ij}(\frac{x}{\varepsilon})\dfrac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j}(s,x) - \sum_{i=1}^d(\frac{1}{\varepsilon}b_i(\frac{x}{\varepsilon}) + c_i(\frac{x}{\varepsilon}))\dfrac{\partial u^\varepsilon}{\partial x_i}(s,x) \\ \qquad -(\frac{1}{\varepsilon}e(\frac{x}{\varepsilon}, u^\varepsilon(s,x)) - f(\frac{x}{\varepsilon}, u^\varepsilon(s,x))) \in \partial\phi(u^\varepsilon(s,x)) \\ \\ u^\varepsilon(0,x) = g(x), \quad u^\varepsilon(s,x) \in Dom(\phi) = \mathbf{cl}(\Theta). \end{cases} \tag{6.12}$$

6.3 The main result

The main result of this section is the following

Theorem 6.1. *Under the above conditions, the family of processes $(X^\varepsilon, Y^\varepsilon, M^\varepsilon, K^\varepsilon)$ converges in law to the family of processes (X, Y, M, K) in $C([0,t], \mathbb{R}^d) \times (D([0,t], \mathbb{R}^k))^2 \times C([0,t], \mathbb{R}^k)$. Moreover, $Y_0^\varepsilon \longrightarrow Y_0$ in $\mathbb{R}$.*

Recall that $\{X_s^\varepsilon,\ 0 \le s \le t\}$ is the solution of the following SDE

$$X_s^\varepsilon = x + \int_0^s (\frac{1}{\varepsilon}b(\frac{X_r^\varepsilon}{\varepsilon}) + c(\frac{X_r^\varepsilon}{\varepsilon}))dr + \int_0^s \sigma(\frac{X_r^\varepsilon}{\varepsilon})dB_r.$$

For each $\varepsilon > 0$, let $\{(Y_s^\varepsilon, Z_s^\varepsilon, U_s^\varepsilon),\ 0 \le s \le t\}$ be the $\mathbb{R}^k \times \mathbb{R}^{k\times d} \times \mathbb{R}^k$–valued progressively measurable process solution of the following reflected BSDE

$$\begin{aligned} Y_s^\varepsilon = g(X_t^\varepsilon) \quad &+ \frac{1}{\varepsilon}\int_s^t e(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon)dr + \int_s^t f(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon)dr - \int_s^t Z_r^\varepsilon dB_r \\ &- \int_s^t U_r^\varepsilon dr. \end{aligned} \tag{6.13}$$

The proof is detailed in several steps.

Proof .

Step 1. Transformation of the systems (6.1) and (6.13)

For every $\varepsilon > 0$, for every $s \le t$, we let $\overline{X}_s^\varepsilon = \frac{X_s^\varepsilon}{\varepsilon}$. From Itô-Krylov formula (see Krylov [45] as well as Pardoux-Veretennikov [72]) we get

$$\begin{aligned} &X_s^\varepsilon + \varepsilon(\widehat{b}(\overline{X}_s^\varepsilon) - \widehat{b}(\tfrac{x}{\varepsilon})) \\ &= x + \int_0^s (I + \nabla\widehat{b})c(\overline{X}_r^\varepsilon)dr + \int_0^s (I + \nabla\widehat{b})\sigma(\overline{X}_r^\varepsilon)dB_r. \end{aligned} \tag{6.14}$$

$$\begin{aligned} &Y_s^\varepsilon + \varepsilon(\widehat{e}(\overline{X}_t^\varepsilon, Y_t^\varepsilon) - \widehat{e}(\overline{X}_s^\varepsilon, Y_s^\varepsilon)) \\ &= g(X_t^\varepsilon) + \int_s^t (\langle\nabla_x\widehat{e}, c\rangle - \frac{\partial\widehat{e}}{\partial y}e + f - \varepsilon\frac{\partial\widehat{e}}{\partial y}f)(\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr \\ &+ \int_s^t \frac{\partial^2\widehat{e}}{\partial x\partial y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)\sigma(\overline{X}_r^\varepsilon)Z_r^\varepsilon dr \\ &+ \int_s^t (\nabla_x\widehat{e}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)\sigma(\overline{X}_r^\varepsilon) - Z_r^\varepsilon)dB_r + \varepsilon\int_s^t \frac{\partial\widehat{e}}{\partial y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon)Z_r^\varepsilon dr \\ &+ \tfrac{\varepsilon}{2}\int_s^t \frac{\partial^2\widehat{e}}{\partial^2 y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon) \mid Z_r^\varepsilon \mid^2 dr - \int_s^t (1 - \varepsilon\frac{\partial\widehat{e}}{\partial y})(\overline{X}_r^\varepsilon, Y_r^\varepsilon)U_r^\varepsilon dr. \end{aligned} \tag{6.15}$$

We put

$$\widetilde{Z}_s^\varepsilon = Z_s^\varepsilon - \nabla_x \widehat{e}(\overline{X}_s^\varepsilon, Y_s^\varepsilon)\sigma(\overline{X}_s^\varepsilon), \quad 0 \le s \le t.$$

Let us note that the difference between $\widetilde{Z}^\varepsilon$ and Z^ε is a uniformly bounded process. Thanks to (6.15), we get

$$\begin{aligned}
&Y_s^\varepsilon + \varepsilon(\widehat{e}(\overline{X}_t^\varepsilon, Y_t^\varepsilon)) - \widehat{e}(\overline{X}_s^\varepsilon, Y_s^\varepsilon)) \\
&= g(X_t^\varepsilon) + \int_s^t (f + \langle \nabla_x \widehat{e}, c\rangle - \frac{\partial \widehat{e}}{\partial y} e - \varepsilon \frac{\partial \widehat{e}}{\partial y} f + \frac{\partial^2 \widehat{e}}{\partial y \partial x} a \nabla_x \widehat{e}^*)(\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr \\
&- \int_s^t \widetilde{Z}_r^\varepsilon (dB_r - \frac{\partial^2 \widehat{e}}{\partial y \partial x}\sigma(\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr) + \varepsilon \int_s^t \frac{\partial \widehat{e}}{\partial y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon) Z_r^\varepsilon dB_r \\
&+ \frac{\varepsilon}{2}\int_s^t \frac{\partial^2 \widehat{e}}{\partial^2 y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon) \mid Z_r^\varepsilon \mid^2 dr - \int_s^t (1 - \varepsilon \frac{\partial \widehat{e}}{\partial y})(\overline{X}_r^\varepsilon, Y_r^\varepsilon) U_r^\varepsilon dr.
\end{aligned}$$

We let

$$\widetilde{B}_s = B_s - \int_0^s (\frac{\partial^2 \widehat{e}}{\partial x \partial y}\sigma)(\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr.$$

It follows from Girsanov's theorem that there exists a new probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ under which $\{\widetilde{B}_s,\ 0 \le s \le t\}$ is a Brownian motion . We obtain

$$\begin{aligned}
&X_s^\varepsilon + \varepsilon(\widehat{b}(\overline{X}_s^\varepsilon) - \widehat{b}(\tfrac{x}{\varepsilon})) \\
&= x + \int_0^s (I + \nabla \widehat{b})(c + a\frac{\partial^2 \widehat{e}^*}{\partial x \partial y})(\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr + \int_0^s (I + \nabla \widehat{b})\sigma(\overline{X}_r^\varepsilon)d\widetilde{B}_r
\end{aligned} \tag{6.16}$$

$$\begin{aligned}
&Y_s^\varepsilon + \varepsilon(\widehat{e}(\overline{X}_t^\varepsilon, Y_t^\varepsilon) - \widehat{e}(\overline{X}_s^\varepsilon, Y_s^\varepsilon)) \\
&= g(X_t^\varepsilon) + \int_s^t (\langle \nabla_x \widehat{e}, c\rangle - \frac{\partial \widehat{e}}{\partial y} e - \varepsilon \frac{\partial \widehat{e}}{\partial y} f + \frac{\partial^2 \widehat{e}}{\partial y \partial x} a \nabla_x \widehat{e}^*)(\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr \\
&- \int_s^t \widetilde{Z}_r^\varepsilon d\widetilde{B}_r + \varepsilon \int_s^t \frac{\partial \widehat{e}}{\partial y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon) Z_r^\varepsilon (d\widetilde{B}_r + (\frac{\partial^2 \widehat{e}}{\partial x \partial y}\sigma)(\overline{X}_r^\varepsilon, Y_r^\varepsilon)dr) \\
&+ \frac{\varepsilon}{2}\int_s^t \frac{\partial^2 \widehat{e}}{\partial^2 y}(\overline{X}_r^\varepsilon, Y_r^\varepsilon) \mid Z_r^\varepsilon \mid^2 dr \\
&- \int_s^t (1 - \varepsilon \frac{\partial \widehat{e}}{\partial y})(\overline{X}_r^\varepsilon, Y_r^\varepsilon) U_r^\varepsilon dr.
\end{aligned} \tag{6.17}$$

The fact that $\{X^\varepsilon;\, \varepsilon > 0\}$ is tight, as a random process of $C([0,t], \mathbb{R}^d)$ equipped with the topology of uniform convergence, is clair, since $\dfrac{\partial^2 \widehat{e}}{\partial x \partial y}$ is bounded and the Radon-Nikodym derivatives $\dfrac{\partial \widetilde{P}}{\partial P} \in L^p$, for every $p > 0$. Hence

$$\sup_\varepsilon \widetilde{\mathbb{E}} \mid X_t^\varepsilon \mid^p < \infty \quad \forall p > 0,$$

from which we deduce

$$\sup_\varepsilon \widetilde{\mathbb{E}} \mid g(X_t^\varepsilon) \mid^k < \infty \quad \forall k > 0.$$

Step 2. Estimates of the processes Y^ε and Z^ε.

We need to prove some estimates of Y^ε and Z^ε under $\widetilde{\mathbb{P}}$, to do that we go back to equation (6.13) and replace B with the new Brownian motion

$$\begin{aligned}
Y_s^\varepsilon = g(X_t^\varepsilon) + \int_s^t (\frac{1}{\varepsilon} e(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon) + f(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon) - Z_r^\varepsilon(\frac{\partial^2 \widehat{e}}{\partial y \partial x}\sigma)(\overline{X}_r^\varepsilon, Y_r^\varepsilon))dr \\
- \int_s^t Z_r^\varepsilon d\widetilde{B}_r - \int_s^t U_r^\varepsilon dr.
\end{aligned}$$

It follows from Itô's formula that

$$\begin{aligned} &e^{\nu t} \mid Y^\varepsilon_s \mid^3 + \int_s^t e^{\nu r}(3 \mid Y^\varepsilon_r \mid \times \mid Z^\varepsilon_r \mid^2 + \nu \mid Y^\varepsilon_r \mid^3) dr \\ &\quad = e^{\nu t} \mid g(X^\varepsilon_s) \mid^3 + \frac{3}{\varepsilon} \int_s^t e^{\nu r} \mid Y^\varepsilon_r \mid Y^\varepsilon_r e(\overline{X}^\varepsilon_r, Y^\varepsilon_r) dr \\ &\quad + 3 \int_s^t e^{\nu r} \mid Y^\varepsilon_r \mid Y^\varepsilon_r f(\overline{X}^\varepsilon_r, Y^\varepsilon_r) dr \\ &\quad - 3 \int_s^t e^{\nu r} \mid Y^\varepsilon_r \mid Y^\varepsilon_r Z^\varepsilon_r (\frac{\partial^2 \widehat{e}}{\partial x \partial y} \sigma)(\overline{X}^\varepsilon_r, Y^\varepsilon_r) dr \\ &\quad - 3 \int_s^t e^{\nu r} \mid Y^\varepsilon_r \mid Y^\varepsilon_r Z^\varepsilon_r d\widetilde{B}_r \\ &\quad - \int_s^t e^{\nu r} \mid Y^\varepsilon_r \mid Y^\varepsilon_r U^\varepsilon_r dr. \end{aligned}$$

The expectation of the above stochastic integral is zero (see Pardoux-Peng [67]). Moreover thanks to (6.9) we have

$$\mid Y^\varepsilon_r \mid Y^\varepsilon_r f(\overline{X}^\varepsilon_r, Y^\varepsilon_r) \leq \mid Y^\varepsilon_r \mid^3 + c.$$

Since $(Y^\varepsilon, U^\varepsilon) \in Gr(\phi)$, we obtain $\langle Y^\varepsilon, U^\varepsilon \rangle \geq 0$ and

$$-3 \mid Y^\varepsilon_r \mid \langle Y^\varepsilon_r, U^\varepsilon_r \rangle \leq 0.$$

Moreover

$$-3 Y^\varepsilon_r \mid Y^\varepsilon_r \mid Z^\varepsilon_r (\frac{\partial^2 \widehat{e}}{\partial x \partial y} \sigma)(\overline{X}^\varepsilon_r, Y^\varepsilon_r) \leq \frac{3}{2} \mid Y^\varepsilon_r \mid\mid Z^\varepsilon_r \mid^2 + \frac{3}{2} \| \frac{\partial^2 \widehat{e}}{\partial x \partial y} \sigma \|_\infty \mid Y^\varepsilon_r \mid^3 .$$

Finally, taking the expectation and using the fact that e is bounded we have

$$\widetilde{\mathbb{E}} \int_s^t e^{\nu r} \mid Y^\varepsilon_r \mid\mid Z^\varepsilon_r \mid^2 dr \leq c(1 + \frac{1}{\varepsilon} \widetilde{\mathbb{E}} \int_s^t \mid Y^\varepsilon_r \mid^2 dr),$$

which is equivalent to

$$\varepsilon \widetilde{\mathbb{E}} \int_s^t \mid Y^\varepsilon_r \mid\mid Z^\varepsilon_r \mid^2 dr \leq c(\varepsilon + \int_s^t \mid Y^\varepsilon_r \mid^2 dr). \tag{6.18}$$

Let us admits for a moment the following

Lemma 6.2. *Under assumption of Theorem 6.1, we have*

$$\sup_\varepsilon \widetilde{\mathbb{E}} \int_s^t \mid U^\varepsilon_r \mid^2 dr < \infty.$$

We go back to equation (6.17) and let $\widehat{Y}^\varepsilon_s = Y^\varepsilon_s - \varepsilon\widehat{e}(\overline{X}^\varepsilon_s, Y^\varepsilon_s)$. From Itô's formula we obtain

$$\begin{aligned}
&| \widehat{Y}^\varepsilon_s |^2 + \int_s^t | \widetilde{Z}^\varepsilon_r - \varepsilon\frac{\partial \widehat{e}}{\partial y}(\overline{X}^\varepsilon_r, Y^\varepsilon_r) Z^\varepsilon_r |^2 \, dr = | g(X^\varepsilon_t) - \varepsilon\widehat{e}(X^\varepsilon_t, Y^\varepsilon_t) |^2 \\
&\quad + 2\int_s^t \widehat{Y}^\varepsilon_r \{ (\langle \nabla_x \widehat{e}, c\rangle - \frac{\partial \widehat{e}}{\partial y} e + (1-\varepsilon\frac{\partial \widehat{e}}{\partial y}) f + \frac{\partial^2 \widehat{e}}{\partial y \partial x} a \nabla_x \widehat{e}^*)(\overline{X}^\varepsilon_r, Y^\varepsilon_r)\} dr \\
&\quad - 2\int_s^t \widehat{Y}^\varepsilon_r Z^\varepsilon_r d\widetilde{B}_r + 2\varepsilon \int_s^t \widehat{Y}^\varepsilon_r \frac{\partial \widehat{e}}{\partial y}(\overline{X}^\varepsilon_r, Y^\varepsilon_r) Z^\varepsilon_r (d\widetilde{B}_r + (\frac{\partial^2 \widehat{e}}{\partial x \partial y}\sigma)(\overline{X}^\varepsilon_r, Y^\varepsilon_r) dr) \\
&\quad + \varepsilon \int_s^t \frac{\partial^2 \widehat{e}}{\partial^2 y}(\overline{X}^\varepsilon_r, Y^\varepsilon_r) \widehat{Y}^\varepsilon_r | Z^\varepsilon_r |^2 \, dr \\
&\quad - 2\int_s^t (1-\varepsilon\frac{\partial \widehat{e}}{\partial y})(\overline{X}^\varepsilon_r, Y^\varepsilon_r) \widehat{Y}^\varepsilon_r U^\varepsilon_r dr.
\end{aligned}$$

Exploiting (6.17) and (6.18), together with the fact that $(1-\varepsilon\frac{\partial \widehat{e}}{\partial y}(\overline{X}^\varepsilon_r, Y^\varepsilon_r) \geq \frac{1}{2})$ for ε small enough, standard inequalities and Lemma 6.2, we deduce that

$$\widetilde{\mathbb{E}} | Y^\varepsilon_s |^2 + \frac{1}{2}\int_s^t | Z^\varepsilon_r |^2 \, dr \leq c(1 + \widetilde{\mathbb{E}} \int_s^t | Y^\varepsilon_r |^2 \, dr).$$

Thanks to Gronwall's lemma, we get

$$\sup_{0\leq s\leq t} \widetilde{\mathbb{E}} | Y^\varepsilon_s |^2 + \widetilde{\mathbb{E}} \int_s^t | \widetilde{Z}^\varepsilon_r |^2 \, dr \leq c.$$

From Burkholder- Davis- Gundy inequality, we have

$$\sup_\varepsilon \widetilde{\mathbb{E}}(\sup_{0\leq s\leq t} | Y^\varepsilon_s |^2 + \int_0^t | \widetilde{Z}^\varepsilon_r |^2 \, dr) < \infty. \tag{6.19}$$

Step 3. Convergence in law
We rewrite (3.5) in the following form

$$Y^\varepsilon_s = g(X^\varepsilon_t) + V^\varepsilon_t - V^\varepsilon_s + M^\varepsilon_t - M^\varepsilon_s + N^\varepsilon_t - N^\varepsilon_s + K^\varepsilon_t - K^\varepsilon_s,$$

where

$$\begin{aligned}
V^\varepsilon_s &= \int_0^s (\langle \nabla_x \widehat{e}, c\rangle - \frac{\partial \widehat{e}}{\partial y} e + f + \frac{\partial^2 \widehat{e}}{\partial x \partial y} a \nabla_x \widehat{e}^*)(\overline{X}^\varepsilon_r, Y^\varepsilon_r)) dr, \\
M^\varepsilon_s &= -\int_0^s \widetilde{Z}^\varepsilon_r d\widetilde{B}_r, \\
N^\varepsilon_s &= -\varepsilon\widehat{e}(\overline{X}^\varepsilon_r, Y^\varepsilon_r) - \varepsilon\int_0^s \frac{\partial \widehat{e}}{\partial y} f(\overline{X}^\varepsilon_r, Y^\varepsilon_r) dr + \varepsilon\int_0^s \frac{\partial \widehat{e}}{\partial y}(\overline{X}^\varepsilon_r, Y^\varepsilon_r) U^\varepsilon_r dr \\
&\quad + \varepsilon\int_0^s \frac{\partial \widehat{e}}{\partial y}(\overline{X}^\varepsilon_s, Y^\varepsilon_s) Z^\varepsilon_r (d\widetilde{B}_r + \big(\frac{\partial^2 \widehat{e}}{\partial x \partial y}\sigma\big)(\overline{X}^\varepsilon_r, Y^\varepsilon_r) dr) + \frac{\varepsilon}{2}\int_0^s \frac{\partial^2 \widehat{e}}{\partial^2 y}(\overline{X}^\varepsilon_r, Y^\varepsilon_r) | Z^\varepsilon_r |^2 \, dr, \\
K^\varepsilon_s &= -\int_0^t U^\varepsilon_r dr.
\end{aligned}$$

It is easy to see that

$$\widetilde{\mathbb{E}}(\sup_{0\leq s\leq t} | N^\varepsilon_s |) \longrightarrow 0,$$

as $\varepsilon \longrightarrow 0$, then $\sup_{0\leq s\leq t} \mid N_s^\varepsilon \mid) \longrightarrow 0$ goes to 0 in probability or in law.
By Lemma (3.1), one can find that for all $0 \leq s \leq t$

$$\widetilde{\mathbb{E}}(\mid K_s^\varepsilon - K_t^\varepsilon \mid^2) \leq C \mid s-t \mid .$$

Thanks to Aldous's criterion [1] (see also [42]), one can prove that the family of process $\{K^\varepsilon; \varepsilon > 0\}$ is tight.
In order to treat the other terms, we adopt the point of view of Meyer- Zheng topology [56] (see also Kurtz [47] or Pardoux [66]) which gives tightness in $D([0,t])$ equipped with the topology of convergence in ds measure (see Theorem 4.1 of Chapter 4).
From (3.7), V^ε and M^ε satisfy the Meyer-Zheng criterion. Therefore $(Y^\varepsilon, M^\varepsilon)$ is tight in the sense of Meyer- Zheng topology [56] - under $\widetilde{\mathbb{P}}$-, since from relation (6.16), $\{X^\varepsilon\}$ is tight "in the usual sense", then there exists a subsequence (which we still denote by $(X^\varepsilon, Y^\varepsilon, M^\varepsilon, K^\varepsilon)$) such that

$$(X^\varepsilon, Y^\varepsilon, M^\varepsilon, K^\varepsilon) \Longrightarrow \ (X, \overline{Y}, \overline{M}, \overline{K}),$$

in $C([0,t],\mathbb{R}^d) \times (D([0,t]))^2 \times C([0,t],\mathbb{R}^d)$.
To complete the proof, we need the following lemma (see Pardoux [65]):

Lemma 6.3. *Let $h : \mathbb{R}^d \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ measurable, periodic with period one in each direction in the first variable and continuous with respect to the second , uniformly with respect to the first . Then*

$$\sup_{0\leq s\leq t} \mid \int_0^s h(\overline{X}_s^\varepsilon, Y_s^\varepsilon)dr - \int_0^s \overline{h}(\overline{Y}_r)dr \mid \longrightarrow 0,$$

in $\widetilde{\mathbb{P}}$-probability as $\varepsilon \longrightarrow \ 0$, with $\overline{h}(y) = \displaystyle\int_{\mathbb{T}^d} h(x,y)\mu(dx)$.

Passing to the limit in (6.16) and (6.17) we get

$$X_s = x + \int_0^s C(\overline{Y}_r)dr + M_s^X,$$

where $\{M_s^X\}$ is a non standard Brownian motion which satisfies

$$<< M^X >>_s = As$$

and

$$\overline{Y}_s = g(X_s) + \int_s^t D(\overline{Y}_r)dr + \overline{M}_t - \overline{M}_s + \overline{K}_t - \overline{K}_s.$$

It follows from similar arguments in Pardoux [66] Section 4.c (see also Pardoux-Veretennikov [71]) Theorem (6.1) that M^X and $\overline{M}$ are $\mathcal{F}_t^{X,Y,K}$-Martingales.

Step 4. Identification of the limit.
Let $(Y, \overline{Z}, U)$ be the unique solution of the reflected BSDE

$$\begin{cases} Y_s = g(X_s) + \displaystyle\int_s^t D(Y_r)dr - \int_s^t \overline{Z_r}dM_r^X - \int_s^t U_r dr \\ (Y,U) \in Gr(\partial\phi), \end{cases}$$

such that

$$\mathbb{E}Tr \int_0^t \overline{Z_r} d < M^X >_r \overline{Z_r}^* < \infty.$$

By virtue of Lemma 6.2, we see that $\{K^\varepsilon, \varepsilon > 0\}$ is bounded in $\mathbb{L}^2(\Omega, \mathbb{H}^1([0,t], \mathbb{R}^d))$ with $\mathbb{H}^1([0,t], \mathbb{R}^d)$ is the Sobolev space with absolutely continuous functions with derivatives in $\mathbb{L}^2([0,t])$. Therefore, $\{\overline{K}_s, 0 \le s \le t\}$ is a process with bounded variation $\{\overline{U}_s, 0 \le s \le t\}$ such that $\int_0^t \mid \overline{U}_r \mid^2 dr < \infty$. Put

$$M_t = \int_0^t \overline{Z_r} dM_r^X \quad \text{and} \quad K_t = -\int_0^t U_s ds.$$

From Itô's formula we obtain

$$\begin{aligned}
\mid Y_s - \overline{Y}_s \mid^2 + [M - \overline{M}]_t - [M - \overline{M}]_s &= 2 \int_s^t (D(Y_r) - D(\overline{Y}_r), Y_r - \overline{Y}_r) dr \\
&\quad + 2 \int_s^t \langle Y_r - \overline{Y}_r, dM_r - d\overline{M}_r \rangle \\
&\quad + 2 \int_s^t \langle Y_r - \overline{Y}_r, dK_r - d\overline{K}_r \rangle.
\end{aligned}$$

Now, since for every $\varepsilon > 0$, we have $(Y^\varepsilon, U^\varepsilon) \in Gr(\partial\phi)$, the Skorohod selection theorem proves that $(\overline{Y}, \overline{U}) \in Gr(\partial\phi)$, where $(\overline{Y}, \overline{M}, \overline{K} = -\int_0^t \overline{U}_r dr)$ is the solution of the following reflected BSDE

$$\overline{Y}_s = g(X_s) + \int_s^t D(\overline{Y}_r) dr + \overline{M}_t - \overline{M}_s + \overline{K}_t - \overline{K}_s.$$

Therefore, $\int_s^t \langle Y_r - \overline{Y}_r, dK_r - d\overline{K}_r \rangle \le 0$. Taking the expectation in the above equality we find that

$$\mathbb{E} \mid Y_s - \overline{Y}_s \mid^2 + \mathbb{E}[M - \overline{M}]_t - \mathbb{E}[M - \overline{M}]_s \le 2\mu \mathbb{E} \int_s^t \mid Y_s - \overline{Y}_s \mid^2 dr.$$

Hence, from Gronwall's lemma $Y_s = \overline{Y}_s, 0 \le s \le t$, $M = \overline{M}$, $U_s = \overline{U}_s, 0 \le s \le t$. ■

Before proving Lemma 6.2, let us recall some properties of the penalization technique (see Menaldi [54]). Let

$$\beta(x) = \frac{1}{2} grad\big(\min\{ \mid x - y \mid^2 \}, y \in \overline{\Theta}\big).$$

Note that there exist $a \in \mathbb{R}^d$ and $\gamma > 0$ such that

$$(x - a)\beta(x) \ge \gamma \mid \beta(x) \mid, \forall x \in \mathbb{R}^d.$$

Therefore

$$\langle A_n(x), x - a \rangle \ge \gamma \mid A_n(x) \mid . \tag{6.20}$$

Let

$$A_n(Y_s^\varepsilon) = n(Y_s^\varepsilon - Pr_{\overline{\Theta}}(Y_s^\varepsilon)),$$

where Pr_Θ is the projection on Θ.

Thanks to a technical approximation (see Gegout [37]), one can suppose that Θ is bounded, convex and smooth i.e. $\rho(x) := d^2(x, \Theta) = \mid x - Pr(x) \mid^2$ is convex, twice differentiable and

$$\Theta = \{x \in \mathbb{R}^d,\ \rho > 0\}; \quad \partial\Theta = \{x \in \mathbb{R}^d,\ \rho = 0\},$$

note that $\nabla\rho(x) = 2\beta(x) = 2(x - Pr(x))^*$.

Proof of Lemma 6.2.

Let $(Y_r^{\varepsilon,n}, Z_r^{\varepsilon,n})$ be the unique solution of the following BSDE

$$Y_s^{\varepsilon,n} = g(X_t^\varepsilon) + \int_s^t (\frac{1}{\varepsilon}e(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^{\varepsilon,n}) + f(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^{\varepsilon,n}) - Z_r^{\varepsilon,n}(\frac{\partial^2\widehat{e}}{\partial y\partial x}\sigma)(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^{\varepsilon,n}))dr$$
$$- \int_s^t Z_r^{\varepsilon,n} d\widetilde{B}_r - \int_s^t A_n(Y_r^{\varepsilon,n})dr.$$

We first prove that

$$\sup_{\varepsilon,n} \widetilde{\mathbb{E}} \int_0^t \mid A_n(Y_r^{\varepsilon,n}) \mid^2 dr < \infty.$$

Let $a \in \mathbb{R}^k$ satisfies (6.20), it follows from Itô's formula that

$$\begin{aligned}
&e^{\nu s} \mid Y_s^{\varepsilon,n} - a \mid^3 + \int_s^t e^{\nu r}(3 \mid Y_r^{\varepsilon,n} - a \mid \times \mid Z_r^{\varepsilon,n} \mid^2 + \nu \mid Y_r^{\varepsilon,n} - a \mid^3)dr \\
=&e^{\nu t} \mid g(X_s^\varepsilon) \mid^3 + \frac{3}{\varepsilon}\int_s^t e^{\nu r} \mid Y_r^{\varepsilon,n} - a \mid (Y_r^{\varepsilon,n} - a)e(\overline{X}_r^\varepsilon, Y_r^{\varepsilon,n})dr \\
&+3\int_s^t e^{\nu r} \mid Y_r^{\varepsilon,n} - a \mid Y_r^{\varepsilon,n} f(\overline{X}_r^\varepsilon, Y_r^{\varepsilon,n})dr \\
&-3\int_s^t e^{\nu r} \mid Y_r^{\varepsilon,n} - a \mid (Y_r^{\varepsilon,n} - a)Z_r^{\varepsilon,n}(\frac{\partial^2\widehat{e}}{\partial x\partial y}\sigma)(\overline{X}_r^\varepsilon, Y_r^{\varepsilon,n})dr \\
&-3\int_s^t e^{\nu r} \mid Y_r^{\varepsilon,n} - a \mid (Y_r^{\varepsilon,n} - a)Z_r^{\varepsilon,n} d\widetilde{B}_r \\
&-3\int_s^t e^{\nu r} \mid Y_r^{\varepsilon,n} - a \mid (Y_r^{\varepsilon,n} - a, A_n(Y_r^{\varepsilon,n}))dr.
\end{aligned}$$

From the above hypothesis we deduce:

$\mid Y_r^{\varepsilon,n} - a \mid (Y_r^{\varepsilon,n} - a)f(\overline{X}_r^\varepsilon, Y_r^\varepsilon) \leq \mid Y_r^{\varepsilon,n} - a \mid^3 + c$

$- \mid Y_r^{\varepsilon,n} - a \mid (Y_r^{\varepsilon,n} - a, A_n(Y_r^{\varepsilon,n})) \leq -\gamma \mid A_n(Y_r^{\varepsilon,n}) \mid\mid Y_r^\varepsilon - a \mid \leq 0.$

Put $\nu = 1 + \frac{3}{2}\|\frac{\partial^2\widehat{e}}{\partial x\partial y}\|_\infty^2$. Since e is bounded, we may take the expectation to obtain

$$\varepsilon\widetilde{\mathbb{E}}\int_s^t \mid Y_r^{\varepsilon,n} \mid\mid Z_r^{\varepsilon,n} \mid^2 dr \leq c(\varepsilon + \int_s^t \mid Y_r^{\varepsilon,n} \mid^2 dr). \tag{6.21}$$

Return to equation (6.17) and let $\widehat{Y}_s^{\varepsilon,n} = Y_s^{\varepsilon,n} - \varepsilon\widehat{e}(\overline{X}_s^\varepsilon, Y_s^{\varepsilon,n})$. It follows from Itô's formula that

$$
\begin{aligned}
&| \widehat{Y}_s^{\varepsilon,n} - a |^2 + \int_s^t | \widetilde{Z}_r^{\varepsilon,n} - \varepsilon \frac{\partial \widehat{e}}{\partial y}(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n}) Z_r^{\varepsilon,n} |^2 \, dr = | g(X_t^{\varepsilon}) - \widehat{e}(X_t^{\varepsilon}, Y_t^{\varepsilon,n}) |^2 \\
&\quad + 2 \int_s^t (\widehat{Y}_r^{\varepsilon,n} - a)(\langle \nabla_x \widehat{e}, c\rangle - \frac{\partial \widehat{e}}{\partial y} e + (1 - \varepsilon \frac{\partial \widehat{e}}{\partial y}) f + \frac{\partial^2 \widehat{e}}{\partial y \partial x} a \nabla_x \widehat{e}^*)(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n}) dr \\
&\quad - 2 \int_s^t (\widehat{Y}_r^{\varepsilon,n} - a) Z_r^{\varepsilon,n} d\widetilde{B} r \\
&\quad + 2\varepsilon \int_s^t \frac{\partial \widehat{e}}{\partial y}(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n}) Z_r^{\varepsilon,n} (d\widetilde{B}_r + \frac{\partial^2 \widehat{e}}{\partial^2 y} \sigma(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n}) dr) \\
&\quad + \varepsilon \int_s^t \frac{\partial^2 \widehat{e}}{\partial y \partial x}(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n})(\widehat{Y}_r^{\varepsilon,n} - a) \mid Z_r^{\varepsilon,n} |^2 \, dr \\
&\quad - 2 \int_s^t (1 - \varepsilon \frac{\partial \widehat{e}}{\partial y})(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n})(\widehat{Y}_r^{\varepsilon,n} - a, A_n(Y_r^{\varepsilon,n})) dr.
\end{aligned}
$$

Since $0 \in \Theta$, $A_n(Y_r^{\varepsilon,n}) = A_n(\widehat{Y}_r^{\varepsilon,n})$, we get

$$
\langle \widehat{Y}_r^{\varepsilon,n} - a, A_n(Y_r^{\varepsilon,n}) \rangle \geq \gamma \mid A_n(\widehat{Y}_r^{\varepsilon,n}) \mid = \gamma \mid A_n(Y_r^{\varepsilon,n}) \mid .
$$

It follows from the fact that $(1 - \varepsilon \frac{\partial \widehat{e}}{\partial y})(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n}) \geq \frac{1}{2}$, standard inequalities and Gronwall's lemma that

$$
\sup_{0 \leq s \leq t} \sup_{\varepsilon,n} \widetilde{\mathbb{E}} \mid Y_s^{\varepsilon,n} - a |^2 + \int_s^t \mid \widetilde{Z}_r^{\varepsilon,n} |^2 \, dr) + \gamma \int_0^t \mid A_n(Y_r^{\varepsilon,n}) \mid dr < \infty. \tag{6.22}
$$

Now, from Itô's formula, the convexity of function ρ and with the notation $\widehat{Y}_r^{\varepsilon,n} = Y_r^{\varepsilon,n} - \varepsilon \widehat{e}(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n})$, we obtain

$$
\begin{aligned}
&\rho(Y_s^{\varepsilon,n} - \varepsilon \widehat{e}(\overline{X}_s^{\varepsilon}, Y_s^{\varepsilon,n})) \leq \rho(g(X_t^{\varepsilon}) - \varepsilon \widehat{e}(\overline{X}_t^{\varepsilon}, Y_t^{\varepsilon,n})) \\
&+ \frac{2}{n} \int_s^t A_n(\widehat{Y}_r^{\varepsilon,n})(\langle \nabla_x \widehat{e}, c\rangle - \frac{\partial \widehat{e}}{\partial y} e + (1 - \varepsilon \frac{\partial \widehat{e}}{\partial y}) f + \frac{\partial^2 \widehat{e}}{\partial y \partial x} a \nabla_x \widehat{e}^*)(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n})) dr \\
&- \frac{2}{n} \int_s^t A_n(\widehat{Y}_r^{\varepsilon,n}) Z_r^{\varepsilon,n} d\widetilde{B}_r \\
&+ \frac{2\varepsilon}{n} \int_s^t \frac{\partial \widehat{e}}{\partial y}(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n}) A_n(\widehat{Y}_r^{\varepsilon,n}) Z_r^{\varepsilon,n} (d\widetilde{B}_r + \frac{\partial^2 \widehat{e}}{\partial^2 y} \sigma(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n})) dr \\
&+ \frac{\varepsilon}{n} \int_s^t \frac{\partial^2 \widehat{e}}{\partial y \partial x}(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n}) A_n(\widehat{Y}_r^{\varepsilon,n}) \mid Z_r^{\varepsilon,n} |^2 \, dr \\
&- \frac{2}{n} \int_s^t (1 - \varepsilon \frac{\partial \widehat{e}}{\partial y})(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n}) A_n(\widehat{Y}_r^{\varepsilon,n})^* A_n(Y_r^{\varepsilon,n}) dr.
\end{aligned}
$$

Thanks to assumption (6.10), ε small enough, we have

$$
\begin{aligned}
n\rho(\quad &\widehat{Y}_s^{\varepsilon,n}) + \int_s^t \mid A_n(Y_r^{\varepsilon,n}) |^2 \, dr \\
&\leq 2 \int_s^t A_n(Y_r^{\varepsilon,n})(\langle \nabla_x \widehat{e}, c\rangle - \frac{\partial \widehat{e}}{\partial y} e + (1 - \varepsilon \frac{\partial \widehat{e}}{\partial y}) f + \frac{\partial^2 \widehat{e}}{\partial y \partial x} a \nabla_x \widehat{e}^*)(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n}) dr \\
&+2\varepsilon \int_s^t \frac{\partial \widehat{e}}{\partial y}(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n}) A_n(Y_r^{\varepsilon,n}) Z_r^{\varepsilon,n} (d\widetilde{B}_r + (\frac{\partial^2 \widehat{e}}{\partial x \partial y} \sigma)(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n})) dr \\
&-2 \int_s^t A_n(Y_r^{\varepsilon,n}) Z_r^{\varepsilon,n} d\widetilde{B}_r + \varepsilon \int_s^t \frac{\partial^2 \widehat{e}}{\partial y \partial x}(\overline{X}_r^{\varepsilon}, Y_r^{\varepsilon,n}) A_n(Y_r^{\varepsilon,n}) \mid Z_r^{\varepsilon,n} |^2 \, dr.
\end{aligned} \tag{6.23}
$$

Now we need to estimate this last integral. We get

$$\rho(\widehat{Y}_s^{\varepsilon,n})+\frac{1}{2}\int_s^t trace(Z_r^{\varepsilon,n}Z_r^{\varepsilon,n*}Hess(\rho(Y_r^{\varepsilon,n}))dr$$
$$=\int_s^t((\frac{1}{\varepsilon}e(\frac{X_r^\varepsilon}{\varepsilon},Y_r^{\varepsilon,n})+f(\frac{X_r^\varepsilon}{\varepsilon},Y_r^{\varepsilon,n}))-Z_r^{\varepsilon,n}(\frac{\partial^2\widehat{e}}{\partial y\partial x}\sigma))(\frac{X_r^\varepsilon}{\varepsilon},Y_r^{\varepsilon,n})\nabla\rho(Y_r^{\varepsilon,n}))dr$$
$$-\int_s^t Z_r^{\varepsilon,n}\nabla\rho(Y_r^{\varepsilon,n})d\widetilde{B}_r-\int_s^t A_n(Y_r^{\varepsilon,n})\nabla\rho(Y_r^{\varepsilon,n})dr.$$

It follows from Itô's formula that

$$e^{\nu s}\rho^{\frac{3}{2}}(Y_s^{\varepsilon,n})+\frac{3}{4}\int_s^t trace(Z_r^{\varepsilon,n}Z_r^{\varepsilon,n*}Hess(\rho^{\frac{1}{2}}(Y_r^{\varepsilon,n}))\rho(Y_r^{\varepsilon,n})dr$$
$$+\frac{3}{8}\int_s^t\mid Z_r^{\varepsilon,n}\mid^2\mid\nabla\rho(Y_r^{\varepsilon,n})\mid^2\rho^{\frac{-1}{2}}(Y_r^{\varepsilon,n})dr+\int_s^t\nu e^{\nu r}\rho^{\frac{3}{2}}(Y_r^{\varepsilon,n})dr$$
$$=\frac{3}{2}\int_s^t e^{\nu r}\big[(\frac{1}{\varepsilon}e(\frac{X_r^\varepsilon}{\varepsilon},Y_r^{\varepsilon,n})+f(\frac{X_r^\varepsilon}{\varepsilon},Y_r^{\varepsilon,n}))-Z_r^{\varepsilon,n}(\frac{\partial^2\widehat{e}}{\partial y\partial x}\sigma)(\frac{X_r^\varepsilon}{\varepsilon},Y_r^{\varepsilon,n})\big]\nabla\rho(Y_r^{\varepsilon,n})\rho^{\frac{1}{2}}(Y_r^{\varepsilon,n})dr$$
$$-\frac{3}{2}\int_s^t Z_r^{\varepsilon,n}\nabla\rho(Y_r^{\varepsilon,n})\rho^{\frac{1}{2}}(Y_r^{\varepsilon,n})d\widetilde{B}_r$$
$$-\frac{3}{2}\int_s^t A_n(Y_r^{\varepsilon,n})^*\nabla\rho(Y_r^{\varepsilon,n})\rho^{\frac{1}{2}}(Y_r^{\varepsilon,n})dr.$$

Since $\mid A_n(x)\mid=\frac{n}{2}\mid\nabla\rho(x)\mid$ et $\rho^{\frac{1}{2}}(x)=\mid x-Pr(x)\mid$, we obtain

$$ne^{\nu s}\rho(Y_s^{\varepsilon,n})^{\frac{3}{2}}+\frac{3}{2}\int_s^t trace(Z_r^{\varepsilon,n}Z_r^{\varepsilon,n*}Hess(\rho(Y_r^{\varepsilon,n}))\rho^{\frac{1}{2}}(Y_r^{\varepsilon,n})dr$$
$$+\frac{3}{2}\int_s^t\mid Z_r^{\varepsilon,n}\mid^2\mid A_n(Y_r^{\varepsilon,n})\mid dr+n\int_s^t\nu e^{\nu r}\rho^{\frac{3}{2}}(Y_r^{\varepsilon,n})dr+\frac{3}{n}\int_s^t\mid A_n(Y_r^{\varepsilon,n})\mid^3 dr$$
$$=\frac{3n}{2}\int_s^t e^{\nu r}\big[(\frac{1}{\varepsilon}e(\frac{X_r^\varepsilon}{\varepsilon},Y_r^{\varepsilon,n})+f(\frac{X_r^\varepsilon}{\varepsilon},Y_r^{\varepsilon,n}))-Z_r^{\varepsilon,n}(\frac{\partial^2\widehat{e}}{\partial y\partial x}\sigma)(X_r^\varepsilon,Y_r^{\varepsilon,n})\big]\nabla\rho(Y_r^{\varepsilon,n})\rho^{\frac{1}{2}}(Y_r^{\varepsilon,n})dr$$
$$-n\int_s^t Z_r^{\varepsilon,n}\nabla\rho(Y_r^{\varepsilon,n})\rho^{\frac{1}{2}}(Y_r^{\varepsilon,n})d\widetilde{B}_r.$$

We also have

$$nZ_r^{\varepsilon,n}\frac{\partial^2\widehat{e}}{\partial y\partial x}\sigma(X_r^\varepsilon,Y_r^{\varepsilon,n})\nabla\rho(Y_r^{\varepsilon,n})\rho(Y_r^{\varepsilon,n})^{\frac{1}{2}}$$
$$\leq\frac{n}{2}\mid Z_r^{\varepsilon,n}\mid\|\frac{\partial^2\widehat{e}}{\partial y\partial x}\sigma\|_\infty\mid\nabla\rho(Y_r^{\varepsilon,n})\mid^2$$
$$\leq\frac{n}{4}\|\frac{\partial^2\widehat{e}}{\partial y\partial x}\sigma\|(\frac{\mid Z_r^{\varepsilon,n}\mid^2\mid\nabla\rho(Y_r^{\varepsilon,n})\mid}{\alpha})+\alpha\mid\nabla\rho(Y_r^{\varepsilon,n})\mid^3)$$
$$\leq\frac{1}{4\alpha}\|(\frac{\partial^2\widehat{e}}{\partial y\partial x}\sigma)(X_r^\varepsilon,Y_r^{\varepsilon,n})\|_\infty\mid Z_r^{\varepsilon,n}\mid^2\mid A_n(Y_r^{\varepsilon,n})\mid+\frac{\alpha n}{4}\|\frac{\partial^2\widehat{e}}{\partial y\partial x}\sigma(X_r^\varepsilon,Y_r^{\varepsilon,n})\|_\infty\mid\nabla\rho(Y_r^{\varepsilon,n})\mid^3$$
$$=\frac{1}{4\alpha}\|\frac{\partial^2\widehat{e}}{\partial y\partial x}\sigma(X_r^\varepsilon,Y_r^{\varepsilon,n})\|_\infty\mid Z_r^{\varepsilon,n}\mid^2\mid A_n(Y_r^{\varepsilon,n})\mid+\frac{2\alpha}{n^2}\|\frac{\partial^2\widehat{e}}{\partial y\partial x}\sigma(X_r^\varepsilon,Y_r^{\varepsilon,n})\|_\infty\mid A_n(Y_r^{\varepsilon,n})\mid^3,$$

where α satisfies the following conditions
$\frac{1}{4\alpha}\|\frac{\partial^2\widehat{e}}{\partial y\partial x}\sigma\|_\infty\leq\frac{3}{8}$ et $\frac{\alpha}{n^2}\|\frac{\partial^2\widehat{e}}{\partial y\partial x}\sigma\|_\infty\leq\frac{3}{2n}$ for n large enough.

The relation $ab \leq \dfrac{a^p}{p} + \dfrac{a^q}{q}$ for $\frac{1}{p} + \frac{1}{q} = 1$ gives

$$\begin{aligned} & nf(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^{\varepsilon,n}) \nabla\rho(Y_r^{\varepsilon,n})\rho^{\frac{1}{2}}(Y_r^{\varepsilon,n}) \\ & \leq nK(\frac{2}{3} \mid \nabla\rho(Y_r^{\varepsilon,n}) \mid^{\frac{3}{2}} + \frac{\rho^{\frac{3}{2}}(Y_r^{\varepsilon,n})}{3}) \\ & \leq \frac{4}{3n^{\frac{1}{2}}}(\mid A_n(Y_r^{\varepsilon,n}) \mid^2 + 1) + \frac{nK}{3}\rho^{\frac{3}{2}}(Y_r^{\varepsilon,n}). \end{aligned}$$

Moreover

$$ne(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^{\varepsilon,n})(\rho^{\frac{1}{2}}\nabla\rho)(Y_r^{\varepsilon,n}) \leq \frac{\mid e \mid_\infty}{n} \mid A_n(Y_r^{\varepsilon,n}) \mid^2 .$$

Hence, thanks to relation (6.22) and for n large enough we have

$$\begin{aligned} & \varepsilon\widetilde{\mathbb{E}} \int_s^t \mid A_n(Y_r^{\varepsilon,n}) \mid\mid Z_r^{\varepsilon,n} \mid^2 dr \\ & \leq C(1 + \frac{1}{n}\widetilde{\mathbb{E}} \int_s^t \mid A_n(Y_r^{\varepsilon,n}) \mid^2 dr + \frac{1}{n^{\frac{1}{2}}}\widetilde{\mathbb{E}} \int_s^t \mid A_n(Y_r^{\varepsilon,n}) \mid^2 dr). \end{aligned}$$

For n large enough and ε small enough, inequalities (6.22) and (6.23) give

$$\sup_{\varepsilon,n} \widetilde{\mathbb{E}} \int_0^t \mid A_n(Y_r^{\varepsilon,n}) \mid^2 dr < \infty. \tag{6.24}$$

Now

$$\lim_{n \longrightarrow \infty} \widetilde{\mathbb{E}}(\sup_{0 \leq s \leq t} \mid K_s^{\varepsilon,n} - K_s^\varepsilon \mid^2) = 0,$$

where $K_s^{\varepsilon,n} = -\int_0^s A_n(Y_r^{\varepsilon,n})dr$ (see [60] or [70]).
Due to inequality (6.24) and Fatou's lemma we get the desired result. ■

6.4 Application to semi-linear variational inequalities.

Let u be the solution of the following SVI

$$\left\{ \begin{array}{l} \forall s \in [0,t],\ x \in \mathbb{R}^d \\ \big[\dfrac{\partial u}{\partial s}(s,x) - \frac{1}{2}\sum_{i,j=1}^d A_{ij}\dfrac{\partial^2 u}{\partial x_i \partial x_j}(s,x) - \sum_{i=1}^d C_i(u(s,x))\dfrac{\partial u}{\partial x_i}(s,x) \\ \qquad\qquad -D(u(s,x))\big] \in \partial I_{\overline{\Theta}}(u(s,x)) \\ u(0,x) = g(x), \quad u(s,x) \in Dom(\phi) = \mathbf{cl}(\Theta). \end{array} \right. \tag{6.25}$$

Theorem 6.4. *Assume $k = 1$. Under the hypothesis of Theorem 6.1 $u^\varepsilon(t,x)$ converges to $u(t,x)$ for all $(t,x) \in [0,t] \times \mathbb{R}^d$ as ε goes to 0.*

Remark 6.5. *Note that in this case Θ is an interval of $\mathbb{R}$. Assume for example that $\Theta =]a,b[$, we get*

$$\partial I_{\overline{\Theta}}(x) = \left\{ \begin{array}{lll} \emptyset & if & x \notin \overline{\Theta} \\ \{0\} & if & x \in \Theta \\ \mathbb{R}_+ & if & x = b \\ \mathbb{R}_- & if & x = a. \end{array} \right.$$

Proof of Theorem 6.4. Let $x \in \mathbb{R}^d$ and $\{X_s^{x,\varepsilon}, 0 \le s \le t\}$ be the solution of the SDE (6.1). For all $t \in \mathbb{R}_+$, we denote by $\{Y_s^{t,x,\varepsilon}, Z_s^{t,x,\varepsilon}, U_s^{t,x,\varepsilon},\ 0 \le s \le t\}$ the solution of the following reflected BSDE

$$Y_s^{t,x,\varepsilon} = g(X_t^{x,\varepsilon}) + \frac{1}{\varepsilon}\int_s^t e(\frac{X_r^{x,\varepsilon}}{\varepsilon}, Y_r^{t,x,\varepsilon})dr + \int_s^t f(\frac{X_r^{x,\varepsilon}}{\varepsilon}, Y_r^{t,x,\varepsilon})dr - \int_s^t Z_r^{t,x,\varepsilon} dB_r - \int_s^t U_r^{t,x,\varepsilon} dr.$$

By virtue of Pardoux and Rascanu [70], the function $u^\varepsilon : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by $u^\varepsilon(t,x) = Y_0^{t,x,\varepsilon}$ is the unique viscosity solution of SVI (6.12). Let $\{Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x},\ 0 \le s \le t\}$ be the solution of the reflected BSDE

$$Y_s = g(X_t^x) + \int_s^t D(Y_r^{t,x})dr - \int_s^t Z_r^{t,x} dB_r - \int_s^t U_r^{t,x} dr.$$

Again, in view of [70] (see also Chapter 1) the function $u : [0,t] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by $u(t,x) = Y_0^{t,x}$ is the unique solution of SVI (6.25). Therefore, the result follows from Theorem 6.1 ∎

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