Reflected backward stochastic differential equation with locally monotone coefficient

K. Bahlali[(a,1)], E.H. Essaky[(b,2)] and Y. Ouknine[(b,3)]

[(a)]UFR Sciences, UVT, B.P. 132, 83957 La Garde Cedex, France.
CPT, CNRS, Luminy. Case 907, 13288 Marseille Cedex 9, France.
E–mail: bahlali@univ-tln.fr
[(b)]Université Cadi Ayyad, Faculté des Sciences Semlalia,
Département de Mathématiques, B.P. 2390, Marrakech, Maroc.
E–mails: essaky@ucam.ac.ma, ouknine@ucam.ac.ma.

Abstract

We study the existence and uniqueness of Reflected Backward Stochastic Differential Equation (RBSDE for short) with both monotone and locally monotone coefficient and squared integrable terminal data. This is done with a polynomial growth condition on the coefficient. An application to the homogenization of multivalued Partial Differential Equations (PDEs for short) is given.

1 Introduction

Let $(W_t)_{0 \leq t \leq T}$ be a r-dimensional Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_t)_{0 \leq t \leq T}$ denote the natural filtration of (W_t) such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, and ξ be an $\mathcal{F}_T$-measurable d-dimensional random variable. Let f be an $\mathbb{R}^d$-valued process defined on $[0,T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times r}$ such that for all $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{d\times r}$, the map $(t,\omega) \longrightarrow f(t,\omega,y,z)$ is $\mathcal{F}_t$-progressively measurable. The BSDE we consider here is of the following type

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dW_s, \qquad 0 \leq t \leq T.$$

Such equations are a relatively recent subject of research, having leapt onto the stage only 12 years ago with the publication of Pardoux and Peng's paper [27]. Originally motivated by questions arising in stochastic control theory, they have since found applications in both mathematical finance theory (e.g. [13]), and in the vast subject of partial differential equations (e.g. [28, 30]). The connection between Brownian motion and diffusions with partial differential equations has been a subject of intensive research for over half a century, only limited to the types of PDEs one can consider (usually linear elliptic type PDEs). Backwards SDEs allows the treatment of a heretofore non treatable type of PDE, by probabilistic methods, and is therefore intrinsically interesting.

In [27], Pardoux and Peng have proved the existence and uniqueness of a solution under globally Lipschitz coefficient by using the Itô's martingale representation theorem and a suitable Picard approximation procedure. Afterwards, many efforts have been done in relaxing the Lipschitz conditions and the growth of the generator function, (see [17, 18, 12, 23, 24, 2, 9]).

The existence and uniqueness of reflected backward stochastic differential equation in a convex domain, via penalization method, have been proved by Gegout-Petit and Pardoux [16] under Lipschitz hypothesis on the coefficient. In the case where the solution is forced to remain above an obstacle, El Karoui et *al.* [14] have derived an existence result for one dimensional reflected BSDE with Lipschitz

[1]Supported by DEF/CNRS, PICS 444
[2]Supported by CMIFM, A.I. MA/01/02
[3]Supported by CNRST Maroc/CNRS France, 8310–2000 and TWAS grant 98–199 RG/MATHS/AF/AC

conditions by using two methods: one uses Picard iteration, the other uses penalization argument (see also [4, 20]). In this case, the solution is a triple (Y, Z, K), where K is an increasing process, satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t.$$

Those equations are also discussed in the case of globally lipschitz in z, globally monotone in y and the generator f has a linear growth [29].

However, in many examples of semilinear PDEs, the nonlinearity is not of linear growth but instead, it is of polynomial growth, see for instance the linear heat equation analyzed by Escobedo et *al.* [15] or the Allen-Cahn equation (see Barles et *al.* [6]). If one attempts to extend those equations to the case of multivalued PDE's, then the problem of solving RBSDE with polynomial growth coefficient comes up naturally.

The subject of the first part of this paper consists to solve this last problem by using the Yosida approximation which is a penalization method. Our result gives, in particular, a probabilistic interpretation of the multivalued PDE $\frac{\partial u}{\partial t} + \triangle u - u^3 \in \partial\phi$, where ϕ is a lower semicontinuous proper and convex function. Noticing that, since we allow the presence of obstacles, our result is also important for applications to finance.

In the second part, we merely assume that the monotonic condition on the variable y as well as the Lipschitz condition on the variable z are satisfied locally in y and z. We then show that if the the monotonic constant μ_N and the Lipschitz constant L_N, of the coefficient f in the ball $B(0, N)$ of $\mathbb{R}^d \times \mathbb{R}^{d\times r}$, are such that $\mu_N^+ + L_N^2 = \mathcal{O}(\log N)$, then the corresponding RBSDE has a unique solution. This is done with an unbounded terminal data. The proof is mainly based on the result of the first part and a suitable approximation of the generator f by a sequence f_n of Lipschitz functions. The idea consists to use the result of the first part inside a fixed ball $B(0, N)$ then to find a good control of the solutions in $\mathbb{R}^d \times \mathbb{R}^{d\times r} \setminus B(0, N)$.

Finally, in the third part, we apply our result to homogenization of multivalued semilinear PDEs.

The paper is organized as follows. In Section 2, we study the existence and uniqueness of RBSDE with monotone generator. The existence and uniqueness of one solution to RBSDE with locally monotone coefficient is proved in Section 3. Section 4 is devoted to the study of an homogenization property for multivalued PDE under locally monotone condition.

2 RBSDE with Monotone Coefficient and polynomial growth

2.1 Formulation of the problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(W_t, t \in [0, 1])$ be a n-dimensional Wiener process defined on it. Let $(\mathcal{F}_t, t \in [0, 1])$ denote the natural filtration of (W_t) augmented with the $\mathbb{P}$-null sets of $\mathcal{F}$. We define the following three objects:

(A.1) A process f defined on $\Omega \times [0, 1] \times I\!R^d \times I\!R^{d\times n}$ with value in $\mathbb{R}^d$ which satisfies the following assumptions:

There exist constants $\gamma \geq 0$, $\mu \in \mathbb{R}$, $C \geq 0$ and $p \geq 1$ such that $\mathbb{P} - a.s.$, we have

$(i)\ \forall (y, z) \in I\!R^d \times I\!R^{d\times n}:\ (\omega, t) \longrightarrow f(\omega, t, y, z)$ is $\mathcal{F}_t$−progressively measurable

$(ii)\ \forall t, \forall y, \forall (z, z'),\quad | f(t, y, z) - f(t, y, z') | \leq \gamma | z - z' |$

$(iii)\ \forall t, \forall z, \forall (y, y'),\quad (y - y')(f(t, y, z) - f(t, y', z)) \leq \mu | y - y' |^2$

$(iv)\ \forall t, \forall y, \forall z,\quad | f(t, y, z) | \leq | f(t, 0, z) | + K(1 + | y |^p)$

$(v)\ \forall t, \forall z,\quad y \longrightarrow f(t, y, z)$ is continuous.

(A.2) A terminal value ξ which is $\mathcal{F}_1$-measurable such that

$$\mathbb{E} | \xi |^{2p} + \mathbb{E}\left(\int_0^1 | f(s, 0, 0) |^2\, ds\right)^p < +\infty.$$

Define

$$
\begin{aligned}
Dom(\phi) &= \{u \in \mathbb{R}^d : \phi(u) < +\infty\} \\
\partial\phi(u) &= \{u^* \in \mathbb{R}^d :< u^*, v - u > +\phi(u) \leq \phi(v), \forall v \in \mathbb{R}^d\} \\
Dom(\partial(\phi)) &= \{u \in \mathbb{R}^d : \partial(\phi) \neq \emptyset\} \\
Gr(\partial\phi) &= \{(u, u^*) \in \mathbb{R}^d \times \mathbb{R}^d : u \in Dom(\partial(\phi)) \quad \text{and} \quad u^* \in \partial\phi(u)\}.
\end{aligned}
$$

(A.3) A proper lower semicontinuous convex function $\phi : \mathbb{R}^d \to]-\infty, +\infty]$.
We also assume that $\xi \in \overline{Dom(\phi)}$ and $\mathbb{E}(\phi(\xi)) < +\infty$.
Before stating our result, we recall some properties of a Yosida approximation of subdifferential operator. For every $x \in \mathbb{R}^d$, we put

$$\phi_n(x) = \min_y \left(\frac{n}{2} \mid x - y \mid^2 + \phi(y)\right).$$

Let $J_n(x)$ be the unique solution of the diffrential inclusion $x \in J_n(x) + \frac{1}{n}\partial\phi(J_n(x))$ (see Barbu, Precupanu [5]). Note that ϕ_n and J_n satisfiy the following:

j) ϕ_n: $\mathbb{R}^d \longrightarrow \mathbb{R}$ is a convex and C^1 class function with with Lipschitz derivative.

jj) For every $x \in \mathbb{R}^d$, $\bigtriangledown\phi_n(x) = n(x - J_n(x)) := A_n(x)$.

jjj) For every $x \in \mathbb{R}^d$, $\inf_{y \in \mathbb{R}^d} \phi(y) \leq \phi(J_n(x)) \leq \phi_n(x) \leq \phi(x)$.

jV) There exist $a \in interior(Dom(\phi))$ and positive numbers R, C such that for every $z \in \mathbb{R}^d$

$$\langle \nabla\phi_n(z)^*, z - a \rangle \geq R \mid A_n(z) \mid -C \mid z \mid -C \text{ for all } n \in \mathbb{N}^*. \tag{2.1}$$

v) For every $x \in \mathbb{R}^d$ $A_n(x) \in A(J_n(x))$.

The map J_n is called the resolvent of the monotone operator $A = \partial\phi$. The operator A_n is called the Yosida approximation of $\partial\phi$. More details can be found in Cépa [10].

Let us introduce our RBSDE. The solution is a triplet (Y_t, Z_t, K_t), $0 \leq t \leq 1$ of progressively measurable processes taking values in $\mathbb{R}^d \times \mathbb{R}^{d\times n} \times \mathbb{R}^d$ and satisfying:

$$
\begin{cases}
(1) Z \text{ is a predictable process and } \mathbb{E}\displaystyle\int_0^1 \|Z_t\|^2 dt < +\infty \\
(2)\, Y_t = \xi + \displaystyle\int_t^1 f(s, Y_s, Z_s)ds - \int_t^1 Z_s dW_s + K_1 - K_t, 0 \leq t \leq 1 \\
(3) \text{ the process } Y \text{ is continuous} \\
(4) K \text{ is absolutely continuous}, K_0 = 0, \text{ and for every progressively measurable} \\
\text{and continuous processes } (\alpha, \beta) \text{ such that } (\alpha_t, \beta_t) \in Gr(\partial\phi), \text{ we have} \\
\qquad \displaystyle\int_0^{\cdot} (Y_t - \alpha_t)(dK_t + \beta_t dt) \leq 0 \\
(5)\, Y_t \in \overline{Dom(\phi)},\ 0 \leq t \leq 1 \text{ a.s.}
\end{cases}
$$

Our goal in this section is to study the RBSDE (1)-(5) when the generator f satisfies the above assumptions.

Consider the following sequence of backward stochastic differential equation

$$Y_t^n = \xi + \int_t^1 (f(s, Y_s^n, Z_s^n) - A_n(Y_s^n))ds - \int_t^1 Z_s^n dW_s, \tag{2.2}$$

where ξ, f satisfy the assumptions stated above and $(A_n)_n$ is the Yosida approximation of the operator $A = \partial\phi$. It is known, since A_n is Lipschitz and f is monotone, that the equation (2.2) has one and only one solution. We set

$$K_t^n := -\int_0^t A_n(Y_s^n)ds \quad \text{for} \quad t \in [0, 1].$$

2.2 Existence and uniqueness of solutions

The main result in this section is the following

Theorem 2.1. *Assume that* ***(A.1), (A.2), (A.3)*** *hold. Then the RBSDE (1)-(5) has a unique solution* $\{(Y_t, Z_t, K_t)\,;\ 0 \le t \le 1\}$. *Moreover,*

$$\lim_{n \longrightarrow +\infty} \mathbb{E} \sup_{0 \le t \le 1} |Y_t^n - Y_t|^2 = 0$$

$$\lim_{n \longrightarrow +\infty} \mathbb{E} \int_0^1 |Z_t^n - Z_t|^2 \, ds = 0$$

$$\lim_{n \longrightarrow +\infty} \mathbb{E} \sup_{0 \le t \le 1} |K_t^n - K_t|^2 = 0,$$

where (Y^n, Z^n) *be the solution of equation 2.2.*

In order to prove Theorem 2.1 we need the following lemmas.

Lemma 2.1. *Let assumptions of Theorem 2.1 hold. Then*

$$\sup_{n \in \mathbb{N}^*} \mathbb{E} \left(\sup_{0 \le t \le 1} |\, Y_t^n \,|^2 + \int_0^1 |Z_s^n|^2 \, ds + \int_0^1 |A_n\left(Y_s^n\right)| \, ds \right) < +\infty. \tag{2.3}$$

Proof . By Itô's formula we get

$$\begin{aligned} |Y_t^n - a|^2 + \int_t^1 |Z_s^n|^2 \, ds \quad &= |\xi - a|^2 + 2 \int_t^1 \left(Y_s^n - a\right)^* f(s, Y_s^n, Z_s^n) ds \\ &-2 \int_t^1 \left(Y_s^n - a\right)^* Z_s^n dW_s - 2 \int_t^1 \left(Y_s^n - a\right)^* A_n\left(Y_s^n\right) ds. \end{aligned} \tag{2.4}$$

We Take expectation and use (2.1) to obtain,

$$\begin{aligned} \mathbb{E} \left|Y_t^n - a\right|^2 + \mathbb{E} \int_t^1 |Z_s^n|^2 \, ds \le \mathbb{E} \left|\xi - a\right|^2 + 2\mathbb{E} \int_t^1 \left(Y_s^n - a\right)^* f(s, Y_s^n, Z_s^n) ds \\ - 2R\mathbb{E} \int_t^1 |\, A_n\left(Y_s^n\right) | \, ds + 2C \int_t^1 |\, Y_s^n \,| \, ds + 2C, \end{aligned}$$

this implies that

$$\begin{aligned} &\mathbb{E} \left|Y_t^n - a\right|^2 + \mathbb{E} \int_t^1 |Z_s^n|^2 \, ds + 2R\mathbb{E} \int_t^1 |A_n\left(Y_s^n\right)| \, ds \\ &\le \mathbb{E} \mid \xi - a \mid^2 + 2C\mathbb{E} \int_t^1 \mid Y_s^n \mid ds + 2C \\ &+ 2\mathbb{E} \int_t^1 \left(Y_s^n - a\right)^* \left(f(s, Y_s^n, Z_s^n) - f(s, a, Z_s^n)\right) ds + 2 \int_t^1 (Y_s^n - a) f(s, a, Z_s^n) ds, \end{aligned}$$

Using assumptions **(A.1)**$(i) - (iii)$, we deduce

$$\begin{aligned} &\mathbb{E} \left(|Y_t^n - a|^2 + \int_t^1 |Z_s^n|^2 \, ds + 2R \int_t^1 |A_n\left(Y_s^n\right)| \, ds \right) \\ &\le \mathbb{E} \mid \xi - a \mid^2 + 2\mu\mathbb{E} \int_t^1 \mid Y_s^n - a \mid^2 ds + 2\mathbb{E} \int_t^1 \mid Y_s^n - a \mid (\gamma \mid Z_s^n \mid + K(1 + \mid a \mid^p)) ds \\ &+ \mathbb{E} \int_t^1 \mid Y_s^n - a \mid^2 ds + \mathbb{E} \int_t^1 \mid f(s, 0, 0) \mid^2 ds + C, \end{aligned}$$

where C is a constant which can change from line to line.
Since $2ab \leq \beta^2 a^2 + \dfrac{1}{\beta^2} b^2$ for each $a, b \geq 0$, we get

$$\mathbb{E}\left(|Y_t^n - a|^2 + \int_t^1 |Z_s^n|^2 \, ds + 2R \int_t^1 |A_n(Y_s^n)| \, ds\right)$$

$$\leq \mathbb{E} \mid \xi - a \mid^2 + (2 \mid \mu \mid + \beta^2 + 1) \mathbb{E} \int_t^1 |Y_t^n - a|^2 \, ds + \frac{2\gamma^2}{\beta^2} \mathbb{E} \int_t^1 \mid Z_s^n \mid^2 ds + C.$$

If we take $\frac{2\gamma^2}{\beta^2} = \frac{1}{2}$, we obtain

$$\mathbb{E}|Y_t^n - a|^2 + \tfrac{1}{2}\mathbb{E}\int_t^1 |Z_s^n|^2 \, ds \leq C\left(1 + \mathbb{E}\int_t^1 |Y_s^n - a|^2 \, ds\right),$$

Hence by Gronwall's lemma we have,

$$\sup_{0 \leq t \leq 1} \mathbb{E}|Y_t^n - a|^2 \leq C, \ \forall n.$$

So that

$$\sup_{0 \leq t \leq 1} \mathbb{E}|Y_t^n|^2 \leq C, \ \forall n.$$

Now, it is not difficult to show that,

$$\sup_{n \in \mathbb{N}^*} \mathbb{E}\left(\int_0^1 |Z_s^n|^2 \, ds + \int_0^1 |A_n(Y_s^n)| \, ds\right) < +\infty. \tag{2.5}$$

We use equation (2.4) and Bulkholder-Davis-Gundy inequality to get,

$$\sup_{n \in \mathbb{N}^*} \mathbb{E} \sup_{0 \leq t \leq 1} |Y_t^n|^2 \leq C. \tag{2.6}$$

Lemma 2.1 is proved. ■

We state the following lemma which is essential for the convergence of the sequence $(Y^n, Z^n)_{n \in IN^*}$.

Lemma 2.2. *Let assumptions of Theorem 2.1 hold. Then*
a) $\sup_{0 \leq t \leq 1} \mathbb{E} \mid Y_t^n \mid^{2p} < +\infty, \ \forall n.$
b) $\sup_{n \in \mathbb{N}^*} \mathbb{E} \int_0^1 |A_n(Y_s^n)|^2 \, ds < +\infty.$

Proof . a) Itô's formula gives

$$|Y_t^n - a|^2 + \int_t^1 |Z_s^n|^2 \, ds = |\xi - a|^2 + 2\int_t^1 (Y_s^n - a)^* f(s, Y_s^n, Z_s^n, U_s^n) ds$$
$$- 2\int_t^1 Y_s^n Z_s^n dW_s - 2\int_t^1 (Y_s^n - a)^* A_n(Y_s^n) \, ds,$$

By assumptions **(A.1)**$(i) - (iii)$, we have

$$|Y_t^n - a|^2 + \int_t^1 |Z_s^n|^2 \, ds + 2R \int_t^1 |A_n(Y_s^n)| \, ds$$

$$\leq \mid \xi - a \mid^2 + 2\mu \int_t^1 \mid Y_s^n - a \mid^2 ds + 2\int_t^1 \mid Y_s^n - a \mid (\gamma \mid Z_s^n \mid + K(1 + \mid a \mid^p)) ds$$

$$+ \int_t^1 \mid Y_s^n - a \mid^2 ds + \int_t^1 \mid f(s, 0, 0) \mid^2 ds + C - \int_t^1 Y_s^n Z_s^n dW_s,$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ of both sides, we get that

$$|Y_t^n - a|^2 \leq \mathbb{E}\big[\mid \xi - a \mid^2 /\mathcal{F}_t\big] + (2\mid \mu \mid + 4\gamma^2 + 1)\mathbb{E}\big[\int_t^1 \mid Y_s^n - a \mid^2 ds/\mathcal{F}_t\big]$$

$$+\mathbb{E}\big[\int_0^1 \mid f(s,0,0) \mid^2 ds/\mathcal{F}_t\big] + 2C\mathbb{E}\int_0^1 (1+\mid a \mid^p)ds + C.$$

Jensen's inequality shows that for every $p > 1$,

$$\begin{aligned}\mathbb{E}\,|Y_t^n - a|^{2p} &\leq C_p\big[\mathbb{E}\big[\mid \xi - a \mid^{2p}\big] + (2\mid \mu \mid + 4\gamma^2 + 1)^p\mathbb{E}\big[\int_t^1 \mid Y_s^n - a \mid^{2p} ds\big] \\ &\quad + \mathbb{E}\big(\int_0^1 \mid f(s,0,0) \mid^2 ds\big)^p + 1\big] \\ &\leq C_p(1 + \mathbb{E}\int_t^1 |Y_s^n - a|^{2p} ds).\end{aligned}$$

Gronwall's lemma implies that

$$\sup_{0\leq t\leq 1} \mathbb{E} \mid Y_t^n \mid^{2p} < +\infty, \quad \forall n. \tag{2.7}$$

Assertion a) is proved.

b) We assume without loss of generality that ϕ is positive and $\phi(0) = 0$[1]. Note that ϕ_n is a convex C^1-function with a lipschitz derivative, and put $\psi_n = \dfrac{\phi_n}{n}$.

By convolution of ψ_n with a smooth function, the convexity of ψ_n and Itô's formula, one can show that,

$$\begin{aligned}\psi_n(Y_t^n) &\leq \psi_n(\xi) + \int_t^1 \nabla\psi_n(Y_r^n)(f(r, Y_r^n, Z_r^n) - A_n(Y_r^n))dr \\ &\quad - \int_t^1 \nabla\psi_n(Y_r^n)Z_r^n dW_r,\end{aligned}$$

thus

$$\begin{aligned}\mathbb{E}\psi_n(Y_s^n) &\leq \mathbb{E}\psi_n(\xi) + \mathbb{E}\int_t^1 \nabla\psi_n(Y_r^n)(f(r, Y_r^n, Z_r^n) - A_n(Y_r^n))dr \\ &= \mathbb{E}\psi_n(\xi) + \mathbb{E}\int_t^1 \nabla\psi_n(Y_r^n)f(r, Y_r^n, Z_r^n)dr - \frac{1}{n}\mathbb{E}\int_t^1 \mid A_n(Y_r^n) \mid^2 dr.\end{aligned}$$

Hence, using the inequality $2ab \leq na^2 + \frac{1}{n}b^2$ we deduce,

$$\begin{aligned}\mathbb{E}\psi_n(Y_s^n) + \frac{1}{n}\mathbb{E}\int_t^1 \mid A_n(Y_r^n) \mid^2 dr &\leq \mathbb{E}\psi_n(\xi) + \frac{1}{2n}\mathbb{E}\int_t^1 \mid A_n(Y_r^n) \mid^2 dr \\ &\quad + \frac{1}{2n}\mathbb{E}\int_t^1 \mid f(s, Y_s^n, Z_s^n) \mid^2 ds.\end{aligned}$$

We use assumptions **(A.1)**$(iv), (ii)$, to get

$$\begin{aligned}&\mathbb{E}\psi_n(Y_s^n) + \frac{1}{n}\mathbb{E}\int_t^1 \mid A_n(Y_r^n) \mid^2 dr \\ &\quad\leq \mathbb{E}\psi_n(\xi) + \frac{1}{2n}\mathbb{E}\int_t^1 \mid A_n(Y_r^n) \mid^2 dr + \frac{2\gamma^2}{n}\mathbb{E}\int_t^1 \mid Z_s^n \mid^2 ds \\ &\quad + \frac{2}{n}\mathbb{E}\int_t^1 \mid f(s,0,0) \mid^2 ds + \frac{2K^2}{n}\mathbb{E}\int_t^1 (1+\mid Y_s^n \mid^{2p})ds.\end{aligned}$$

[1] This assumption is not a restriction since we can replace $\phi(y)$ by $\phi(y + y_0) - \phi(y_0) - \langle y_0^*, y\rangle$ where $\langle y, y_0^*\rangle \in Gr(\partial\phi)$.

The relations (2.5), (2.6) and (2.7) allowed us to prove that

$$\mathbb{E}\psi_n(Y_s^n) + \frac{1}{n}\mathbb{E}\int_t^1 \mid A_n(Y_r^n) \mid^2 dr \leq \frac{C}{n},$$

which implies that

$$\sup_n \mathbb{E}\int_0^1 \mid A_n(Y_r^n) \mid^2 dr < +\infty. \tag{2.8}$$

Lemma 2.2 is proved ■

Lemma 2.3. *Let assumptions of Theorem 2.1 hold. Then*

$$\mathbb{E}\sup_{0\leq t\leq 1} \mid Y_t^n - Y_t^m \mid^2 + \mathbb{E}\int_t^1 \mid Z_s^n - Z_s^m \mid^2 ds \leq C(\frac{1}{n} + \frac{1}{m})$$

Proof . Using Itô's formula, we get

$$\begin{aligned}
&|Y_t^n - Y_t^m|^2 + \int_t^1 |Z_s^n - Z_s^m|^2 \, ds \\
&= 2\int_t^1 (Y_s^n - Y_s^m)^* \left[f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m)\right] ds \\
&+ 2\int_t^1 (Y_s^n - Y_s^m)^* (Z_s^n - Z_s^m) \, dW_s \\
&- 2\int_t^1 (Y_s^n - Y_s^m)^* A_n(Y_s^n) \, ds + 2\int_t^1 (Y_s^n - Y_s^m)^* A_m(Y_s^m) \, ds.
\end{aligned}$$

and then,

$$\begin{aligned}
&|Y_t^n - Y_t^m|^2 + \int_t^1 |Z_s^n - Z_s^m|^2 \, ds \\
&= 2\int_t^1 (Y_s^n - Y_s^m)^* \left[f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^n)\right] ds \\
&+ 2\int_t^1 (Y_s^n - Y_s^m)^* \left[f(s, Y_s^m, Z_s^n) - f(s, Y_s^m, Z_s^m)\right] ds \\
&+ 2\int_t^1 (Y_s^n - Y_s^m)^* (Z_s^n - Z_s^m) \, dW_s \\
&- 2\int_t^1 (Y_s^n - Y_s^m)^* A_n(Y_s^n) \, ds + 2\int_t^1 (Y_s^n - Y_s^m)^* A_m(Y_s^m) \, ds.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{E}\,|Y_t^n - Y_t^m|^2 + \mathbb{E}\int_t^1 |Z_s^n - Z_s^m|^2 \, ds \\
&\leq 2\mu\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 \, ds + 2\gamma\mathbb{E}\int_t^1 |Y_s^n - Y_s^m| \mid Z_s^n - Z_s^m \mid ds \\
&-2\mathbb{E}\int_t^1 (Y_s^n - Y_s^m)^* \left(A_n(Y_s^n) - A_m(Y_s^m)\right) ds.
\end{aligned}$$

Since, $Id = J_n + \frac{1}{n}A_n = J_m + \frac{1}{m}A_m$, $(A_m(Y_s^m), A_n(Y_s^n)) \in A(J_m(Y_s^m)) \times A(J_n(Y_s^n))$ and $xy \leq \frac{1}{4}x^2 + y^2$,$\forall\ x \geq 0\ \forall\ y \geq 0$, we can show that

$$-\langle Y_s^n - Y_s^m, A_n(Y_s^n) - A_m(Y_s^m)\rangle \leq \frac{1}{4m}\,|A_n(Y_s^n)|^2 + \frac{1}{4n}\,|A_m(Y_s^m)|^2 \,.$$

Hence

$$\mathbb{E}\left|Y_t^n - Y_t^m\right|^2 + \mathbb{E}\int_t^1 \left|Z_s^n - Z_s^m\right|^2 ds$$

$$\leq (2 \mid \mu \mid + \beta^2)\mathbb{E}\int_t^1 \left|Y_s^n - Y_s^m\right|^2 ds + \frac{\gamma^2}{\beta^2}\mathbb{E}\int_t^1 \left|Z_s^n - Z_s^m\right|^2 ds$$

$$+\mathbb{E}\int_t^1 \left(\frac{1}{4m}\left|A_n\left(Y_s^n\right)\right|^2 + \frac{1}{4n}\left|A_m\left(Y_s^m\right)\right|^2\right) ds.$$

If we choose β such that $\frac{\gamma^2}{\beta^2} < \frac{1}{2}$, we obtain

$$\sup_{0\leq t\leq 1} \mathbb{E}\left|Y_t^n - Y_t^m\right|^2 + \frac{1}{2}\mathbb{E}\int_t^1 \left|Z_s^n - Z_s^m\right|^2 ds \leq C\left(\frac{1}{n} + \frac{1}{m}\right).$$

Using the Burkholder-Davis-Gundy inequality, we get

$$\mathbb{E}\sup_{0\leq t\leq 1} \left|Y_t^n - Y_t^m\right|^2 + \frac{1}{2}\mathbb{E}\int_0^1 \left|Z_s^n - Z_s^m\right|^2 ds \leq C\left(\frac{1}{n} + \frac{1}{m}\right).$$

Lemma 2.3 is proved. ■

Lemma 2.4. *(see Saisho [31]) Let $(k^n)_{n\in\mathbb{N}}$ be a sequence of continuous and bounded variation functions from $[0,1]$ to $\mathbb{R}^d$, such that :*
(i) $\sup_n Var\left(k^n\right) \leq C < +\infty$.
(ii) $\lim_{n\to\infty} k^n = k$ *uniformly on* $[0,1]$.
(iii) Let $(f^n)_{n\in\mathbb{N}}$ be a sequence of càdlàg functions from- $[0,1]$ to R^d, such that $\lim_{n\to\infty} f^n = f$ uniformly on $[0,1]$.
Then for every $t\in[0,1]$ we have:

$$\lim_{n\to\infty}\int_0^t \left\langle f^n\left(s\right), dk^n\left(s\right)\right\rangle = \int_0^t \left\langle f\left(s\right), dk\left(s\right)\right\rangle.$$

Proof of Theorem 2.1
Existence. By Lemma 2.3, $(Y^n, Z^n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in the Banach space of progressively measurable processes $\mathbb{L}$ defined by,

$$\mathbb{L} = \left\{(Y,Z)\ /\ \mathbb{E}(\sup_{0\leq t\leq 1}\left|Y_t\right|^2) + \frac{1}{2}\mathbb{E}\int_0^1 \left|Z_s\right|^2 ds) < \infty\right\}.$$

Let (Y,Z) be the limit of (Y^n, Z^n) in $\mathbb{L}$.
Coming back to the equation satisfied by $(Y^n, Z^n)_{n\in\mathbb{N}}$, we can show that $(K^n)_{n\in\mathbb{N}}$ converges uniformly in $L^2(\Omega)$ to the process $K_. = \lim_{n\longrightarrow+\infty}\int_0^. A_n\left(Y_s^n\right) ds$, that is

$$\mathbb{E}\sup_{0\leq t\leq 1}\left|K_t^n - K_t\right|^2 = 0.$$

The relation (2.8) can be written in the form

$$\sup_{n\in\mathbb{N}^*}\mathbb{E}\left\|K_n\right\|^2_{H^1(0,1;\mathbb{R}^d)} < +\infty,$$

where $H^1\left(0,1;\mathbb{R}^d\right)$ is the classical Sobolev space consisting of all absolutely continuous functions with derivative in $L^2(0,1)$. Hence the sequence (K^n) is strongly bounded in the Hilbert space $L^2\left(\Omega; H^1\left(0,1;\mathbb{R}^d\right)\right)$, and there exists then a subsequence of (K^n) which converges weakly. The limiting process K belongs to $L^2\left(\Omega; H^1\left(0,1;\mathbb{R}^d\right)\right)$ and for a.s.ω $K_.(\omega) \in H^1\left(0,1;\mathbb{R}^d\right)$. Hence K is absolutely continuous and

$\dfrac{dK_t}{dt} = V_t$, where $-V_t \in \partial\phi(Y_t)$.
We shall prove that (Y, Z, K) is the unique solution to our equation. Taking a subsequence, if necessary, we can suppose that:

$$\sup_{t\in[0,1]} |K_t^n - K_t| \longrightarrow 0, \ a.s.$$

$$\sup_{t\in[0,1]} |Y_t^n - Y_t| \longrightarrow 0, \ a.s.$$

It follows that K_t and Y_t are continuous. Let (α, β) be a continuous processes with values in $Gr(\partial\phi)$. It holds that

$$\langle J_n(Y_t^n) - \alpha\,(t)\,, dK_t^n + \beta_t dt\rangle \leq 0.$$

Since $J_n(Y_t^n)$ converge to $\mathbf{pr}(Y_t)$, where $\mathbf{pr}$ denotes the projection on $\overline{Dom(\phi)}$, then we use Lemma 2.4 to show that $\langle \mathbf{pr}(Y_t) - \alpha\,(t)\,, dK\,(t) + \beta_t dt\rangle \leq 0$.
Since the process $(Y_t, 0 \leq t \leq 1)$ is continuous, the proof of existence will complete if we show that

$$\mathbb{P}\left\{Y_t \in \overline{Dom(\phi)}\right\} = 1 \qquad \forall\ t \geq 0.$$

Assume that there exist $0 < t_0 < \infty$ and $B_0 \in\mathcal{F}$ such that $\mathbb{P}\,(B_0) > 0$ and $Y_{t_0}(\omega) \notin \overline{Dom(\phi)}$ $\forall$ $\omega \in B_0$. By the continuity, there exist $\delta > 0, B_1 \in\mathcal{F}$ such that $\mathbb{P}\,(B_1) > 0$, $Y_t\,(\omega) \notin \overline{Dom(\phi)}$ for every $(\omega, t) \in B_1 \times [t_0, t_0 + \delta]$. Using the fact that $\sup_{n\in\mathbb{N}^*} \mathbb{E}\displaystyle\int_0^1 |A_n\,(Y_s^n)|\,ds < +\infty$, and Fatou's lemma, we obtain

$$\int_{B_1}\int_{t_0}^{t_0+\delta} \liminf_{n\longrightarrow+\infty} |A_n\,(Y_s^n)|\,ds\,d\mathbb{P} < +\infty,$$

which contradict the fact that $\liminf\limits_{n\longrightarrow+\infty} |A_n\,(Y_s^n)| = +\infty$ on the set $B_1 \times [t_0, t_0 + \delta]$. This complete the existence proof. ■

Uniqueness. Let $\{(Y_t, Z_t, K_t)\,;\ 0 \leq t \leq 1\}$ and $\{(Y_t', Z_t', K_t')\,;\ 0 \leq t \leq 1\}$ denote two solutions of our BSDE. Define

$$\{(\Delta Y_t, \Delta Z_t, \Delta K_t)\ \ ; 0 \leq t \leq 1\} = \{(Y_t - Y_t', Z_t - Z_t', K_t - K_t')\ \ ; 0 \leq t \leq 1\}.$$

It follows from Itô's formula that,

$$\mathbb{E}\left[|\Delta Y_t|^2 + \int_t^1 |\Delta Z_s|^2\,ds\right] = 2\mathbb{E}\int_t^1 \langle \Delta Y_s, f(s, Y_s, Z_s) - f(s, Y_s', Z_s)\rangle\,ds$$

$$+2\mathbb{E}\int_t^1 \langle \Delta Y_s, f(s, Y_s', Z_s) - f(s, Y_s', Z_s')\rangle\,ds + 2\mathbb{E}\int_t^1 \langle \Delta Y_s, d\Delta K_s\rangle\,.$$

By assumptions $\mathbf{(A.1)}(ii) - (iii)$, we get

$$\mathbb{E}\left[|\Delta Y_t|^2 + \int_t^1 |\Delta Z_s|^2\,ds\right]$$

$$= (2\mu + \beta^2)\mathbb{E}\int_t^1 \mid \Delta Y_s \mid^2\,ds + \frac{\gamma^2}{\beta^2}\mathbb{E}\int_t^1 \mid \Delta Z_s \mid^2\,ds + 2\mathbb{E}\int_t^1 \langle Y_s, d\Delta K_s\rangle\,.$$

Since $\partial\phi$ is a monotone operator and $\dfrac{-dK_t}{dt} \in \partial\phi(Y_t)$, $\dfrac{-dK_t'}{dt} \in \partial\phi(Y_t')$, then

$$\mathbb{E}\int_t^1 \langle \Delta Y_s, d\Delta K_s\rangle \leq 0.$$

Hence, taking $\dfrac{\gamma^2}{\beta^2}=\dfrac{1}{2}$, we have

$$\mathbb{E}\left[|\Delta Y_t|^2+\int_t^1|\Delta Z_s|^2\,ds\right]\leq C\mathbb{E}\int_t^1|\Delta Y_s|^2\,ds+\frac{1}{2}\mathbb{E}\int_t^1|\Delta Z_s|^2\,ds.$$

The result follows from Gronwall's lemma. ■

3 Reflected Backward Stochastic Differential Equation with Locally monotone Coefficient

The aim of this section is to extend the previous results to the case where the generator f is locally monotone on the y-variable and locally lipschitz on the z-variable. Similar result on existence and uniqueness (without reflection) has been proved in Pardoux [26] for BSDE in the case where the generator f is globally monotone w.r.t. the variable y and Lipschitz w.r.t. the variable z, and more recently in Bahlali et *al.* [2] for BSDE with reflection and jumps in the case where the generator is locally Lipschitz w.r.t. the variables y and z. Our result is, in particular, an extension of these two works.
Consider the following assumptions:
(i) f is continuous in (y,z) for almost all (t,ω),
(ii) There exist $M>0$ and $0\leq\alpha<1$ such that $\mid f(t,\omega,y,z)\mid\leq M(1+\mid y\mid^{\alpha}+\mid z\mid^{\alpha})$.
(iii) There exists μ_N such that:

$$\langle y-y',f(t,y,z)-f(t,y',z)\rangle\leq\mu_N\mid y-y'\mid^2;\ \mathbb{P}-a.s.,\ a.e.t\in[0,1]\text{ and}$$
$$\forall y,\ z\text{ such that }\mid y\mid\leq N,\ \mid y'\mid\leq N,\ \mid z\mid\leq N.$$

(iv) For each $N>0$, there exists L_N such that:

$$\mid f(t,y,z)-f(t,y,z')\mid\leq L_N\mid z-z'\mid;\mid z\mid,\mid z'\mid\leq N;\ \mathbb{P}-a.s.,\ a.e.t\in[0,1]\text{ and}$$
$$\forall y,\ z,\ z'\text{ such that }\mid y\mid\leq N,\ \mid z\mid\leq N,\ \mid z'\mid\leq N.$$

When the assumptions $(i),(ii)$, are satisfied, we can define the family of semi norms $(\rho_n(f))_n$

$$\rho_n(f)=(\mathbb{E}\int_0^1\sup_{|y|,|z|\leq n}\mid f(s,y,z)\mid^2\,ds)^{\frac{1}{2}}.$$

The main result of this section is the following

Theorem 3.1. *Let (i)-(iv) hold and ξ be a square integrable random variable.*
-(a)- Assume moreover that

$$\lim_{N\to+\infty}\frac{\exp(L_N^2+2\mu_N^+)}{(L_N^2+2\mu_N^+)N^{2(1-\alpha)}}=0,\tag{3.1}$$

where $\mu_N^+:=\sup(\mu_N,0)$. Then equation $(1)-(5)$ has a unique solution.
-(b)- If there exists a constant $L\geq 0$ such that

$$L_N^2+2\mu_N^+\leq L+2logN,$$

then equation $(1)-(5)$ has also a unique solution.

Remark 3.1. *It should be noted that there is existence and uniqueness if we replace condition (ii) by the following*
(ii') There exists $M>0$ and $0\leq\alpha<1$ such that $\mid f(t,\omega,y,z)\mid\leq M(1+\mid y\mid+\mid z\mid^{\alpha})$.

To prove Theorem 3.1 we need the following lemmas.

Lemma 3.1. *Let f be a function which satisfies $(i),(ii),(iii),(iv)$. Then there exists a sequence of functions (f_n) such that,*
-(a)- For each n, f_n is globally Lipschitz in (y,z) a.e. t and P-a.s.ω.
-(b)- For every $N\in\mathbb{N}^*$, $|f_n(t,\omega,y,z)-f_n(t,\omega,y,z')|\leq L_{(N+\frac{1}{n})}|z-z'|$, for n large enough and for each (y,z,z') such that $|y|\leq N$, $|z|\leq N$, $|z'|\leq N$.
-(c)- For every $N\in\mathbb{N}^*$, $<y-y', f_n(t,\omega,y,z)-f_n(t,\omega,y',z)>\leq \mu_{(N+\frac{1}{n})}|y-y'|^2$, for n large enough and for each (y,y',z) such that $|y|\leq N$, $|y'|\leq N$, $|z|\leq N$.
-(d)- For every N, $\rho_N(f_n-f)\longrightarrow 0$ as $n\longrightarrow\infty$.

Proof . Let $\rho_n:\mathbb{R}^d\longrightarrow\mathbb{R}_+$ be a sequence of smooth functions with compact support which approximate the Dirac measure at 0 and which satisfy $\int\rho_n(u)du=1$. Let $\varphi_n:\mathbb{R}^d\longrightarrow\mathbb{R}_+$ be a sequence of smooth functions such that $0\leq\varphi_n\leq 1$, $\varphi_n(u)=1$ for $|u|\leq n$ and $\varphi_n(u)=0$ for $|u|\geq n+1$. Likewise we define the sequence ψ_n from $\mathbb{R}^{d\times r}$ to $\mathbb{R}_+$. We put, $f_{q,n}(t,y,z)=\int f(t,y-u,z)\rho_q(u)du\varphi_n(y)\psi_n(z)$. For $n\in\mathbb{N}^*$, let $q(n)$ be an integer such that $q(n)\geq M[n+n^\alpha]$. It is not difficult to see that the sequence $f_n:=f_{q(n),n}$ satisfies all the assertions (a)-(d). ■

Consider, for fixed (t,ω) the sequence $f_n(t,\omega,y,z)$ associated to f by Lemma 3.1. We get from the previous section that there exists a unique triplet $\{(Y_t^n,Z_t^n,K_t^n;0\leq t\leq 1)\}$ of progressively measurable processes which satisfy:

$$\begin{cases}(1')Z^n \text{ is adapted process and } \mathbb{E}\displaystyle\int_0^1|Z_t^n|^2dt<+\infty,\\ (2')\,Y_t^n=\xi+\displaystyle\int_t^1 f_n(s,Y_s^n,Z_s^n)ds-\int_t^1 Z_s^n dW_s+K_1^n-K_t^n, 0\leq t\leq 1,\\ (3')\text{ the process } Y^n \text{ is continuous}\\ (4')\,K^n \text{ is absolutely continuous}, K_0^n=0, \text{ and for every progressively measurable}\\ \text{and continuous processes}\,(\alpha,\beta)\,\text{such that}\,(\alpha_t,\beta_t)\in Gr(\partial\phi),\text{ we have}\\ \qquad\qquad\displaystyle\int_0^{\cdot}(Y_t^n-\alpha_t)(dK_t^n+\beta_t dt)\leq 0.\\ (5')\,Y_t^n\in\overline{Dom(\phi)},\ 0\leq t\leq 1\ \text{ a.s.}\end{cases}$$

Lemma 3.2. *There exists a constant C depending only in M and $\mathbb{E}\mid\xi\mid^2$, such that*

$$\mathbb{E}(\sup_{0\leq t\leq 1}|Y_t^n|^2+\int_0^1|Z_s^n|^2\,ds+\mid K_1^n\mid^2)\leq C,\ \forall\, n\in\mathbb{N}^*.$$

Proof : Since $|x|^\alpha\leq 1+|x|\ \forall\alpha\in[0,1]$, the proof follows by standard arguments for BSDE. ■

Lemma 3.3. *There exist (Y,Z,K) such that*

$$\lim_{n\to\infty}\mathbb{E}\{\sup_{0\leq t\leq 1}\mid Y_t^n-Y_t\mid^2+\sup_{0\leq t\leq 1}\mid K_t^n-K_t\mid^2+\int_0^1|Z_s^n-Z_s|^2\,ds\}=0.$$

Proof . By Itô's formula we have,

$$\begin{aligned}\mathbb{E}(|Y_t^n-&\ Y_t^m|^2)+E\int_t^1|Z_s^n-Z_s^m|^2ds\\ &=2E\int_t^1\langle Y_s^n-Y_s^m,\quad f_n(s,Y_s^n,Z_s^n)-f_m(s,Y_s^m,Z_s^m)\rangle ds\\ &\quad+2\int_t^1(Y_s^n-Y_s^m)d(K_s^n-K_s^m)\\ &=I_0(n,m)+I_1(n,m)+I_2(n,m)+I_3(n,m)+2\int_t^1(Y_s^n-Y_s^m)d(K_s^n-K_s^m)\end{aligned}\tag{3.2}$$

where

$$I_0(n,m)=2E\int_t^1\langle Y_s^n-Y_s^m,\quad f_n(s,Y_s^n,Z_s^n)-f_m(s,Y_s^m,Z_s^m)\rangle\mathbf{1}_{A_{n,m}^N}ds$$

$$I_1(n,m) = 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f_n(s,Y_s^n,Z_s^n) - f(s,Y_s^n,Z_s^n)\rangle \mathbf{1}_{\overline{A}_{n,m}^N}\, ds$$

$$I_2(n,m) = 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f(s,Y_s^n,Z_s^n) - f(s,Y_s^m,Z_s^m)\rangle \mathbf{1}_{\overline{A}_{n,m}^N}\, ds$$

$$I_3(n,m) = 2E\int_t^1 \langle Y_s^n - Y_s^m, \quad f(s,Y_s^m,Z_s^m) - f_m(s,Y_s^m,Z_s^m)\rangle \mathbf{1}_{\overline{A}_{n,m}^N}\, ds.$$

Since K^n, K^m are absolutely continuous, $Y^n, Y^m \in \overline{Dom(\phi)}$ and the measures $\langle Y_t^n - \alpha_t, dK_t^n - \beta_t dt\rangle$ $\langle Y_t^m - \alpha_t, dK_t^m - \beta_t dt\rangle$, are negatives, we deduce from Lemma 4.1 in Cépa [10] that $\langle Y_s^n - Y_s^m, d(K_s^n - K_s^m)\rangle$, is also negative.

We shall estimate $I_0(n,m)$, $I_1(n,m)$, $I_2(n,m)$, $I_3(n,m)$. Let β be a strictly positive number. For a given $N > 1$, we put $A_{n,m}^N := \{(s,\omega); \quad |Y_s^n|^2 + |Z_s^n|^2 + |Y_s^m|^2 + |Z_s^m|^2 \geq N^2\}$, $\overline{A}_{n,m}^N := \Omega \setminus A_{n,m}^N$ and denote by $\mathbf{1}_E$ the indicator function of the set E. It is not difficult to see that,

$$\begin{aligned} I_0(n,m) &\leq \beta^2 \mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 \mathbf{1}_{A_{n,m}^N}\, ds \\ &\quad + \frac{1}{\beta^2}\mathbb{E}\int_t^1 \mid f_n(s,Y_s^n,Z_s^n) - f_m(s,Y_s^m,Z_s^m)\mid^2 \mathbf{1}_{A_{n,m}^N} ds \end{aligned}$$

We use Hölder's inequality (since $\alpha < 1$) and Chebychev's inequality to get,

$$I_0(n,m) \leq \beta^2 \mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 \mathbf{1}_{A_{n,m}^N}\, ds + \frac{K_2(M,\xi)}{\beta^2 N^{2(1-\alpha)}}. \tag{3.3}$$

In another hand we have

$$I_1(n,m) \leq \mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 ds + \mathbb{E}\int_t^1 \mid f_n(s,Y_s^n,Z_s^n) - f(s,Y_s^n,Z_s^n)\mid^2 \mathbf{1}_{\overline{A}_{n,m}^N} ds,$$

and then

$$I_1(n,m) \leq \mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 ds + \rho_N^2(f_n - f). \tag{3.4}$$

Likewise we show that,

$$I_3(n,m) \leq \mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 ds + \rho_N^2(f_m - f). \tag{3.5}$$

We use assumptions (iii) and (iv) to prove that,

$$\begin{aligned} I_2(n,m) &\leq 2E\int_t^1 < Y_s^n - Y_s^m, \quad f(s,Y_s^n,Z_s^n) - f(s,Y_s^m,Z_s^n) > \mathbf{1}_{\overline{A}_{n,m}^N} ds \\ &\quad + 2\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|\, |f(s,Y_s^m,Z_s^n) - f(s,Y_s^m,Z_s^m)| \mathbf{1}_{\overline{A}_{n,m}^N} ds \\ &\leq (2\mu_N + \gamma^2)\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 \mathbf{1}_{\overline{A}_{n,m}^N} ds + \frac{L_N^2}{\gamma^2}\mathbb{E}\int_t^1 |Z_s^n - Z_s^m|^2 ds. \end{aligned}$$

We choose β and γ such that $\beta^2 = L_N^2 + 2\mu_N^+$ and $\gamma^2 = L_N^2$ then we use (3.3), (3.4), (3.5) and the last inequality to show that,

$$\begin{aligned} \mathbb{E}(|Y_t^n - Y_t^m|^2) + \mathbb{E}\int_t^1 |Z_s^n - Z_s^m|^2 ds &\leq (L_N^2 + 2\mu_N^+ + 2)\mathbb{E}\int_t^1 |Y_s^n - Y_s^m|^2 ds \\ &\quad + [\rho_N^2(f_n - f) + \rho_N^2(f_m - f)] + \frac{K_3(M,\xi)}{(L_N^2 + 2\mu_N^+)N^{2(1-\alpha)}}. \end{aligned}$$

Hence Gronwall Lemma implies that,

$$\mathbb{E}(|Y_t^n - Y_t^m|^2) \leq \left[[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)] + \frac{K_4(M,\xi)}{(L_N^2 + 2\mu_N^+)N^{2(1-\alpha)}}\right] \exp(L_N^2 + 2\mu_N^+ + 2).$$

Using the Burkholder-Davis-Gundy inequality, we show that a universal positive constant C exists such that,

$$\mathbb{E}(\sup_{0\leq t\leq 1} \mid Y_t^n - Y_t^m \mid^2) \leq C\Big[[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)] + \frac{K_4(M,\xi)}{(L_N^2 + 2\mu_N^+)N^{2(1-\alpha)}}\Big] \exp(L_N^2 + 2\mu_N^+ + 2).$$

It follows from equation 3.2 that

$$\mathbb{E}\int_0^1 |Z_s^n - Z_s^m|^2 ds \leq C(M,\xi)\big[\mathbb{E}\int_0^1 \mid Y_t^n - Y_t^m \mid^2 ds\big]^{\frac{1}{2}}$$

Passing to the limit successively on n, m and on N, we show that (Y^n, Z^n) is a Cauchy sequence in the Banach space $\mathbb{L}$.
Now, if we return to the equation satisfied by (Y^n, Z^n), we obtain that

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq 1} \mid K_t^n - K_t^m \mid^2 \leq \mathbb{E}\sup_{0\leq t\leq 1} \mid Y_t^n - Y_t^m \mid^2 &\\ + C\mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m) \mid^2 ds &\\ + \mathbb{E}\int_0^1 \mid Z_s^n - Z_s^m \mid^2 ds.&\end{aligned}$$

In order to complete the proof, we need to show that the sequence of processes $f_n(., Y^n, Z^n)_n$ converges to $f(., Y, Z)$ in $\mathbb{L}^2$.
We have

$$\begin{aligned}&\mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 ds\\ &= \mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 \mathbf{1}_{A_n^N} ds\\ &+ 2\mathbb{E}\int_0^1 \mid f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n) \mid^2 \mathbf{1}_{\overline{A}_n^N} ds\\ &+ 2\mathbb{E}\int_0^1 \mid f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 \mathbf{1}_{\overline{A}_n^N} ds\\ &\leq \frac{K_1}{N^{2(1-\alpha)}} + 2\rho_N^2(f_n - f) + I(n),\end{aligned}$$

where

$$I(n) = 2\mathbb{E}\int_0^1 \mid f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^2 \mathbf{1}_{\overline{A}_n^N} ds.$$

Since (Y^n, Z^n) converges to (Y,Z) in $\mathbb{L}$, we get for a subsequence, which still denote (Y^n, Z^n), that

$$f(s, Y_s^n, Z_s^n) \longrightarrow f(s, Y_s, Z_s), \quad d\mathbb{P} \times dt - a.e. \text{ as } n \text{ goes to } +\infty.$$

But for $\varepsilon = \dfrac{2-2\alpha}{\alpha}$, we have

$$\begin{aligned}&\mathbb{E}\int_0^1 \mid f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s) \mid^{2+\varepsilon} ds\\ &\leq \mathbb{E}\int_0^1 (2 + \mid Y_s \mid^2 + \mid Y_s^n \mid^2 + \mid Z_s \mid^2 + \mid Z_s^n \mid^2) ds\\ &< +\infty.\end{aligned}$$

Hence

$$\lim_{n\to+\infty} I(n) = 0.$$

Therefore

$$\lim_{n\to+\infty} \mathbb{E}\int_0^1 \mid f(s,Y_s^n,Z_s^n) - f(s,Y_s,Z_s) \mid^2 ds = 0.$$

Lemma 3.3 is proved. ■

Proof of Theorem 3.1.

Existence. The proof of assertion (a) can be derived from Lemma 3.3 by passing to the limit successively on n, m and N.
Let us prove assertion (b). Arguing as in assertion (a) and assume first that $L_N^2 + 2\mu_N^+ \leq L + 2(1-\alpha)\log(N)$, we show that

$$\mathbb{E}(\sup_{0\leq t\leq 1} \mid Y_t^n - Y_t^m \mid^2) \leq C\left[[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)]N^{2(1-\alpha)} + \frac{K_5(M,\xi)}{L_N^2 + 2\mu_N^+}\right] e^{(2+L)}$$

$$\big(\mathbb{E}\int_0^1 |Z_s^n - Z_s^m|^2 ds\big)^2 \leq C\left[[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)]N^{2(1-\alpha)} + \frac{K_5(M,\xi)}{L_N^2 + 2\mu_N^+}\right] e^{(2+L)}.$$

We can assume that L_N or μ_N goes to infinity (see Remark 3.2). Passing to the limit we get the desired result. Assume now that $L_N^2 + 2\mu_N^+ \leq L + \sqrt{\log(N)}$. Let δ be a strictly positive number such that $\delta < \frac{(1-\alpha)}{2}$. Let $([t_{i+1}, t_i])$ be a subdivision of $[0,1]$ such that $|t_{i+1} - t_i| \leq \delta$. Applying Lemma 3.3 in all the subintervals $[t_{i+1}, t_i]$ we get the existence proof.

Uniqueness. Let $\{(Y_t, Z_t, K_t)\ \ 0 \leq t \leq 1\}$ and $\{(Y_t', Z_t', K_t')\ \ 0 \leq t \leq 1\}$ be two solutions of our BSDE, we put

$$\{(\Delta Y_t, \Delta Z_t, \Delta K_t)\ \ 0 \leq t \leq 1\} = \{(Y_t - Y_t', Z_t - Z_t', K_t - K_t')\ \ 0 \leq t \leq 1\}$$

It follows from Itô's formula that

$$\mathbb{E}\left[|\Delta Y_t|^2 + \int_t^1 |\Delta Z_s|^2\, ds)\right] = 2\mathbb{E}\int_t^1 \langle \Delta Y_s, f(s,Y_s,Z_s) - f(s,Y_s',Z_s')\rangle\, ds$$
$$+ 2\mathbb{E}\int_t^1 \langle \Delta Y_s, d\Delta K_s\rangle$$

By Lemma 2.4 we get

$$\mathbb{E}\int_t^1 \langle \Delta Y_s, d\Delta K_s\rangle \leq 0.$$

For $N > 1$, let μ_N denote the monotony constant of f in the balls $B(0,N)$, $\mathbf{1}_{A^N} := \{(s,w); \mid Y_s \mid^2 + \mid Y_s' \mid^2 + \mid Z_s \mid^2 + \mid Z_s' \mid^2 \geq N\}$, $\overline{A}^N := \Omega \setminus A^N$.

$$\mathbb{E}\left[|\Delta Y_t|^2 + \int_t^1 |\Delta Z_s|^2\, ds\right] \leq I_1(N) + I_2(N),$$

where

$$I_1(N) = 2\mathbb{E}\int_t^1 \langle \Delta Y_s, f(s,Y_s,Z_s) - f(s,Y_s',Z_s)\rangle \mathbf{1}_{\overline{A}^N}\, ds$$
$$+ 2\mathbb{E}\int_t^1 \langle \Delta Y_s, f(s,Y_s',Z_s) - f(s,Y_s',Z_s')\rangle \mathbf{1}_{\overline{A}^N}\, ds,$$

and

$$I_2(N) = 2\mathbb{E}\int_t^1 \langle \Delta Y_s, f(s,Y_s,Z_s) - f(s,Y_s',Z_s')\rangle \mathbf{1}_{A^N}\, ds.$$

We shall estimate $I_1(N)$ and $I_2(N)$. As above we obtain

$$I_1(N) \leq (2\mu_N^+ + \gamma^2)\mathbb{E}\int_t^1 \mid \Delta Y_s \mid^2 \mathbf{1}_{\overline{A}^N} ds + \frac{L_N^2}{\gamma^2}\mathbb{E}\int_t^1 \mid \Delta Z_s \mid^2 ds,$$

and

$$I_2(N) \leq \beta^2\mathbb{E}\int_t^1 \mid \Delta Y_s \mid^2 \mathbf{1}_{A^N} ds + \frac{C}{\beta^2 N^{2(1-\alpha)}}$$

Taking $\beta^2 = L_N^2 + 2\mu_N^+$ and $\gamma^2 = L_N^2$ and using the estimates for $I_1(N)$ and $I_2(N)$, we have

$$\mathbb{E}\,|\Delta Y_t|^2 \leq (L_N^2 + 2\mu_N^+)\mathbb{E}\int_t^1 |\Delta Y_s|^2\, ds + \frac{C}{(L_N^2 + 2\mu_N^+)N^{2(1-\alpha)}}.$$

Using Gronwall's and Burkholder-Davis-Gundy inequalities, we get

$$\mathbb{E}\sup_{0\leq t\leq 1} |\Delta Y_t|^2 \leq \frac{C}{(L_N^2 + 2\mu_N^+)N^{2(1-\alpha)}}\exp(L_N^2 + 2\mu_N^+),$$

$$\mathbb{E}\int_0^1 |\Delta Z_s|^2\, ds \leq \frac{C}{(L_N^2 + 2\mu_N^+)N^{2(1-\alpha)}}\exp(L_N^2 + 2\mu_N^+),$$

the uniqueness follows by passing to the limit on N. ■

Suppose now that f is globally Lipschitz with respect to z, that is

$$\mid f(t,y,z) - f(t,y,z') \mid \leq L \mid z - z' \mid . \tag{iv'}$$

Remark 3.2. *Theorem 3.1 remains true under assumptions (i), (ii), (iii), (iv') and* $2\mu_N^+ \leq L + 2(1-\alpha)logN$*, for* $L > 0$.
Indeed, if μ_N *is also bounded the result of Theorem 3.1 follows from Pardoux [26]. Else, arguing as in the proof of Theorem 3.1 we obtain*

$$\mathbb{E}(\sup_{0\leq t\leq 1} \mid Y_t^n - Y_t^m \mid^2) \leq C\bigg(\big[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)\big]N^{2(1-\alpha)} + \frac{K_6(M,\xi)}{2\mu_N^+}\bigg)e^L$$

and

$$\mathbb{E}\int_0^1 |Z_s^n - Z_s^m|^2 ds \leq C\bigg(\big[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)\big]N^{2(1-\alpha)} + \frac{K_6(M,\xi)}{2\mu_N^+}\bigg)e^L.$$

Passing to the limit, we get the desired result.

Corollary 3.1. *Assume that (i), (ii), (iii) and (iv') hold. If* $\lim_N \dfrac{\exp^{2\mu_N^+}}{2\mu_N^+ N^{2(1-\alpha)}} = 0$*, then the RBSDE (1)-(5) has a unique solution. In particular, if* $\mu_N^+ \leq \log(N)$*, then (1)-(5) has also a unique solution.*

Proof of Corollary 3.1. Arguing as in the proof of Theorem 3.1, we show that

$$\mathbb{E}(\sup_{0\leq t\leq 1} \mid Y_t^n - Y_t^m \mid^2) \leq C\bigg(\big[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)\big] + \frac{K_6(M,\xi)}{2\mu_N^+ N^{2(1-\alpha)}}\bigg)e^{2\mu_N^+}$$

and

$$\mathbb{E}\int_0^1 |Z_s^n - Z_s^m|^2 ds \leq C\bigg(\big[\rho_N^2(f_n - f) + \rho_N^2(f_m - f)\big] + \frac{K_6(M,\xi)}{2\mu_N^+ N^{2(1-\alpha)}}\bigg)e^{2\mu_N^+}.$$

Passing to the limit on n, m, N and using the same arguments as in the proof of Theorem 3.1, one has the desired result. ■

4 Application to the perturbations of multivalued PDEs

Let $\{X_t^\varepsilon; t \geq 0\}$ be a diffusion process with values in $\mathbb{R}^d$, such that $X^\varepsilon \Longrightarrow X$ weakly in $\mathbb{C}([0,t],\mathbb{R}^d)$ equipped with the topology of convergence on compact subsets of $\mathbb{R}_+$, where X itself is a diffusion with generator L. We suppose that the martingale problem associated to X is well posed, and there exist $p, q \geq 0$ such that

$$\sup_\varepsilon \mathbb{E}(\mid X_t^\varepsilon \mid^{2p} + \int_0^t \mid X_s^\varepsilon \mid^{2q} ds) < \infty. \tag{4.1}$$

Moreover, we assume that $g : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ and $f : \mathbb{R}^d \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ are continuous, and that

$$\mid g(x) \mid \leq C(1 + \mid x \mid^p) \tag{4.2}$$

$$\mid f(x,y) \mid \leq C(1 + \mid x \mid^q + \mid y \mid^\alpha) \tag{4.3}$$

$$\langle f(x,y) - f(x,y'), y - y' \rangle \leq \mu_N \mid y - y' \mid^2, \tag{4.4}$$

Let $\{(Y_s^\varepsilon, Z_s^\varepsilon, K_s^\varepsilon); 0 \leq s \leq t)\}$ be the unique solution of the reflected BSDE

$$\left\{ \begin{array}{l} Y_s^\varepsilon = g(X_t^\varepsilon) + \int_s^t f(X_r^\varepsilon, Y_r^\varepsilon)dr - \int_s^t Z_r^\varepsilon dB_r + K_t^\varepsilon - K_s^\varepsilon \\ K_t^\varepsilon = -\int_0^t U_s^\varepsilon ds, \quad (Y^\varepsilon, U^\varepsilon) \in \mathbf{Gr}(\partial\phi), \end{array} \right. \tag{4.5}$$

where $\{B_s, 0 \leq s \leq t\}$ is a Brownian motion. Next, we shall prove that the family of processes $(X^\varepsilon; Y^\varepsilon; Z^\varepsilon; K^\varepsilon)$ converges in law to the unique solution (X, Y, Z, K) of the RBSDE

$$\left\{ \begin{array}{l} Y_s = g(X_t) + \int_s^t f(X_r, Y_r)dr - \int_s^t Z_r dB_r + K_t - K_s \\ K_t = -\int_0^t U_s ds, \quad (Y, U) \in \mathbf{Gr}(\partial\phi), \end{array} \right.$$

and then we shall apply this result to the homogenization of a class of multivalued PDE's.

Theorem 4.1. *(See Meyer-Zheng [22] or Kurtz [21]).*
The sequence of quasi-martingale $\{V_s^n; 0 \leq s \leq t\}$ defined on the filtred probability space $\{\Omega; \mathcal{F}_s, 0 \leq s \leq t; \mathbb{P}\}$ is tight if

$$\sup_n \big(\sup_{0 \leq s \leq t} \mathbb{E} \mid V_s^n \mid + CV_t(V^n) \big) < +\infty,$$

where $CV_t(V^n)$, denotes the "conditional variation of V^n on $[0,t]$" defined by

$$CV_t(V^n) = \sup \mathbb{E}(\sum_i \mid \mathbb{E}\big(V_{t_{i+1}}^n - V_{t_i}^n / \mathcal{F}_{t_i}) \mid \big),$$

with "sup" meaning that the supremum is taken over all partitions of the interval $[0,t]$.

We put

$$M_t^\varepsilon = -\int_0^t Z_s^\varepsilon dB_r.$$

We denote by:
- $\mathbb{C}([0,t],\mathbb{R}^d)$ the space of functions defined on $[0,t]$ with values in $\mathbb{R}^d$, equipped with the topology of uniform convergence.
- $\mathbb{D}([0,t],\mathbb{R}^k)$ the space of càdlàg functions defined on $[0,t]$ with values in $\mathbb{R}^k$, equipped with Meyer-Zheng topology.

The main result is the following.

Theorem 4.2. *Under the conditions quoted in the beginning of this section. If $\lim_{N\to\infty} \frac{e^{\mu_N^+}}{N^{2(1-\alpha)}} = 0$, then the family of processes $(X^\varepsilon, Y^\varepsilon, M^\varepsilon, K^\varepsilon)$ converges in law to (X, Y, M, K) on $\mathbb{C}([0,t],\mathbb{R}^d) \times \mathbb{D}([0,t],\mathbb{R}^{2k}) \times \mathbb{C}([0,t],\mathbb{R}^k)$.*

To do the proof of this theorem, we need the following lemmas

Lemma 4.1. *Let U^ε be a family of random variables defined on the same probability spaces. For each $\varepsilon \geq 0$, we assume the existence of a family of random variables $(U^{\varepsilon,n})_n$, such that*
- $U^{\varepsilon,n} \overset{\text{dist}}{\Longrightarrow} U^{0,n}$ *as ε goes to zero.*
- $U^{\varepsilon,n} \Longrightarrow U^\varepsilon$ *as $n \to +\infty$, uniformly in ε.*
- $U^{0,n} \Longrightarrow U^0$ *as $n \to +\infty$*

then, U^ε converge in distribution to U^0.

Proof : This lemma is a simplified version of Theorem 4.2 in [Billingsley [7], p.25]. ■

Consider the following backward stochastic differential equation

$$Y_s^{\varepsilon,n} = g(X_s^\varepsilon) + \int_s^t f(X_s^\varepsilon, Y_r^{\varepsilon,n})dr - \int_s^t Z_r^{\varepsilon,n} dB_r - \int_s^t A_n(Y_r^{\varepsilon,n})dr, \tag{4.6}$$

where $A_n(y)$ is defined as above.
Let (Y^n, Z^n) be the unique solution of the backward stochastic differential equation

$$Y_s^n = g(X_s) + \int_s^t f(X_s, Y_r^n)dr - \int_s^t Z_r^n dB_r - \int_s^t A_n(Y_r^n)dr.$$

We set

$$M_t^{\varepsilon,n} = -\int_0^t Z_r^{\varepsilon,n} dB_r \quad \text{and} \quad M_t^n = -\int_0^t Z_r^n dB_r.$$

Lemma 4.2. *Under assumptions of Theorem 4.2, for every n the family of processes $(Y^{\varepsilon,n}, M^{\varepsilon,n})$ converges in law to the the family of processes (Y^n, M^n) on $\mathbb{D}([0,t], \mathbb{R}^{2k})$.*

Proof . **Step1. A priori estimates**. Using standard arguments and (2.1) to show that

$$\sup_\varepsilon \mathbb{E}\big(\sup_{0\leq s\leq t} \mid Y_s^{\varepsilon,n} - a \mid^2 + \int_0^t \mid Z_r^{\varepsilon,n} \mid^2 dr + 2\gamma \int_s^t \mid A_n(Y_r^{\varepsilon,n}) \mid dr\big) < +\infty. \tag{4.7}$$

Step2. Tightness.
Clearly, we have

$$CV_t(Y^{\varepsilon,n}) \leq \int_0^t \mid f(X_r^\varepsilon, Y_r^{\varepsilon,n}) \mid dr + \int_0^t \mid A_n(Y_r^{\varepsilon,n}) \mid dr.$$

It follows from step 1 and (4.3) that

$$\sup_\varepsilon (CV_t\big((Y^{\varepsilon,n}) + \mathbb{E} \sup_{0\leq s\leq t} \mid Y_s^{\varepsilon,n} - a \mid^2 + \int_0^t \mid Z_r^{\varepsilon,n} \mid^2 dr)\big) < +\infty, \tag{3.4}$$

hence the sequence $\{(Y_s^{\varepsilon,n}, M_s^{\varepsilon,n}); 0 \leq s \leq t\}$ satisfies the Meyer-Zheng tightness criterion under $\mathbb{P}$.
Step3. Convergence in law.
By step2 there exists a subsequence (which we still denote $(Y^{\varepsilon,n}, M^{\varepsilon,n})$) such that

$$(Y^{\varepsilon,n}, M^{\varepsilon,n}) \Longrightarrow (Y^n, M^n),$$

on $(\mathbb{D}([0,t], \mathbb{R}^k))^2$, where the first factor is equipped with the topology of convergence in ds measure, and the second with the topology of uniform convergence.
Clearly, for each $0 \leq s \leq t$, $(x,y) \longrightarrow \int_s^t f(x(r), y(r))dr$ is continuous for $\mathbb{C}([0,t], \mathbb{R}^d) \times \mathbb{D}([0,t], \mathbb{R}^k)$ equipped with the same topology as above, and $y \longrightarrow \int_s^t A_n(y(r))dr$ is continuous in $\mathbb{C}([0,t], \mathbb{R}^k)$. We can now take the limit in (4.6), yielding as ε goes to 0

$$Y_s^n = g(X_s) + \int_s^t f(X_r, Y_r^n)dr + M_t^n - M_s^n - \int_s^t A_n(Y_r^n)dr.$$

Moreover, for any $0 \leq s_1 < s_2 \leq t$, $\phi \in \mathbb{C}_b^\infty$ and ψ_s a function of $X_r^\varepsilon, Y_r^{\varepsilon,n}$ $0 \leq r \leq t$, bounded and continuous in $\mathbb{C}([0,t], \mathbb{R}^d) \times \mathbb{D}([0,t], \mathbb{R}^k) \times \mathbb{C}([0,t], \mathbb{R}^d)$, we have

$$\mathbb{E}\big(\psi_{s_1}(X^\varepsilon, Y^{\varepsilon,n})(\phi(X_{s_2}^\varepsilon) - \phi(X_{s_1}^\varepsilon) - \int_{s_1}^{s_2} L\phi(X_r^\varepsilon)dr)\big) \longrightarrow 0 \quad \text{as} \quad n \to +\infty,$$

and for each $n \in \mathbb{N}$,

$$\mathbb{E}(\psi_{s_1}(X^\varepsilon, Y^{\varepsilon,n}) \int_0^\alpha (M^{\varepsilon,n}_{s_2+r} - M^{\varepsilon,n}_{s_1+r})dr) = 0.$$

From the weak convergence of $(X^\varepsilon, Y^{\varepsilon,n}, M^{\varepsilon,n})$ and the fact that $\mathbb{E}(\sup_{0\le s\le t} \mid M^{\varepsilon,n}_s \mid^2) < +\infty$, by dividing the second identity by α and letting α go to zero, we have

$$\mathbb{E}(\psi_{s_1}(X, Y^n)(\phi(X_{s_2}) - \phi(X_{s_1}) - \int_{s_1}^{s_2} L\phi(X_r)dr)\big) \longrightarrow 0,$$

$$\mathbb{E}(\psi_{s_1}(X, Y^n)(M^n_{s_2} - M^n_{s_1}) = 0.$$

Therefore, both M^n and M^X -the martingale part of X- are $\mathcal{F}^{X,Y^n}_t$ martingales.

Step4. Identification of the limit.

Let $(\overline{Y}^n, \overline{U}^n)$ denote the unique solution of the BSDE

$$\overline{Y}^n_s = g(X_t) + \int_s^t f(X_r, \overline{Y}^n_r)dr - \int_s^t \overline{U}^n_r dM^X_r - \int_s^t A_n(\overline{Y}^n_r)dr,$$

which satisfies $\mathbb{E}Tr\int_s^t \overline{U}^n_r < M^X >_r \overline{U}^n_r < +\infty$. Set also $\widetilde{M}^n_s = \int_0^s \overline{U}^n_r dM^X_r$. Since $\overline{Y}^n$ and $\overline{U}^n$ are $\mathcal{F}^X_t$ adapted, and M^X is $\mathcal{F}^{X,Y^n}_t$ martingale, hence so is $\widetilde{M}^n$.

Let β be a strictly positive number. For a given $N > 1$, we put $A^N_{n,m} := \{(s, \omega);\quad |Y^n_s|^2 + |\overline{Y}^m_s|^2 \ge N^2\}$, $\overline{A}^N_n := \Omega \setminus A^N_n$ and denote by $\mathbf{1}_E$ the indicator function of the set E. From Itô's formula, it follows that

$$\begin{aligned}
&\mathbb{E} \mid \overline{Y}^n_s - Y^n_s \mid^2 + \mathbb{E}[M^n - \widetilde{M}^n]_t - \mathbb{E}[M^n - \widetilde{M}^n]_s \\
&= 2\int_s^t \langle f(X_r, \overline{Y}^n_r) - f(X_r, Y^n_r), \overline{Y}^n_r - Y^n_r\rangle(\mathbf{1}_{\overline{A}^N_{n,m}} + \mathbf{1}_{A^N_{n,m}})dr \\
&- 2\int_s^t \langle A_n(\overline{Y}^n_r) - A_n(Y^n_r), \overline{Y}^n_r - Y^n_r\rangle dr.
\end{aligned}$$

Since A_n is monotone, we have for every $x, z \in \mathbb{R}^d$

$$\langle A_n(x) - A_n(z), x - z\rangle \ge 0.$$

Thus

$$\mathbb{E} \mid \overline{Y}^n_s - Y^n_s \mid^2 + \mathbb{E}[M^n - \widetilde{M}^n]_t - \mathbb{E}[M^n - \widetilde{M}^n]_s \le 2\mu^+_N \mathbb{E}\int_s^t \mid \overline{Y}^n_r - Y^n_r \mid^2 dr + \frac{C}{N^{2(1-\alpha)}}.$$

We conclude from Gronwall's lemma that

$$\mathbb{E} \mid \overline{Y}^n_s - Y^n_s \mid^2 + \mathbb{E}[M^n - \widetilde{M}^n]_t - \mathbb{E}[M^n - \widetilde{M}^n]_s \le \frac{C}{N^{2(1-\alpha)}} e^{2\mu^+_N t}.$$

Passing to the limit on N we obtain, $\overline{Y}^n_r = Y^n_r, 0 \le s \le t$, and $M^n = \widetilde{M}^n$. ■

Using the same argument as in the proof of Theorem 4.2 to show the following lemmas

Lemma 4.3. *Under the assumptions of Theorem 4.2 the family of processes $(Y^{\varepsilon,n}, M^{\varepsilon,n}, K^{\varepsilon,n})_n$ converges uniformly in $\varepsilon \in]0, 1]$ in probability to the family of processes $(Y^\varepsilon, M^\varepsilon, K^\varepsilon)$ as n goes to $+\infty$.*

Lemma 4.4. *Under the assumption of the above lemma, the family of processes (Y^n, M^n, K^n) converges in probability to (Y, M, K) as n goes to $+\infty$.*

Proof of Theorem 4.2

Combining the above lemmas, we find that $(X^\varepsilon, Y^\varepsilon, M^\varepsilon, K^\varepsilon)$ converge in law to (X, Y, M, K) in the sense defined as above, where

$$Y_s = g(X_t) + \int_s^t f(X_r, Y_r)dr - \int_s^t Z_r dB_r + K_t - K_s.$$

■

Corollary 4.1. *Under the assumptions of Theorem 4.2, $\{Y_0^\varepsilon\}$ converge to Y_0 as ε goes to 0.*

Proof : Since Y_0^ε is deterministic, we have

$$Y_0^\varepsilon = \mathbb{E}(g(X_t^\varepsilon) + \int_0^t f(X_s^\varepsilon, Y_s^\varepsilon)dr - K_t^\varepsilon).$$

Put

$$A_\varepsilon = g(X_t^\varepsilon) + \int_0^t f(X_s^\varepsilon, Y_s^\varepsilon)ds - K_t^\varepsilon,$$

we have

$$\mathbb{E} \mid A_\varepsilon \mid^2 \leq C(1+ \mid X_t^\varepsilon \mid^{2p}) + \mathbb{E}\int_0^t \mid Y_s^\varepsilon \mid^2 ds + \mathbb{E}\int_0^t \mid X_s^\varepsilon \mid^{2q} ds + \mathbb{E} \mid K_t^\varepsilon \mid^2 .$$

According to Lemma 4.3 and using assumption 4.1, we obtain

$$\sup_\varepsilon \mathbb{E} \mid A_\varepsilon \mid^2 < \infty.$$

Since Theorem 4.2 states that A_ε converge in law, as ε goes to 0, toward

$$g(X_t) + \int_0^t f(X_r, Y_r)dr + K_t,$$

the uniform integrability of A_ε implies that

$$\lim_{\varepsilon\to 0} \mathbb{E}(A_\varepsilon) = \mathbb{E}(\lim_{\varepsilon\to 0} A_\varepsilon).$$

This means that Y_ε converges to

$$Y_0 = g(X_t) + \int_0^t f(X_r, Y_r)dr + K_t.$$

■

Now, we apply our result to the proof of a stability result for PDEs.

5 Application to the viscosity solutions of multivalued PDEs

Let u^ε be the solution of the PDE

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial s}(s,x) - L_\varepsilon u^\varepsilon(s,x) - f(x, u^\varepsilon(s,x)) \in \partial\phi(u^\varepsilon(s,x)), for \quad s \in [0,t] \\ u^\varepsilon(0,x) = g(x), u^\varepsilon(t,x) \in \overline{Dom(\phi)}, x \in \mathbb{R}^d, \end{cases} \tag{4.8}$$

and u be the solution the following variational inequality

$$\begin{cases} \dfrac{\partial u}{\partial s}(s,x) - Lu(s,x) - f(x, u(s,x)) \in \partial\phi(u(s,x)), for \quad s \in [0,t] \\ u(0,x) = g(x), u(t,x) \in \overline{Dom(\phi)}, x \in \mathbb{R}^d. \end{cases} \tag{4.9}$$

Theorem 5.1. *Assume $k = 1$. Then, under conditions of Theorem 4.2, $u^\varepsilon(t,x)$ converge to $u(t,x)$ for all $(t,x) \in [0,t] \times \mathbb{R}^d$ as ε goes to 0.*

Proof . Let $x \in \mathbb{R}^d$ and $\{X_s^{x,\varepsilon}; 0 \leq s \leq t\}$ the diffusion process defined as above, starting at x. for all $t \in \mathbb{R}^+$, we denote by $(\{Y_s^{t,x,\varepsilon}, Z_s^{t,x,\varepsilon}, K_s^{t,x,\varepsilon}\}; 0 \leq s \leq t)$ the solution of the reflected BSDE

$$Y_s^{t,x,\varepsilon} = g(X_t^{x,\varepsilon}) + \int_s^t f(X_r^{x,\varepsilon}, Y_r^{t,x,\varepsilon})dr - \int_s^t Z_r^{t,x,\varepsilon} dB_r + K_t^{t,x,\varepsilon} - K_s^{t,x,\varepsilon}.$$

By virtue of Pardoux, Rascanu [29] (see also [25]), the function $u^\varepsilon : \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by $u^\varepsilon(t,x) = Y_0^{t,x}$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, is the unique viscosity solution of the PDE (4.8). Let $\{X_s^x; s \geq 0\}$ be the diffusion process with infinitesimal generator L, starting at $x \in \mathbb{R}$ and $(\{Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x}\}; 0 \leq s \leq t)$ be the unique solution of the RBSDE

$$Y_s^{t,x} = g(X_t^x) + \int_s^t f(X_r^x, Y_r^{t,x})dr - \int_s^t Z_r^{t,x} dB_r + K_t^{t,x} - K_s^{t,x}.$$

Again, in view of [29] (see also [25]) the function $u : [0,t] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by $u(t,x) = Y_0^{t,x}$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, is the unique viscosity solution of the PDE (4.9). Therefore, the result follows from corollary(4.1). ■

References

[1] K. Bahlali, Existence and uniqueness of solutions for backward stochastic differential equations with locally Lipschitz coefficient. *Elect. Comm. Proba.* 7, 169–179, (2002).

[2] K. Bahlali, E.H. Essaky and Y. Ouknine, Reflected backward stochastic differential equation with jumps and locally Lipschitz coefficient. Random Operators and Stochastic Equations, **10**, no. 4, pp. 335-350, (2002).

[3] K. Bahlali, B. Mezerdi and Y. Ouknine, Some generic properties in backward stochastic differential equations with continuous coefficient. Monte Carlo Method and applications. Monte Carlo 2000 conference at Monte Carlo, France, 3-5 jul. 2000. Monte Carlo Methods Appl. 7, no. 1-2, 15–19, (2001).

[4] V, Bally, M.E. Caballero, N. El Karoui, B. Fernandez, Reflected BSDE's, PDE's and variational inequality, prépublication de l'université Paris VI, Laboratoire de probabilités.

[5] V. Barbu, T. Precupanu, Convexity and Optimization in Banach Spaces (Mathematics and Its Applications. East European Series) 1986.

[6] G. Barles, H. M. Soner, P. E. Souganidis, Front propagation and phase field theory, SIAM J. Control Optim. **31**, 2, 439-469, (1993).

[7] P. Billingsley, Convergence of probability measures. Willey 1968.

[8] J.M. Bismut, Conjugate Convex Functions in Optimal Stochastic Control, J. Math. Anal. Apl., vol. **44**, 384-404, (1973).

[9] P. Briand, R. Carmona, BSDEs with polynomial growth generators, J. Appl. Math. Stochastic Anal. **13**, 207-238, (2000).

[10] E. Cépa, Equations Différentielles Multivoques, Thése de doctorat de l'université d'Orleans, (1994).

[11] R. Darling, É. Pardoux, (1997), Backward SDE with monotonicity and random terminal time, *Ann. of Probab.* **2**5, 1135-1159.

[12] A. Dermoune, S. Hamadène and Y. Ouknine, Backward stochastic differential equation with local time. *Stoc. Stoc. Reports.* 6, 103-119, (1999).

[13] N. El Karoui, S. Peng and M.C. Quenez, Backward stochastic differential equations in finance. *Mathematical Finance.* , 1-71, (1997).

[14] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng and M. C. Quenez, Reflected solutions of backward sde's, and related obstacle problems for pde's, Ann. Probab, **25**, 702–737, (1997).

[15] M. Escobedo, O. Kavian, H. Manato, Large time behavior of solutions of a dissipative semilinear heat equation, Comm. Partial Differential Equations **20**, 5, 645-659, (1996).

[16] A. Gegout-Petit, E. Pardoux, Equations differentielles stochastiques rétrogrades refléchies dans un convexe, Stochastics and Stochastics Reports, **57**, 111-128, (1996).

[17] S. Hamadène, Multidimentional Backward SDE's with uniformly continuous coefficients. (preprint).

[18] S. Hamadène, Equations differentielles stochastiques retrogrades, le cas localement lipschitzien. *Ann. Inst. Henri Poincaré.* **32**, 645-660, (1996).

[19] S, Hamadène, J.P, Lepeltier, Zero-sum stochastic differential games and BSDEs, Systems and control letteres, **24**, 259-263, (1995).

[20] S. Hamadène, Y. Ouknine, Reflected backward stochastic differential equations with jumps and random obstacle. Submitted (1999).

[21] T.G. Kurtz, Random time changes and convergence in distribution under the Meyer-Zheng conditions, Annals of Prob. **19**, 1010-1034, (1991).

[22] P.A Meyer, W.A. Zheng, Tightness criteria for laws of semimartingales, Ann. Inst. Henri Poincaré **20**, 353-372, (1984).

[23] M. N'zi, Multivalued backward stochastic differential equations with local lipschitz drift. Stochastic and Stoch. Reports **60**, pp. 205-218 (1998).

[24] M. N'zi, Y. Ouknine, Multivalued backward stochastic differential equations with continuous coefficient. Random. Oper. Stoch. Equations Vol.5, No 1, pp. 59-68, (1997).

[25] Y. Ouknine, Reflected BSDE with jumps, Stoch and Stoch Reports, **65**, pp.111-125, (1998).

[26] E. Pardoux, Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic PDEs of second order. The Geilo Workshop, 1996, L. Decreusefond, J. Gjerde, B. Oksendal, A.S. Ustünel eds., Birkhäuser, 79-127, (1998).

[27] E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation. System Control Lett. **14**, pp. 55-61, (1990).

[28] E. Pardoux, S. Peng, Backward stochastic differential equations and quasilinear parabolic partial differential equations. In: B. L. Rozovski, R. B. Sowers (eds). Stochastics partial equations and their applications, Lect. Notes control Inf. Sci. **176**, 200-217, Springer, Berlin, (1992).

[29] E. Pardoux, A. Rascanu, Backward SDE's with maximal monotone operator. Stoch. Proc. Appl.**76**, 191-215, (1998).

[30] S. Peng, Probabilistic interpretation for systems of quasilinear parabolic equations. Stochastics, **37**, 61-74, (1991).

[31] Y. Saisho, Stochastic differential equations for multidimensional domains with reflecting boundary. Prob. Theory and Rel. Fields, **74**, p.455-477, (1987).