BACKWARD STOCHASTIC DIFFERENTIAL EQUATION WITH TWO REFLECTING BARRIERS AND JUMPS

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Abstract

In this paper, by using a penalization as well as a fixed point methods, we prove existence and uniqueness of the solution for the one-dimensional reflected backward stochastic differential equation when the noise is driven by a Brownian motion and an independent Poisson point process.

Keys Words: Backward stochastic differential equation; Reflecting barriers; Penalization; Poisson point process; Martingale representation theorem; fixed point theorem.

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1. **Introduction**

Backward stochastic differential equations (BSDE's in short) is an interesting subject of present interest in stochastic calculus developed during the last decade from the pioneering works of Pardoux and Peng [19, 20]. The application of such equations to finance theory and nonlinear partial differential equations has motivated many efforts to establish existence and uniqueness of the solution (see [1, 2, 3, 4, 7, 16, 10, 17, 15, 12, 18] and the references given there).

In [9], El Karoui et *al* have introduced the notion of one barrier reflected BSDE, which is a backward equation but the solution is forced to stay above a given continuous obstacle. Moreover, the authors have established the existence and uniqueness of the solution via a penalization as well as a Picard's iteration methods. Carrying on this work, Hamadène and Ouknine [14] have generalized this result to one barrier reflected BSDE with jumps when the noise is driven by a Brownian motion and an independent Poisson random measure. They proved the existence and uniqueness of the solution if the barrier is no longer continuous but just right continuous left limited (*rcll* in short).

The notion of double barriers reflected BSDE has been introduced by Civitanic and Karatzas [5] where the solution is forced to remain between two described upper and lower barriers U and L. They proved the existence and uniqueness of the solution if either the barriers are regular or they satisfy the so-called Mokobodski condition which turns into the existence of a difference of a non-negative supermartingales between L and U.

In the present work, we wish to consider a more general equations: two barriers reflected BSDE with jumps when the solution is forced to stay between an upper and lower obstacles.

This can be formulated as follows:

$$(1.1)\quad \begin{cases} (i) \quad Y_t = \xi + \int_t^1 f(s, Y_s, Z_s, V_s)ds - \int_t^1 Z_s dW_s + (K_1^+ - K_t^+) - (K_1^- - K_t^-) \\ \qquad\qquad - \int_t^1 \int_\Lambda V_s(e)\,\tilde{\mu}(de, ds), \qquad t \leq 1 \\ (ii) \quad \forall t \leq 1,\ L_t \leq Y_t \leq U_t \text{ and } \int_0^1 (Y_t - L_t)dK_t^+ = \int_0^1 (U_t - Y_t)dK_t^- = 0;\ \mathbb{P}-a.s. \end{cases}$$

The obstacles L and U are given, as are the random variable ξ and the function f, and the unknowns are (Y, Z, K^+, K^-, V). Such equations appear when one studies the notion of zero-sum mixed problems [11] or American game options [6]. They also provide a probabilistic formulae to variational inequalities with two obstacles of differential-integral type.

In this paper, our aim is to show the existence and uniqueness of the solution for the reflected BSDE with jumps (1.1) if the upper barrier U is smooth and the lower barrier L is only right continuous left limited. In the proof of our result, we use a penalization method to show the existence of a solution when the function f does not depend on the solution and then, in the general case, we construct a contraction which has a fixed point which is the solution of our reflected BSDE with jumps (1.1).

The paper is organized as follows. The BSDE problem with reflection barriers and jumps as well as some preliminary results are described in Section 2. In Section 3 a standard penalization method is applied in order to prove existence and uniqueness of the solution when the coefficient does not depend on the solution. The general case is treated in Section 4 by using the result of Section 3 and a fixed point argument.

2. **Reflected backward stochastic differential equation with jumps**

2.1. **Notations and assumptions.** Let $(\Omega, F, I\!P, \mathcal{F}_t, W_t, \mu_t, t \in [0,1])$ be a complete Wiener-Poisson space in $I\!R^d \times I\!R^m \backslash \{0\}$, with Lévy measure λ, i.e. $(\Omega, F, I\!P)$ is a complete probability space, $(\mathcal{F}_t, t \in [0,1])$ is a right continuous increasing family of complete sub $\sigma-$algebras of F, $(W_t, t \in [0,1])$ is a standard Wiener process in $I\!R^d$ with respect to $(\mathcal{F}_t, t \in [0,1])$, and $(\tilde{\mu}_t,\ t \in [0,1])$ is a martingale measure in $I\!R^m \backslash \{0\}$ independent of $(W_t,\ t \in [0,1])$, corresponding to a standard Poisson random measure $p(t, A)$, namely, for any Borel measurable subset A of $I\!R^m \backslash \{0\}$ such that $\lambda(A) < \infty$, it holds :

$$\tilde{\mu}_t(A) = p(t, A) - t\lambda(A),$$

where

$$I\!E(p(t, A)) = t\lambda(A)$$

λ is assumed to be a $\sigma-$finite measure on $I\!R^m \backslash \{0\}$ with its Borel field, satisfying

$$\int_{I\!R^m \backslash \{0\}} \left(1 \wedge |x|^2\right) \lambda(dx) < +\infty.$$

In the sequel Λ stands for $I\!R^m \backslash \{0\}$ and $\mathcal{U}$ its Borel field. We assume that

$$\mathcal{F}_t = \sigma\left[\int_{A \times (0,s]} p(ds, dx); s \leq t, A \in \mathcal{U}\right] \vee \sigma[W_s, s \leq t] \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of $I\!P$-null sets and $\sigma_1 \vee \sigma_2$ denotes the σ-field generated by $\sigma_1 \cup \sigma_2$.

Let us introduce the following spaces:

• L^2 of $\mathcal{F}_1$-measurable random variables $\xi : \Omega \longrightarrow \mathbb{R}$ with $\mathbb{E} \mid \xi \mid^2 < +\infty$.

• S^2 of $\mathcal{F}_t$-adapted right continuous with left limit (*rcll* in short) processes $(Y_t)_{t \leq 1}$ with values in $\mathbb{R}$ and $\mathbb{E}[\sup_{t \leq 1} |Y_t|^2] < \infty$.

• $H^{2,k}$ of $\mathcal{F}_t$-progressively measurable processes with values in $\mathbb{R}^k$ such that $\mathbb{E}[\int_0^1 |Z_s|^2 ds] < \infty$.
• $\mathcal{L}^2$ of mappings $V : \Omega \times [0,1] \times \Lambda \to \mathbb{R}$ which are $\mathcal{P} \otimes \mathcal{U}$-measurable and $\mathbb{E}[\int_0^1 ds \int_\Lambda (V_s(e))^2 \lambda(de)] < \infty$; $\mathcal{P}$ is the $\sigma-$algebra of predictable sets in $\Omega \times [0,1]$.
• $\mathcal{A}^2$ of continuous, increasing, $\mathcal{F}_t$-adapted process $K : [0,1] \times \Omega \longrightarrow [0, +\infty($ with $K(0) = 0$ and $\mathbb{E}(K_1)^2 < +\infty$.
Finally, for a given *rcll* process $(w_t)_{t\le 1}$, $w_{t-} = \lim_{s \nearrow t} w_s, t \le 1$ $(w_{0-} = w_0)$; $w_- := (w_{t-})_{t\le 1}$.

Let ξ be a given random variable in L^2, and a map $f : \Omega \times [0,1] \times \mathbb{R}^{1+d} \times L^2(\Lambda, \mathcal{U}, \lambda; \mathbb{R}) \longrightarrow \mathbb{R}$ which is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^{1+d}) \times \mathcal{B}(L^2(\Lambda, \mathcal{U}, \lambda; \mathbb{R}))$-measurable and satisfies:
(i) $(f(t,0,0,0))_{t\le 1}$ belongs to $L^2(\Omega \times [0,1], dP \otimes dt)$ i.e., $\mathbb{E}\int_0^1 (f(t,0,0,0))^2 dt < +\infty$

(ii) f is uniformly Lipschitz with respect to (y,z,v), i.e., there exists a constant $k \ge 0$ such that for any y, y', z, $z' \in \mathbb{R}$ and v, $v' \in L^2(\Lambda, U, \lambda; \mathbb{R})$,

$$P-a.s., \; |f(\omega,t,y,z,v) - f(\omega,t,y',z',v')| \le k(|y-y'| + |z-z'| + \|v-v'\|).$$

Consider also two reflecting barriers L, U which are real valued and $\mathcal{P}$-measurable processes satisfying:

(j) $I\!\!E[\sup_{0\le t\le 1}\{(U_t^-)^2 + (L_t^+)^2\}] < +\infty, \quad L_t^+ := \max\{L_t, 0\}, \quad U_t^- := \max\{-U_t, 0\}$
(jj) $L_t \le U_t, \quad \forall\ 0 \le t \le 1, \quad L_1 \le \xi \le U_1,\ \mathbb{P}-a.s.$
(jjj) $\{L_t, 0 \le t \le 1\}$ is *rcll* and its jumping times are inaccessible stopping times
(jv) $\{U_t, 0 \le t \le 1\}$ is regular enough, i.e., it satisfies the following:

$$\begin{cases} \text{There exists a sequence of processes } (U^n)_{n\ge 0} \text{ such that} \\ (i)\forall t \le 1, U_t^n \ge U_t^{n+1} \text{ and } \lim_{n\to\infty} U_t^n = U_t, \mathbb{P}-a.s \\ (ii)\forall n \ge 0 \,\text{and}\, t \le 1, U_t^n = U_0^n + \int_0^t u_s^n ds + \int_0^t v_s^n dW_s + \int_0^t \int_\Lambda w_s^n(e)\tilde{\mu}\,(de, ds) \\ \text{where the processes}\, u^n, v^n, w^n \text{ are } \mathcal{F}_t\text{-adapted such that} \\ \sup_{n\ge 0} \sup_{0\le t\le 1} \mid u_t^n \mid \le M, \mathbb{E}\{\int_0^1 \mid v_s^n \mid^2 ds\}^{\frac{1}{2}} < \infty \,\text{and}\, \mathbb{E}\{\int_0^1 \int_\Lambda \mid w_s^n \mid^2 \lambda(de)ds\}^{\frac{1}{2}} < \infty, \forall n \ge 1. \end{cases}$$

We recall the Itô formula for *rcll* semimartingales.

2.2. **Itô's formula.** Let $X = \{X_t : t \in [0,T]\}$ be a *rcll* semimartingale, its quadratic variation is denoted by $[X] = \{[X]_t : t \in [0,T]\}$ and let F be a $\mathcal{C}^2$ real valued function, then $F(X)$ is also a semimartingale, and the following formula holds:

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t F'(X_{s-})dX_s + \frac{1}{2}\int_0^T F''(X_s)d[X]_s^c \\ &\quad + \sum_{0<s\le t} \{F(X_s) - F(X_{s-}) - F'(X_{s-})\Delta X_s\}. \end{aligned} \tag{2.1}$$

where $[X]^c$ (sometimes denoted by $\langle X \rangle$) is the continuous part of the quadratic variation $[X]$ and $\Delta X_s = X_s - X_{s-}$. We also note that in the case where $F(x) = x^2$, the formula (2.1) takes the form

$$X_t^2 = X_0^2 + \int_0^t 2X_{s-}dX_s + \int_0^t d[X]_s. \tag{2.2}$$

Moreover if X and Y are two càdlàg semimartingales then we have

$$X_tY_t = X_0Y_0 + \int_0^t X_{s-}dY_s + \int_0^t Y_{s-}dX_s + \int_0^t d[X,Y]_s. \tag{2.3}$$

where $[X,Y]$ stands for the quadratic covariation of X , Y also called the bracket process. For a complete survey in this topic we refer to Protter [21].

2.3. **One barrier reflected BSDE with jumps.** In this subsection, we present a result for existence and uniqueness for one single reflected BSDE with jumps.

Definition 2.1. *A solution for one barrier reflected BSDE with jumps is a quadruple* $(Y,Z,K,V) := (Y_t,Z_t,K_t,V_t)_{t\le 1}$ *of processes with values in* $\mathbb{R}^{1+d}\times\mathbb{R}^+\times L^2(\Lambda,\mathcal{U},\lambda;\mathbb{R})$ *and which satisfies :*

$$\begin{cases} (i) & Y\in S^2,\ Z\in H^{2,d} \text{ and } V\in\mathcal{L}^2;\ K\in S^2\,(K_0=0), \text{ is continuous and non-decreasing} \\ (ii) & Y_t = \xi + \int_t^1 f(s,Y_s,Z_s,V_s)ds + K_1 - K_t - \int_t^1 Z_sdW_s - \int_t^1\int_\Lambda V_s(e)\tilde{\mu}(ds,de)\,, t\le 1 \\ (iii) & \forall t\le 1, Y_t\ge L_t \text{ and } \int_t^1 (Y_t-L_t)dK_t = 0. \end{cases}$$

The following result established by Hamadène and Ouknine [14] is concerned with the existence and uniqueness of a solution for a single barrier reflected BSDE with jumps:

Theorem 2.1. *Under the above assumptions on* f*,* ξ *and* $(L_t)_{t\le 1}$*, the one barrier reflected BSDE with jumps associated with* (f,ξ,L) *has a unique solution.*

2.4. **Double barriers reflected BSDE with jumps.** Let us now introduce our double barriers reflected BSDE with jumps (in short, RDBSDE; "D" for discontinuous):

Definition 2.2. *The process* $\left(Y_t,Z_t,K_t^+,K_t^-,V_t\right)_{t\le 1}$*, with value in* $\mathbb{R}^{1+d}\times\mathbb{R}^+\times\mathbb{R}^+\times L^2\left(\Lambda,\mathcal{U},\lambda;I\!R^d\right)$*, is called a solution for the double barriers reflected BSDE with jumps if*

(2.4)

$$\begin{cases} (i) & Y\in S^2,\ Z\in H^{2,d},\ V\in\mathcal{L}^2;\ \text{ and } K^\pm\in\mathcal{A}^2 \\ (ii) & Y_t = \xi + \int_t^1 f(s,Y_s,Z_s,V_s)ds - \int_t^1 Z_sdW_s + (K_1^+-K_t^+) - (K_1^- - K_t^-) \\ & \qquad - \int_t^1\int_\Lambda V_s\,(e)\,\tilde{\mu}\,(de,ds)\,, \qquad t\le 1 \\ (iii) & \forall t\le 1,\ L_t\le Y_t\le U_t \text{ and } \int_0^1 (Y_t-L_t)dK_t^+ = \int_0^1 (U_t-Y_t)dK_t^- = 0;\ \mathbb{P}-a.s. \end{cases}$$

The main purpose of this paper is to show that equation (2.4) has a unique solution. To begin with, we assume that the generator f does not depend on (y,z,v), i.e., P-*a.s.*, $f(t,\omega,y,z,v)\equiv f(t,\omega)$, for any t,y,z and v.

3. The (y,z,v)-independent case

In this section, we are going to show the existence and uniqueness, under the above assumptions on f, ξ, L and U, of the solution of the following RDBSDE

$$Y_t = \xi + \int_t^1 f(s)ds - \int_t^1 Z_s dW_s + (K_1^+ - K_t^+) - (K_1^- - K_t^-) - \int_t^1 \int_\Lambda V_s(e)\,\tilde{\mu}(de,ds). \tag{3.1}$$

The main result is the following

Theorem 3.1. *The RDBSDE (3.1) has a unique solution (Y, Z, K^+, K^-, V).*

Let $(Y^n, Z^n, K^{+,n}, V^n)$ be the solution of the single barrier RDBSRE associated with $(f(s) - n(y-U_s)^+, \xi, L)$:

$$Y_t^n = \xi + \int_t^1 f(s)ds - \int_t^1 Z_s^n dW_s + (K_1^{+,n} - K_t^{+,n}) - n\int_t^1 (Y_s^n - U_t)^+ ds - \int_t^1 \int_\Lambda V_s^n(e)\,\tilde{\mu}(de,ds).$$

We have divided the proof of Theorem 3.1 into sequence of Lemmas.

Lemma 3.1. *For each $n \geq 0$, there exists a constant $M > 0$ such that*

$$\sup_{0\leq t\leq 1} n(Y_t^n - U_t)^+ \leq M,\ \mathbb{P} - a.s.$$

Proof. For each $n \geq 0$ and $k \geq 0$, let $(Y^{n,k}, Z^{n,k}, V^{n,k})$ be the solution of the following BSDE

$$\begin{aligned} Y_t^{n,k} = \xi &+ \int_t^1 f(s)ds - \int_t^1 Z_s^{n,k} dW_s - n\int_t^1 \left(Y_s^{n,k} - U_s\right)^+ ds + k\int_t^1 \left(Y_s^{n,k} - L_s\right)^- ds \\ &- \int_t^1 \int_\Lambda V_s^{n,k}(e)\,\tilde{\mu}(ds,de), \quad \forall t \leq 1. \end{aligned}$$

Set $\overline{Y}^{n,k} := Y^{n,k} - U^k$, then

$$\begin{aligned} \overline{Y}_t^{n,k} =& \xi - U_1^k + \int_t^1 u_s^k ds + \int_t^1 f(s)ds - \int_t^1 \left(Z_s^{n,k} - v_s^k\right) dW_s - n\int_t^1 \left(\overline{Y}_s^{n,k} - (U_s - U_s^k)\right)^+ ds \\ &+ k\int_t^1 \left(\overline{Y}_s^{n,k} - (L_s - U_s^k)\right)^- ds - \int_t^1 \int_\Lambda \left(V_s^{n,k} - w_s^k\right)(e)\,\tilde{\mu}(ds,de). \end{aligned}$$

For each $n \in \mathbb{N}$, let $\mathcal{D}^n$ denote the class of $\mathcal{P}$-measurable processes $\nu : [0,1]\times\Omega \longrightarrow [0,n]$. Let $\nu \in \mathcal{D}^n$ and $\mu \in \mathcal{D}^k$ then by applying Itô's formula to the product $\overline{Y}^{n,k}$ and $\exp(-\int_0^\cdot (\mu(r)+\nu(r))dr)$ and using the same arguments as in Cvitanic and Karatzas [5] (see also Matoussi et *al* [12]) one can show that

$$\begin{aligned} \overline{Y}_t^{n,k} =& essup_{\mu\in\mathcal{D}^k} essinf_{\nu\in\mathcal{D}^n} \mathbb{E}\{(\xi - U_1^k)\exp(-\int_t^1 (\mu(r)+\nu(r))dr) \\ &+ \int_t^1 \exp(-\int_t^s (\mu(r)+\nu(r))dr)\big(u_s^k + f(s) + \nu(s)(U_s - U_s^k) + \mu(s)(L_s - U_s^k)\big)ds/\mathcal{F}_s\} \end{aligned}$$

Therefore

$$\begin{aligned} \overline{Y}_t^{n,k} &= essup_{\mu\in\mathcal{D}^k} essinf_{\nu\in\mathcal{D}^n} \mathbb{E}\{\int_t^1 \exp(-\int_t^s (\mu(r)+\nu(r))dr) \mid u_s^k \mid /\mathcal{F}_s\} \\ &\leq essup_{\mu\in\mathcal{D}^k} \mathbb{E}\{\int_t^1 \exp(-\int_t^s (\mu(r)+n)dr) \mid u_s^k \mid /\mathcal{F}_s\} \leq \frac{M}{n}, \end{aligned}$$

from which the result follows. Lemma 3.1 is proved. ■

Lemma 3.2. *There exist two processes Y and K^+ such that*

$$\lim_{n\to+\infty} \mathbb{E}[\int_0^1 |Y_s^n - Y_s|^2 ds] = 0,$$
$$\lim_{n\to+\infty} \mathbb{E}[\sup_{0\le t\le 1} |K_s^{+,n} - K_s^+|^2] = 0.$$

Proof. Let $(\overline{Y}, \overline{Z}, \overline{K}, \overline{V})$ be the solution of the following BSDE associated with $f(t) - M, \xi, L$

$$\overline{Y}_t = \xi + \int_t^1 f(s)ds - \int_t^1 \overline{Z}_s dW_s + \overline{K}_1 - \overline{K}_t - \int_t^1 M ds - \int_t^1 \int_\Lambda \overline{V}_s(e)\,\tilde{\mu}(ds, de), \forall t \le 1.$$

Comparison theorem for ordinary BSDE with jumps implies that $(Y^n)_{n\ge 1}$ (resp. $(dK^n)_{n\ge 1}$) is non-increasing (resp. non-decreasing) sequence of processes and $\forall n \ge 1$, $\mathbb{P} - a.s$, $Y^n \ge \overline{Y}$ (resp. $dK^n \le d\overline{K}$). Hence there exist $\mathcal{P}$-measurable processes Y and K^+ such that $\mathbb{P} - a.s.$ $\forall t \le 1, Y_t^n \searrow Y_t$ and $K_t^n \nearrow K_t$ pointwisely as $n \to +\infty$. Now according to Hamadène and Ouknine [14] we have

$$\mathbb{E}\left((K_1^{+1})^2 + (\overline{K}_1)^2 + \int_0^1 (| Z_s^1 |^2 + | \overline{Z}_s |^2) ds + \sup_{0\le t\le 1} ((Y_t^1)^2 + \overline{Y}_t^2) \right) < +\infty. \tag{3.2}$$

Since $\forall n \ge 1$, $\mathbb{P} - a.s$, $Y^1 \ge Y^n \ge \overline{Y}$, by dominated convergence theorem we obtain that

$$\lim_{n\to+\infty} \mathbb{E}[\int_0^1 |Y_s^n - Y_s|^2 ds] = 0.$$

By Dini's theorem, since the process K^+ is continuous, we get also that

$$\lim_{n\to+\infty} \mathbb{E}[\sup_{0\le t\le 1} |K_s^{+,n} - K_s^+|^2] = 0.$$

Lemma 3.2 is proved. ■

Lemma 3.3. *There exists a constant $C \ge 0$ such that*

$$I\!E[\sup_{0\le t\le 1} | Y_t^n |^2 + (K_1^{+n})^2 + \int_0^1 |Z_t^n|^2\, dt + \int_0^1 \int_\Lambda |V_s^n(e)|^2 \lambda(de)\, ds] \le C, \quad \forall n \ge 1.$$

Proof. Since, for each $n \ge 1$, $\mathbb{P} - a.s.$, $Y^1 \ge Y^n \ge \overline{Y}_t$ and $K_1^{+1} \le K_1^{+n} \le \overline{K}_t$ and Thanks to (3.2) we get

$$I\!E[\sup_{0\le t\le 1} | Y_t^n |^2 + (K_1^{+n})^2] \le C, \quad \forall n \ge 1.$$

Now, it follows from Itô's formula that

$$\begin{aligned} Y_t^{n2} &+ \int_t^1 |Z_s^n|^2 ds + \int_{]t,1]} ds \int_\Lambda (V_s^n(e))^2 \lambda(de) + \sum_{t<s\le 1} (\Delta_s Y^n)^2 \\ &= \xi^2 + 2\int_{]t,1]} Y_s^n f(s) ds + 2\int_{]t,1]} Y_s^n dK_s^{+n} + 2\int_{]t,1]} n Y_s^n (Y_s^n - U_s)^+ ds - 2\int_{]t,1]} Y_{s-}^n Z_s^n dW_s \\ &-2\int_{]t,1]} Y_{s-}^n \int_\Lambda V_s^n(e) \tilde{\mu}(ds, de), \qquad t \le 1. \end{aligned}$$

Since $\int_0^{\cdot} Y_{s-}^n Z_s^n dW_s - \int_0^{\cdot} Y_{s-}^n \int_\Lambda V_s^n(e)\tilde{\mu}(ds,de)$ is a martingale we obtain,

$$\begin{aligned}
&\mathbb{E}[\int_t^1 |Z_s^n|^2 ds + \int_{]t,1]} ds \int_\Lambda (V_s^n(e))^2\lambda(de)] \\
&\le \mathbb{E}[\xi^2] + \mathbb{E}[\int_{]t,1]} (Y_s^n)^2 ds] + \mathbb{E}[\int_{]t,1]} (f(s))^2 ds] + \mathbb{E}[\sup_{t\le s\le 1} (Y_s^n)^2] + \mathbb{E}[(K_1^{+n})^2] \\
&+\alpha^2\mathbb{E}[\sup_{t\le s\le 1} (Y_s^n)^2] + \frac{1}{\alpha^2}\mathbb{E}[\int_{]t,1]} n(Y_s^n - U_s)^+ ds]^2;
\end{aligned} \tag{3.3}$$

But

$$n\int_0^1 (Y_s^n - U_s)^+ ds = \xi + K_1^{+n} - Y_0^n + \int_0^1 f(s)ds - \int_0^1 Z_s^n dW_s - \int_0^1\int_\Lambda V_s^n(e)\tilde{\mu}(ds,de),$$

Therefore

$$\mathbb{E}[\int_0^1 n(Y_s^n - U_s)^+ ds]^2 \le C(1 + \mathbb{E}[\int_0^1 |Z_s^n|^2 ds + \int_0^1 ds \int_\Lambda (V_s^n(e))^2\lambda(de)]).$$

Coming back to equation (3.3) and choosing $\alpha^2 = 2C$, we obtain

$$I\!\!E[\int_0^1 |Z_t^n|^2\, dt + \int_0^1 \int_\Lambda |V_s^n\,(e)|^2\,\lambda\,(de)\,ds] \le C, \quad \forall n \ge 1.$$

Lemma 3.3 is proved. ■

Lemma 3.4.

$$\lim_{n\to\infty} \mathbb{E}[\sup_{t\le 1} |(Y_t^n - U_t)^+|^2] = 0$$

Proof. Let $\left(\widehat{Y}_t^n, \widehat{Z}_t^n, \widehat{K}_t^n, \widehat{V}_t^n\right)_{t\le 1}$ be the solution of the following BSDE associated with $(f(t)-n(y-U_t), \xi, L)$

$$\widehat{Y}_t^n = \xi + \int_t^1 \{f(s) - n(\widehat{Y}_s^n - U_s)\}ds - \int_t^1 \widehat{Z}_s^n dW_s + \widehat{K}_1^n - \widehat{K}_t^n - \int_t^1\int_\Lambda \widehat{V}_s^n\,(e)\,\tilde{\mu}\,(ds,de)$$

By comparison theorem we have $\mathbb{P}-a.s.$, $\forall t \le 1$, $Y^n \le \widehat{Y}^n$ and $d\widehat{K}^n \le dK^{+,n} \le d\overline{K}$. Now let τ be an $\mathcal{F}_t$-stopping time such that $\tau \le 1$. Then,

$$\widehat{Y}_\tau^n = E[\xi \exp -n\,(1-\tau) + \int_\tau^1 (f(s) + nU_s)\exp -n\,(s-\tau)\,ds + \int_\tau^1 \exp[-n\,(s-\tau)]d\widehat{K}_t^n\,|\mathcal{F}_\tau].$$

Since U is regular and $\mathbb{E}[\sup_{0\le t\le 1} \mid U_t\mid^2] < +\infty$ we get

$$\xi\exp[-n\,(1-\tau)] + n\int_\tau^1 U_s \exp[-n\,(s-\tau)]ds \to \xi 1_{[\tau=1]} + U_\tau 1_{[\tau<1]} \text{ as } n\to\infty$$

$\mathbb{P}-a.s.$, and in $L^2(\Omega,\mathbb{P})$. Henceforth we have also the convergence of the conditional expectation in $L^2(\Omega,\mathbb{P})$. In addition

$$\left|\int_\tau^1 f(s)exp\{-n\,(s-\tau)\}ds\right| \le \frac{1}{\sqrt{n}}\left(\int_\tau^1 f^2(s)ds\right)^{\frac{1}{2}}$$

then

$$\int_\tau^1 f(s)exp - n\,(s-\tau)\,ds \longrightarrow 0 \text{ in } L^1(\Omega,P) \text{ as } n\to\infty.$$

Since

$$0 \le \int_\tau^1 \exp[-n\,(s-\tau)]d\widehat{K}_t^n \le \int_\tau^1 \exp[-n\,(s-\tau)]dK_t^{+n} \le \int_\tau^1 \exp[-n\,(s-\tau)]d\overline{K}_t \to 0,$$

in $L^1(\Omega,\mathbb{P})$ as $n \to +\infty$, we have

$$\widehat{Y}^n_\tau \longrightarrow \xi 1_{[\tau=1]} + U_\tau 1_{[\tau<1]} \text{ in } L^1(\Omega,\mathbb{P}) \text{ as } n\to\infty.$$

Therefore $Y_\tau \le U_\tau$ $P-a.s.$ From that and the section theorem ([8], p.220), we deduce that $Y_t \le U_t$, $\forall t\le 1$, $P-a.s.$ and then $(Y^n_t - U_t)^+ \searrow 0$, $\forall t \le 1$, P-$a.s.$

Now since $Y^n \searrow Y$ then, if we denote ${}^pX$ the predictable projection of any process X, ${}^pY^n \searrow {}^pY$ and ${}^pY \le U$. But for any n the jumping times of the process $(\int_0^t \int_\Lambda \bar{V}^n_s(e)\,\tilde{\mu}(ds,de))_{0\le t\le 1}$ are inaccessible since μ is a Poisson random measure. It follows that the jumping times of Y^n are also inaccessible. Then for any predictable stopping time δ we have $Y^n_\delta = Y^n_{\delta-}$, henceforth the predictable projection of Y^n is Y^n_-, i.e., ${}^pY^n = Y^n_-$.

So we have proved that ${}^pY^n \searrow {}^pY \le U$, i.e., $Y^n_- \searrow {}^pY \le U$, hence $Y^n_- - U \searrow {}^pY - U \le 0$. It follows that $\left(Y^n_{t-} - U\right)^+ \searrow 0$, $\forall t\le 1$ $\mathbb{P}-a.s.$ as $n\to\infty$. Consequently, from a weak version of the Dini's theorem ([8], p.202), we deduce that $\sup_{t\le 1}(Y^n_t - U_t)^+ \searrow 0$ $\mathbb{P}-a.s.$ as $n\to\infty$. Therefore the dominated convergence theorem implies

$$E[\sup_{t\le 1}|(Y^n_t - U_t)^+|^2] \longrightarrow 0\, a.s. \text{ as } n\to\infty$$

since for any $n\ge 0$, $(Y^n_t - U_t)^+ \le |\, Y^0_t \,| + |\, U_t\,|$. Lemma 3.4 is proved. ■

Set

$$K^{-n}_t := \int_0^t n\,(Y^n_s - U_s)^+\,ds$$

Lemma 3.5. *There exist $\mathcal{F}_t$-adapted processes $Z = (Z_t)_{t\le 1}$, $K^- = (K^-_t)_{t\le 1}$ (K^- non-decreasing and $K^-_0 = 0$) and $V = (V_t)_{t\le 1}$ such that*

$$\mathbb{E}[\int_0^1 |Z^n_s - Z_s|^2 ds + \sup_{t\le 1}|K^{-,n}_t - K^-_t|^2 + \int_0^1 ds \int_\Lambda |V^n_s(e) - V_s(e)|^2\lambda(de)] \to 0 \text{ as } n\to\infty.$$

Moreover

$$\lim_{n\to\infty}\mathbb{E}[\sup_{t\le 1}|Y^n_t - Y_t|^2] = 0.$$

Proof. Using Itô's formula we have for any $p\ge n\ge 0$ and $t\le 1$,
(3.4)

$$\begin{aligned}
&(Y^n_t - Y^p_t)^2 + \int_t^1 |Z^n_s - Z^p_s|^2 ds + \int_t^1 ds\int_\Lambda |V^n_s(e) - V^p_s(e)|^2\lambda(de) + \sum_{t<s\le 1}\Delta_s(Y^n - Y^p)^2\\
&= 2\int_t^1 (Y^n_s - Y^p_s)(dK^{+,n}_s - dK^{+p}_s) - 2\int_t^1 (Y^n_s - Y^p_s)(dK^{-,n}_s - dK^{-p}_s)\\
&-2\int_t^1 (Y^n_{s-} - Y^p_{s-})(Z^n_s - Z^p_s)dW_s - 2\int_t^1\int_\Lambda ds(Y^n_{s-} - Y^p_{s-})(V^n_s(e) - V^p_s(e))\tilde{\mu}(ds,de)\\
&\le 2\int_t^1 (Y^n_s - Y^p_s)(n(Y^n_s - U_s)^+ - p(Y^p_s - U_s)^+) - 2\int_t^1 (Y^n_{s-} - Y^p_{s-})(Z^n_s - Z^p_s)dW_s\\
&-2\int_t^1\int_U ds(Y^n_{s-} - Y^p_{s-})(V^n_s(e) - V^p_s(e))\tilde{\mu}(ds,de),
\end{aligned}$$

since $\mathbb{P}-a.s.$, $\int_t^1 (Y_s^n - Y_s^p)(dK_s^{+,n} - dK_s^{+p}) \leq 0$. Therefore

$$\begin{aligned}
&\mathbb{E}[\int_t^1 |Z_s^n - Z_s^p|^2 ds + \int_t^1 ds \int_\Lambda |V_s^n(e) - V_s^p(e)|^2 \lambda(de)] \\
&\leq 2\mathbb{E}[\int_t^1 (Y_s^p - U_s)^+ n(Y_s^n - U_s)^+ ds] + 2\mathbb{E}[\int_t^1 (Y_s^n - U_s)^+ p(Y_s^p - U_s)^+) ds] \\
&\leq \{\mathbb{E}\sup_{0\leq t\leq 1}[Y_s^p - U_s)^+]^2\}^{\frac{1}{2}} (\mathbb{E}\{\int_t^1 n(Y_s^n - U_s)^+ ds\}^2)^{\frac{1}{2}} \\
&+ \{\mathbb{E}\sup_{0\leq t\leq 1}[Y_s^n - U_s)^+]^2\}^{\frac{1}{2}} (\mathbb{E}\{\int_t^1 p(Y_s^p - U_s)^+ ds\}^2)^{\frac{1}{2}}
\end{aligned}$$

Using Lemma 3.4 and the fact that for each $n \geq 1$ $\mathbb{E}\{\int_t^1 n(Y_s^n - U_s)^+ ds\}^2 < +\infty$ we obtain

$$\mathbb{E}[\int_0^1 |Z_s^n - Z_s|^2 ds + \int_0^1 ds \int_\Lambda |V_s^n(e) - V_s^p(e)|^2 \lambda(de)] \to 0 \text{ as } n \to \infty.$$

It follows that $(Z^n)_{n\geq 0}$ and $(V^n)_{n\geq 0}$ are Cauchy sequences in complete spaces then there exist processes Z and V, respectively $\mathcal{F}_t$-progressively measurable and $\mathcal{P}\otimes\mathcal{U}$-measurable such that the sequences $(Z^n)_{n\geq 0}$ and $(V^n)_{n\geq 0}$ converge respectively toward Z and V in $L^2(d\mathbb{P}\otimes dt)$ and $L^2(d\mathbb{P}\otimes dt\lambda(de))$ respectively.

Now going back to (3.4), taking first the supremum then the expectation and using the BDG's inequality ([8], p.304) yields,

$$\begin{aligned}
&\mathbb{E}[\sup_{t\leq s\leq 1} (Y_s^n - Y_s^p)^2 + \int_t^1 |Z_s^n - Z_s^p|^2 ds + \int_0^1 ds \int_\Lambda |V_s^n(e) - V_s^p(e)|^2 \lambda(de)] \\
&\leq \{\mathbb{E}\sup_{0\leq t\leq 1}[Y_s^p - U_s)^+]^2\}^{\frac{1}{2}} (\mathbb{E}\{\int_t^1 n(Y_s^n - U_s)^+ ds\}^2)^{\frac{1}{2}} \\
&+ \{\mathbb{E}\sup_{0\leq t\leq 1}[Y_s^n - U_s)^+]^2\}^{\frac{1}{2}} (\mathbb{E}\{\int_t^1 p(Y_s^p - U_s)^+ ds\}^2)^{\frac{1}{2}} + \alpha\mathbb{E}[\sup_{t\leq s\leq 1} (Y_s^n - Y_s^p)^2] \\
&+ \alpha^{-1}\mathbb{E}[\int_t^1 |Z_s^n - Z_s^p|^2 ds] + \alpha^{-1}\mathbb{E}[\int_t^1 \int_\Lambda ds |V_s^n(e) - V_s^p(e)|^2 \lambda(de)], \quad t \leq 1,
\end{aligned}$$

where α is a universal real non-negative constant. Henceforth choosing $\alpha < 1/2$ implies that $\mathbb{E}[\sup_{0\leq s\leq 1}(Y_s^n - Y_s^p)^2] \to 0$ as $p, n \to \infty$ and then $\mathbb{E}[\sup_{0\leq s\leq 1}(Y_s^n - Y_s)^2] \to 0$ as $n \to \infty$, moreover $Y = (Y_t)_{t\leq 1}$ is an $\mathcal{F}_t$-adapted *rcll* process.
Set

$$K_t^- = Y_t - Y_0 + \int_0^t f(s)ds + K_t^+ - K_0^+ - \int_0^t Z_s dW_s - \int_0^t \int_\Lambda V_s(e)\tilde{\mu}(ds, de),$$

one can show easily, at least for a subsequence (which we still denote n), that

$$\mathbb{E}[\sup_{0\leq t\leq 1} | \int_t^1 n(Y_s^n - U_s)^+ ds - K_t^- |^2] \to 0 \text{ as } n \to +\infty. \tag{3.5}$$

Lemma 3.5 is proved. ■

Proof of Theorem 3.1. It remains to prove that the process (Y, Z, K^+, K^-, V) is a solution to the double barriers reflected backward stochastic differential equation. Obviously the process (Y, Z, K, V) satisfies

$$Y_t = \xi + \int_t^1 f(s)ds + (K_1^+ - K_t^+) - (K_1^- - K_t^-) - \int_t^1 Z_s dW_s - \int_t^1 ds \int_\Lambda V_s(e)\tilde{\mu}(ds, de), \forall t \leq 1.$$

On the other hand since $Y_t^n \leq L_t$ and $\lim_{n\to\infty} \mathbb{E}[\sup_{t\leq 1}((Y_t^n - U_t)^+)^2] = 0$ then $\mathbb{P}-a.s.$, $\forall t \leq 1$, $L_t \leq Y_t \leq U_t$.

Let us show that $\int_0^1 (Y_s - L_s)dK_s^+ = \int_0^1 (U_s - Y_s)dK_s^- = 0, \quad \mathbb{P} - a.s.$

$$\begin{aligned}\int_0^1 (Y_s - L_s)dK_s^+ &= \int_0^1 (Y_s - Y_s^n)dK_s^+ + \int_0^1 (Y_s^n - L_s)dK_s^+ \\ &= \int_0^1 (Y_s - Y_s^n)dK_s^+ + \int_0^1 (Y_s^n - L_s)(dK_s^+ - dK_s^{+n})\end{aligned}$$

Let ω be fixed. It follows from Lemma 3.5 that for any $\epsilon > 0$, there exists $n_0(\omega)$ such that for any $n \geq n_0(\omega)$, $\forall t \leq 1$, $Y_t(\omega) \leq Y_t^n(\omega) + \epsilon$. Hence

$$\int_0^1 (Y_s - Y_s^n)dK_s^+ \leq \varepsilon K_1^+(\omega). \tag{3.6}$$

On the other hand, since the function $Y(\omega) - L(\omega) : t \in [0,1] \longmapsto Y_t(\omega) - L_t(\omega)$ is *rcll* then there exists a sequence of step functions $(f^m(\omega))_{m\geq 0}$ which converges uniformly on $[0,1]$ to $Y(\omega) - L(\omega)$, i.e. there exists $m_0(\omega) \geq 0$ such that for $m \geq m_0(\omega)$ we have $\forall t \leq 1$, $|Y_t(\omega) - L_t(\omega) - f_t^m(\omega)| < \epsilon$. It follows that

$$\begin{aligned}\int_0^1 (Y_s - L_s)d(K_s^+ - K_s^{+,n}) &= \int_0^1 (Y_s - L_s - f_s^m(\omega))d(K_s^+ - K_s^{+,n}) + \int_0^1 f_s^m(\omega)d(K_s^+ - K_s^{+,n}) \\ &\leq \int_0^1 f_s^m(\omega)d(K_s^+ - K_s^{+,n}) + \epsilon(K_1^+(\omega) + K_1^{+n}(\omega)).\end{aligned}$$

But the right-hand side converge to $2\epsilon K_1^+(\omega)$, as $n \to \infty$, since $f^m(\omega)$ is a step function and then $\int_0^1 f_s^m(\omega)d(K_s^+ - K_s^{+,n}) \to 0$. Therefore we have

$$\limsup_{n\to\infty} \int_0^1 (Y_s - L_s)d(K_s^+ - K_s^{+,n}) \leq 2\epsilon K_1^+(\omega). \tag{3.7}$$

Now from (3.6) and (3.7) we deduce that

$$\int_0^1 (Y_s - L_s)dK_s^+ \leq 3\epsilon K_1(\omega).$$

As ϵ is whatever and $Y \geq L$ then

$$\int_0^1 (Y_s - L_s)dK_s^+ = 0.$$

Moreover, thanks to Lemma 3.5, equation (3.5) and Saisho lemma (see [22], p.465), at least for a subsequence, we get

$$\int_0^1 (U_s - Y_s^n)n(Y_s - U_s)^+ds \to \int_0^1 (U_s - Y_s)dK_s^-,$$

$\mathbb{P} - a.s.$ as $n \to +\infty$. Therefore $\int_0^1 (U_s - Y_s)dK_s^- \leq 0$. But, since $Y \leq U$, $P - a.s$, $\int_0^1 (U_s - Y_s)dK_s^- = 0$.

The other properties are satisfied by construction of processes (Y, Z, K^+, K^-, V) and the proof is over.

Now let us prove the uniqueness of the solution. If $(Y', Z', K^{+'}, K^{-'}, V')$ is another solution then using Itô's formula we obtain that

$$\begin{aligned}&\left|Y_t - Y'_t\right|^2 + \int_t^1 |Z_s - Z'_s|^2 ds + \int_t^1 \int_\Lambda \left(V_s(e) - V'_s(e)\right)^2 \lambda(de)ds + \sum_{t<s\leq 1} \Delta_s(Y-Y')^2 \\ &= 2\int_t^1 (Y_s - Y'_s)(dK_s^+ - dK_s^{+'}) - 2\int_t^1 (Y_s - Y'_s)(dK_s^- - dK_s^{-'}) \\ &- 2\int_t^1 (Y_s - Y'_s)(Z_s - Z'_s)dW_s - \int_t^1 \int_\Lambda [(Y_{s-} - Y'_{s-})(V_s - V'_s)(e)\tilde{\mu}(ds, de).\end{aligned}$$

But, $\int_t^1 (Y_s - Y'_s)(Z_s - Z'_s)dW_s - \int_t^1 \int_\Lambda [(Y_{s-} - Y'_{s-})(V_s - V'_s)(e)\tilde{\mu}(ds, de)$ is a martingale and $\int_t^1 (Y_s - Y'_s)(dK_s^+ - dK_s^{+'}) \leq 0$ and $\int_t^1 (Y_s - Y'_s)(dK_s^- - dK_s^{-'}) \geq 0$ then

$$\left|Y_t - Y'_t\right|^2 + \int_t^1 |Z_s - Z'_s|^2 ds + \int_t^1 \int_\Lambda \left(V_s(e) - V'_s(e)\right)^2 \lambda(de)ds \leq 0.$$

Put $K = K^+ - K^-$ and $K' = K^{+'} - K^{-'}$, we get $Y = Y', Z = Z', K = K'$ and $V = V'$. Finally let us show that $K^+ = K^{+'}$ and $K^- = K^{-'}$. $\forall t \leq 1$, $\int_0^t (L_s - Y_s)dK_s = \int_0^t (L_s - Y_s)dK'_s$. But, $\int_0^t (L_s - Y_s)dK_s = -\int_0^t (L_s - Y_s)dK_s^- = -\int_0^t (U_s - L_s)dK_s^-$. In the same way, $\int_0^t (L_s - Y_s)dK'_s = -\int_0^t (U_s - L_s)dK_s^{-'}$ and then $\int_0^t (U_s - L_s)dK_s^- = \int_0^t (U_s - L_s)dK_s^{-'}$. Since $K_0^- = K_0^{-'}$ and $L_t < U_t$, we get $K^- = K^{-'}$. In the same way we obtain also that $K^+ = K^{+'}$. Theorem 3.1 is proved. ■

Now we prove existence and uniqueness of the following RDBSDE associated with $(f(t, y, z, v), \xi, L, U)$

$$Y_t = \xi + \int_t^1 f(s, Y_s, Z_s, V_s)ds - \int_t^1 Z_s dW_s + (K_1^+ - K_t^+) - (K_1^- - K_t^-) - \int_t^1 \int_\Lambda V_s(e)\,\mu(de, ds). \tag{3.8}$$

In the proof of our result, we construct a contraction which has a fixed point which is the solution of our RBSDE with jumps (3.8).

4. **The general case**

We are now in position to give the main theorem of this section.

Theorem 4.1. *The reflected BSDE with jumps (3.8) associated with (f, ξ, L, U) has a unique solution (Y, Z, K^+, K^-, V).*

Proof. It remains to show the existence which will be obtained via a fixed point of the contraction of the function Φ defined as follows:

Let $\mathcal{D} := S^2 \times H^{2,d} \times \mathcal{L}^2$ the space of $\mathcal{P}$-measurable processes (Y, Z, V) endowed with the norm,

$$\|(Y, Z, V)\|_\alpha = \{\mathbb{E}[\int_0^1 e^{\alpha s}(|Y_s|^2 + |Z_s|^2 + \int_\Lambda |V_s(e)|^2 \lambda(de))ds]\}^{1/2};\ \alpha > 0.$$

Let Φ be the map from $\mathcal{D}$ into itself which to (Y,Z,V) associates $\Phi(Y,Z,V)=(\tilde{Y},\tilde{Z},\tilde{V})$ where $(\tilde{Y},\tilde{Z},\tilde{K}^+,\tilde{K}^-,\tilde{V})$ is the solution of the reflected DBSDE associated with $(f(t,y,z,v),\xi,L,U)$. Let (Y',Z',V') be another triple of $\mathcal{D}$ and $\Phi(Y',Z',V')=(\tilde{Y}',\tilde{Z}',\tilde{V}')$, then using Itô's formula we obtain, for any $t\le 1$,

$$\begin{aligned}
&e^{\alpha t}(\tilde{Y}_t-\tilde{Y}'_t)^2+\alpha\int_t^1 e^{\alpha s}(\tilde{Y}_s-\tilde{Y}'_s)^2ds+\int_t^1 e^{\alpha s}|\tilde{Z}_s-\tilde{Z}'_s|^2ds+\\
&\int_t^1 e^{\alpha s}ds\int_\Lambda(\tilde{V}_s(e)-\tilde{V}'_s(e))^2\lambda(de)+\sum_{t<s\le 1}e^{\alpha s}(\Delta_s\tilde{Y}-\Delta_s\tilde{Y}')^2\\
&=(M_1-M_t)+2\int_t^1 e^{\alpha s}(\tilde{Y}_s-\tilde{Y}'_s)(d\tilde{K}^+_s-d\tilde{K}^{+'}_s)-2\int_t^1 e^{\alpha s}(\tilde{Y}_s-\tilde{Y}'_s)(d\tilde{K}^-_s-d\tilde{K}^{-'}_s)\\
&+2\int_t^1 e^{\alpha s}(\tilde{Y}_s-\tilde{Y}'_s)(f(s,Y_s,Z_s,V_s)-f(s,Y'_s,Z'_s,V'_s))ds
\end{aligned}$$

where $(M_t)_{t\le 1}$ is a martingale. But $\int_t^1 e^{\alpha s}(\tilde{Y}_s-\tilde{Y}'_s)(d\tilde{K}^+_s-d\tilde{K}^{+'}_s)\le 0$ and $\int_t^1 e^{\alpha s}(\tilde{Y}_s-\tilde{Y}'_s)(d\tilde{K}^-_s-d\tilde{K}^{-'}_s)\ge 0$ then for any $\varepsilon>0$ we have

$$\begin{aligned}
&\alpha\mathbb{E}[\int_t^1 e^{\alpha s}(\tilde{Y}_s-\tilde{Y}'_s)^2ds]+\mathbb{E}[\int_t^1 e^{\alpha s}|\tilde{Z}_s-\tilde{Z}'_s|^2ds]+\mathbb{E}[\int_t^1 e^{\alpha s}ds\int_\Lambda(\tilde{V}_s(e)-\tilde{V}'_s(e))^2\lambda(de)]\\
&\le 2\mathbb{E}[\int_t^1 e^{\alpha s}(\tilde{Y}_s-\tilde{Y}'_s)(f(s,Y_s,Z_s,V_s)-f(s,Y'_s,Z'_s,V'_s))ds]\\
&\le k\epsilon\mathbb{E}[\int_t^1 e^{\alpha s}(\tilde{Y}_s-\tilde{Y}'_s)^2ds]+\frac{k}{\epsilon}\mathbb{E}[\int_t^1 e^{\alpha s}\{|Y_s-Y'_s|^2+|Z_s-Z'_s|^2+\\
&\qquad\int_\Lambda|V_s(e)-V'_s(e)|^2\lambda(de)\}ds].
\end{aligned}$$

It implies that,

$$\begin{aligned}
&(\alpha-k\epsilon)\mathbb{E}[\int_t^1 e^{\alpha s}(\tilde{Y}_s-\tilde{Y}'_s)^2ds]+\mathbb{E}[\int_t^1 e^{\alpha s}(\tilde{Z}_s-\tilde{Z}'_s)^2ds]+\\
&\mathbb{E}[\int_t^1 e^{\alpha s}ds\int_\Lambda(\tilde{V}_s(e)-\tilde{V}'_s(e))^2\lambda(de)]\le\\
&\tfrac{k}{\epsilon}\mathbb{E}[\int_t^1 e^{\alpha s}\{|Y_s-Y'_s|^2+|Z_s-Z'_s|^2+\int_\Lambda|V_s(e)-V'_s(e)|^2\lambda(de)\}ds].
\end{aligned}$$

Now if α large enough and ϵ such that $k<\epsilon<\frac{\alpha-1}{k}$, then Φ is a contraction on $\mathcal{D}$ and it has a unique fixed point on $\mathcal{D}$ which is, with K^+ and K^-, the unique solution of RDBSDE associated with (f,ξ,L,U). ■

Remark 4.1. *(Regularity of processes K^- and K^+) Let $(Y^n,Z^n,K^{+,n},V^n)$ be the solution of the single barrier RDBSRE associated with $(f(s)-n(y-U_s)^+,\xi,L)$. From Lemma 3.3 we obtain that*

$$\mathbb{E}[\int_0^1(Y^n_s-U_s)^{+2}ds]\le\frac{C}{n^2}.$$

This inequality can be written as

$$\sup_{n\in\mathbb{N}^*}\mathbb{E}[\|K^{-,n}\|_{H^1(0,1;\mathbb{R}^d)}]<\infty$$

where $K^{-,n}_t=n\int_0^t(Y^n_s-U_s)^+ds, t\le 1$, and $H^1(0,1;\mathbb{R}^d)$ is the usual Sobolev space consisting of all absolutely continuous functions with derivative in $L^2(0,1)$. Hence the sequence $(K^{-,n})_n$ is bounded in the Hilbert space $L^2(\Omega;H^1(0,1;\mathbb{R}^d))$ and then there exists a subsequence *of* $(K^{-,n})_n$ *which converges weakly. The limiting process, which is actually* K^-*, belongs to* $L^2(\Omega; H^1(0,1;\mathbb{R}^d))$ *and then* $\mathbb{P} - a.s.$*,* $K^-_\cdot(\omega) \in H^1(0,1;\mathbb{R}^d)$ *i.e.* K^- *is absolutely continuous with respect to Lebesgue measure dt. If we suppose moreover that the lower barrier L is smooth one can prove that the process* K^+ *is also absolutely continuous with respect to Lebesgue measure.*

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