Quasi–linear parabolic SPDEs with continuous coefficients

K. Bahlali[a,1], M. Eddahbi[b,1] and E. Essaky[c,1]

[a]UFR Sciences, UVT, B.P. 132, 83957 La Garde Cedex, France.CPT, CNRS Luminy. Case 907, 13288 Marseille Cedx 9, France.

[b]Université Cadi Ayyad Département de Maths et Info. Faculté des Sciences et Techniques, Marrakech B.P. 618, Marrakech, Maroc

[c]Université Cadi Ayyad Faculté des Sciences Semlalia Département de Mathématiques, B.P. 2390 40000 Marrakech, Morocco.

Abstract

We deal with quasi–linear parabolic stochastic partial differential equations. We prove that in the sense of Baire category, almost all quasi–linear parabolic stochastic partial differential equations (SPDE) with continuous coefficient have the properties of existence and uniqueness of solutions, as well as the continuous dependence of solutions on the coefficient and the L^2-convergence of their Picard's approximations.

Key words. Parabolic stochastic partial differential equation, space–time white noise, Baire category theorem, meager set, G_δ–set.

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1 Introduction

We consider the following quasi–linear parabolic stochastic partial differential equation, driven by space–time white noise in one space dimension on $]0,+\infty[\times]0,1[$

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + b\left(t,x,u(t,x)\right) + \sigma\left(t,x,u(t,x)\right)\frac{\partial^2 W\left(t,x\right)}{\partial t \partial x} \tag{1.1}$$

with initial condition $u(0,x)=u_0(x)$ and either Neumann or Dirichlet boundary conditions, where $\frac{\partial^2 W(t,x)}{\partial t\partial x}$ is the formal derivative of the Brownian sheet.
We assume that $u_0 \in \mathcal{C}_0[0,1]$, the space of continuous functions $v:[0,1]\longrightarrow\mathbb{R}$ vanishing at

[1]bahlali@univ-tln.fr eddahbi@fstg-marrakech.ac.ma essaky@ucam.ac.ma

0 and 1.
As shown by Walsh (1986), one can give a rigorous meaning to equation (1.1), by means of a weak formulation which can be written in integrated form, given by the following evolution equation :

$$\begin{aligned} u(t,x) &= G_t(x,u_0) + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(s,y,u(s,y))W(dy,ds) \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x,y)b(s,y,u(s,y))\, dy\, ds, \end{aligned} \tag{1.2}$$

where $G_t(x,u_0) = \int_0^1 G_t(x,y)u_0(y)dy$ and $G_t(x,y)$ is the fundamental solution of the heat equation on $[0,T] \times [0,1]$ with the boundary conditions specified before, that means $G_t(\cdot,\cdot)$ is the Green kernel associated with the partial differential equation

$$\frac{\partial v(t,x)}{\partial t} = \frac{\partial^2 v(t,x)}{\partial x^2},$$

with the same boundary condition as those of $u(t,x)$. We shall refer in the sequel the equation (1.2) as $Eq(\sigma,b)$. More developments on the field of SPDEs can be found for instance in Pardoux (1993) and in Nualart (1995).

It is well–known that $Eq(\sigma,b)$ admits a unique strong solution when the coefficients $b(t,x,r)$ and $\sigma(t,x,r)$ are Lipschitz continuous in their third argument and satisfy the linear growth condition. In the two last decades a lot of works have tried to relax the Lipschitz assumptions on the coefficients, see for instance Gyöngy and Pardoux (1993a, 1993b), Bally and Gyöngy and Pardoux (1994), Gyöngy (1995), Donati and Pardoux (1993), Shiga (1994), Kotelenz (1992) and Eddahbi and Erraoui (1998). These results can be obtained as a consequence of stability properties and by means of Malliavin calculus. Almost all these papers have deal with the case of non degenerate diffusion coefficient.

In this paper, we deal with quasi–linear SPDE with continuous coefficients, for which we give a topological approach. We are concerned with the prevalence, in the sense of Baire categories, of SPDE which have the properties of existence and uniqueness of solutions, as well as the continuous dependence of solutions on the coefficient and the L^2-convergence of their associated successive approximations.

Prevalence problems are usually studied in many areas of mathematics (see e.g., Orlicz (1932), Halmos (1944), Rohlin (1948), Lasota-York (1973), De Blasi-Myjack (1978), Skorohod (1980), Heunis (1984), Simon (1994), Bahlali-Mezerdi-Ouknine (1996), Alibert-Bahlali (2001) and takes its origin from the earlier paper of Orlicz (1932) where it is shown that "most" ordinary differential equations with continuous coefficient have unique solutions. In the theory of SDE, the first result in this direction is due to Skorohod (1980) where the author has used it also to study the dependence of solutions on a parameter. The method developed in Skorohod (1980) is quite probabilistic and seems not to be related to those of deterministic equations. We can not give an analogue to Skorokhod's approach for SPDE since, this approach needs the solutions in the law sense and, unfortunately the notion of weak solution is actually not clear in SPDE's theory.

Here, we give an analytic approach by adapting some ideas used in [18, 15, 13, 2, 1] to our situation. We consider the space of bounded measurable functions $\big(\sigma(t,x,r),\, b(t,x,r)\big)$

which are continuous in r for almost all (t, x) and measurable in (t, x) for all r. We define an appropriate complete metric on it and then look at the prevalence, in the sense of Baire categories, of the set of all those $\big(\sigma(t,x,r),\, b(t,x,r)\big)$ such that:

(1) The corresponding SPDE $Eq(\sigma, b)$ have unique solution.

(2) The approximate solutions, given by the successive approximations associated to $Eq(\sigma, b)$, converge to the unique solution of $Eq(\sigma, b)$.

(3) The solutions of equation $Eq(\sigma, b)$ [When they exist] are continuous with respect to the coefficient $\big(\sigma(t,x,r),\, b(t,x,r)\big)$.

It is shown, by using the Baire categories theorem, that the set of coefficients $\big(\sigma(t,x,r),\, b(t,x,r)\big)$ having the three above properties is a set of a second category of Baire. See definition 2.2 below for the Baire category sets and Oxtoby's book [Ox] for more details on this subject. Since a set of the second category in a Baire space contains "almost all" the points of the space (it may be thought of as the topological analogue of the measure theoritical concept of a set whose complement is of measure zero), our results state that in some sense almost all SPDE with bounded continuous coefficient have solutions which satisfy the above properties (1), (2), (3).

The paper is organized as follows. Some definitions and notations, which we have needed, are given in Section 2. The existence and uniqueness of solutions are studied in Section 3. In Section, we deal with the dependence of the solution on its coefficients. In Section 5, we study the Picard's approximation.

2 Preliminaires and notations

Let $(\Omega,\ \mathcal{F},\ \mathbb{P})$ be a complete probability space. Let W be a space–time white noise on $[0,T]\times[0,1]$ i.e. $W = (W(A)\ ;\ A \in \mathcal{B}([0,T]\times[0,1]))$ is a centered gaussian process defined on $(\Omega,\ \mathcal{F},\ \mathbb{P})$ whose covariance function is given by

$$\mathbb{E}W(A)W(B) = \lambda\left(A \cap B\right),$$

where λ denotes Lebesgue measure on $[0,T]\times[0,1]$ and $\mathcal{B}(E)$ denotes the Borel field of subsets of the topological space E. We also denote by $\mathcal{P}$: the progressively measurable subsets of $\mathbb{R}_+ \times \Omega$. The two–parameter stochastic process $(W(t,x),\ (t,x)\in[0,T]\times[0,1])$ defined by

$$W(t,x) = W([0,t]\times[0,x]),$$

is a mean zero gaussian process with covariance function given by

$$\mathbb{E}W(t,x)W(s,y) = \lambda\left(R_{t\wedge s, x\wedge y}\right) = (t\wedge s)\,(x\wedge y)\,.$$

This process is called the Brownian sheet or the two–parameter Wiener process in $[0,T]\times[0,1]$. Note that we can choose a continuous version of $W(t,x)$ such that $W(t,x) = 0$ on the axes.

For each $t \in [0,T]$ we denote by $\mathcal{F}_t$ the $\sigma-$field generated by the random variables $\{W(s,x),\ (s,x)\in[0,t]\times[0,1]\}$, that is :

$$\mathcal{F}_t = \sigma\left(W\left(A\right), A \in \mathcal{B}\left([0,t]\times[0,1]\right)\right) \vee \mathcal{N},$$

where $\mathcal{N}$ is the class of $\mathbb{P}$–nul sets of $\mathcal{F}$.
We consider the following parabolic SPDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &= \dfrac{\partial^2 u}{\partial x^2}(t,x) + b\,(u)\,(t,x) + \sigma\,(u)\,(t,x)\dfrac{\partial^2 W}{\partial t \partial x}(t,x) \\ u(0,x) &= u_0(x) \text{ initial condition }, \end{cases}$$

where $b\,(u)\,(s,y) = b(s,y,u(s,y))$ and $\sigma\,(u)\,(s,y) = \sigma(s,y,u(s,y))$ and $u_0 \in \mathcal{C}_0[0,1]$, which we shall refer also as $Eq(\sigma, b)$.
We are looking for a continuous random field $(u(t,x)\ ;\ (t,x) \in \mathbb{R}_+ \times [0,1])$ such that $u(t,x)$ is $\mathcal{F}_t$ adapted and satisfies the following integral equation

$$\begin{aligned} u(t,x) &= G_t(x,u_0) + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma\,(u)\,(s,y)W(dy,ds) \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x,y)b\,(u)\,(s,y)dy\;ds. \end{aligned}$$

The definition of the kernel $G_t(x,y)$ shows that the following explicit expansion holds

$$G_t(x,y) = \varphi_t(y-x) + \gamma\varphi_t(y+x)$$

where

$$\varphi_t(x) = \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{+\infty} \exp\left(-\frac{(x-2n)^2}{4t}\right).$$

with $\gamma = 1$ if $\frac{\partial u}{\partial x}\,(t,0) = \frac{\partial u}{\partial x}\,(t,1) = 0$ (Neumann conditions) and $\gamma = -1$ if $u\,(t,0) = u\,(t,1) = 0$ (Dirichlet conditions).
Let us first recall some well–known properties of the Green kernel $G_t(x,y)$.
(**P.1**) For any $t \in\,]0,+\infty[\ ,\ x \in [0,1]$

$$\int_0^1 G_t(x,y)dy = 1.$$

(**P.2**) For any $s,t \in \mathbb{R}_+$ and $x,y \in [0,1]$

$$\int_0^1 G_t(x,y)G_s(y,z)\;dy = G_{t+s}(x,z).$$

(**P.3**) There exists a constant C such that for any $t \in I\!R_+$and $x,y \in [0,1]$

$$G_t(x,y) \leq \frac{C}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{4t}\right).$$

We shall states some preliminary lemma which will be useful in the sequel and gives information about the increments of the Green function $G_t(x,y)$ (see Walsh (1986) or Bally *et al.* (1995) for the proof).

Lemma 2.1 *For any* $\beta > 0$

$$\int_0^1 |G_t(x,y)|^\beta \, dy \le C(T,\beta)\ t^{\frac{1-\beta}{2}}. \tag{2.3}$$

Let $\beta \in \left]\frac{3}{2}, 3\right[$. *For any* $x, y \in [0,1]$ *and* $t \in [0,T]$

$$I(\beta) := \int_0^t \int_0^1 |G_{t-s}(x,z) - G_{t-s}(y,z)|^\beta \, dz\, ds \le C(T,\beta)\, |x-y|^{3-\beta}. \tag{2.4}$$

For any $\beta \in]1,3[,\ s,t \in [0,T],\ x \in [0,1]$ *with* $s \le t$

$$J(\beta) := \int_0^s \int_0^1 |G_{t-r}(x,y) - G_{s-r}(x,y)|^\beta \, dy\, dr \le C(T,\beta)\, |t-s|^{\frac{3-\beta}{2}}, \tag{2.5}$$

and

$$K(\beta) := \int_s^t \int_0^1 |G_{t-r}(x,y))|^\beta \, dy\, dr \le C(T,\beta)\, |t-s|^{\frac{3-\beta}{2}} \tag{2.6}$$

where $C(T,\beta)$ *is a constant depending only in* T *and* β.

We denote by $\mathcal{E}$ the set of $\mathbb{R}$–valued field u defined on $\mathbb{R}$ which are $\mathcal{F}_t$–adapted and such that

$$\|u\|^p = \sup_{t\in[0,1]} \mathbb{E} \int_0^1 |u(t,x)|^p dx < +\infty.$$

$(\mathcal{E}, \|.\|)$ is a Banach space.
Define a metric d on $\mathcal{E}$ by

$$d(u,v) = \|u - v\|.$$

Definition 2.1 *A solution of equation* $Eq(\sigma,0)$ *is a random field* u *which belongs to the space* $(\mathcal{E}, \|.\|)$ *and satisfies* $Eq(\sigma,0)$.

Throughout the paper the solutions of equation $Eq(\sigma,0)$ will be denoted by u^σ. For a given real number $M > 0$, we denote by $\mathcal{C}$ the set of functions $\sigma(s,y,r)$, defined on $[0,T]\times[0,1]\times\mathbb{R}$ with values in $\mathbb{R}$, which are continuous in r for almost all (s,y), measurable in (s,y) for all r and bounded. Let **Lip** be the subset of $\mathcal{C}$ consisting of functions f which are Lipschitz in r with linear growth condition uniformly in s and y.

Definition 2.2 *A Baire space is a separated topological space in which all countable intersection of dense open subsets is dense also. Let* $\mathbf{B}$ *be a Baire space. A subset* $\mathbf{F}$ *of* $\mathbf{B}$ *is said to be meager (or a set of first category in the Baire sense), if it is contained in a countable union of closed nowhere dense subsets of* $\mathbf{B}$. *The complement of a meager set is called a comeager (or residual or a set of second category).*

3 Existence and uniqueness

We denote by $\mathcal{R}_e$ the set of functions $\sigma \in \mathcal{C}$ for which equation $Eq(\sigma, 0)$ has a, not necessarily unique, solution and by $\mathcal{R}_u$ the subset of $\mathcal{C}$ which consists to all functions σ for which equation $Eq(\sigma, 0)$ has a unique solution.

Theorem 3.1 *$\mathcal{R}_u$ is a residual set in the Baire space $(\mathcal{C}, \rho)$.*

To prouve this theorem we need some lemmas and the following Lemma is not difficult to prove.

Lemma 3.1 *For $p > 6$, let $\rho_m(f) := (\int_0^1 \int_0^1 \sup_{|r| \leq m} |f(s, y, r)|^p dyds)^{\frac{1}{p}}$. Endowed with the distance*

$$\rho(f - g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(f - g)}{1 + \rho_n(f - g)}, \tag{3.7}$$

$(\mathcal{C}, \rho)$ is a complete metric space in which **Lip** *is dense.*

Lemma 3.2 *Let σ be an element of* **Lip**. *Let $(\sigma_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{R}_e$ and u_0^n be a sequence of $\mathcal{F}_0$–adapted, random field which belongs to $L^p(\Omega)$ for some $p > 6$. We assume that*

$$\rho(\sigma_n - \sigma) \longrightarrow 0 \quad \text{and} \quad \mathbb{E}\int_0^1 |u_0^n(x) - u_0(x)|^p dx \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Then u^{σ_n} converges to u^σ in $(\mathcal{E}, \|.\|)$.

Proof. Without loss of generality we may suppose that $u_0 = u_0^n = 0$ for each n. Let u^σ (resp. u^{σ_n}) be a solution of equation $Eq(\sigma, 0)$ (resp. $Eq(\sigma_n, 0)$), we have

$$\begin{aligned}
\mathbb{E}|u^n(t,x) - u(t,x)|^{2p} &\leq C_p \mathbb{E}\left(\int_0^t \int_0^1 G_{t-s}(x,y)^2 |\sigma_n(u^n)(s,y) - \sigma(u)(s,y)|^2 dyds\right)^p \\
&\leq C_p \int_0^t \int_0^1 \mathbb{E}|\sigma_n(u^n)(s,y) - \sigma(u)(s,y)|^{2p} dyds.
\end{aligned}$$

Set $H_t^n := \int_0^1 \mathbb{E}|u^n(t,x) - u(t,x)|^{2p} dx$, and $A_N = \{(s, y, \omega) : |u^n(s,y)|^2 + |u(s,y)|^2 > N^2\}$ and A_N^c denotes the complemretary set of A_N in Ω. Let us first note that by the linear growth condition one can prove that

$$\sup_{n>0} \sup_{t \in [0,T]} \int_0^1 \mathbb{E}[|u^n(t,x)|^p + |u(t,x)|^p] dx \ = C(p) \ < +\infty \quad \text{for all } p > 6 \tag{3.8}$$

Hence

$$\begin{aligned} H_t^n &\leq C_p \int_0^t \int_0^1 \mathbb{E}|\sigma_n(u^n)(s,y) - \sigma(u)(s,y)|^{2p} \mathbb{1}_{A_N} dyds \\ &+ C_p \int_0^t \int_0^1 \mathbb{E}|\sigma_n(u^n)(s,y) - \sigma(u)(s,y)|^{2p} \mathbb{1}_{A_N^c} dyds \\ &\leq \frac{C_p}{N^{2p}} + C_p \int_0^t \int_0^1 \mathbb{E}|\sigma_n(u^n)(s,y) - \sigma(u^n)(s,y)|^{2p} \mathbb{1}_{A_N^c} dyds \\ &+ C_p \int_0^t \int_0^1 \mathbb{E}|\sigma(u^n)(s,y) - \sigma(u)(s,y)|^{2p} \mathbb{1}_{A_N^c} dyds \\ &\leq \frac{C_p}{N^{2p}} + C_p \int_0^t \int_0^1 \sup_{|r|\leq N} |\sigma_n(s,y,r) - \sigma(s,y,r)|^{2p} dyds \\ &+ C_p L^{2p} \int_0^t H_s^n ds \end{aligned}$$

Therefore,

$$H_t^n \leq \frac{C_p}{N^{2p}} + C_p \rho_N^{2p}(\sigma_n - \sigma) + C_p L^{2p} \int_0^t H_s^n ds$$

Hence by Gronwall's lemma, we have

$$\sup_{t\in[0,1]} H_t^n \leq C_p \left(\frac{1}{N^{2p}} + \rho_N^{2p}(\sigma_n - \sigma) \right) \exp(C_p L^{2p})$$

Lemma 3.2 follows by passing to the limit successively on n and N. ∎

Now, we define the oscillation function $\mathrm{Osc} : \mathcal{C} \longrightarrow \mathbb{R}_+$ as follow,

$$\mathrm{Osc}(f) = \lim_{\delta\to 0} \sup\{d(u^{\sigma_1}, u^{\sigma_2}); \quad \sigma_i \in \mathbf{Lip} \quad \text{and} \quad \rho(\sigma - \sigma_{\mathbf{i}}) < \delta \quad \text{for} \quad \mathbf{i = 1, 2}\})$$

We then have the following

Lemma 3.3 *(i) If σ belongs to* **Lip** *then* $\mathrm{Osc}(\sigma) = 0$.
(ii) The function Osc *is upper semicontinuous on* **Lip**.
(iii) If $\mathrm{Osc}(\sigma) = 0$ *for a σ in $\mathcal{C}$, then equation $Eq(\sigma, 0)$ has at least one solution in $\mathcal{E}$.*

Remark 3.1 *Lemma 3.3 (iii) is a sufficient condition to ensure existence of solutions.*

Proof of Lemma 3.3. Assertion (i) is a consequence of Lemma 3.2.
Proof of (ii) Let (σ_n) be a sequence in $\mathcal{C}$ converging to a limit σ, which belongs to **Lip**. Assume that $lim_{n\to\infty}\mathrm{Osc}(\sigma_n) > 0$. Then there exists an $\varepsilon > 0$ and a subsequence (n_k) such that, for each k there exists two sequences $(\sigma_{n_k}^1)$ and $(\sigma_{n_k}^2)$ in **Lip** which satisfy:

$$\rho(\sigma_{n_k} - \sigma_{n_k}^1) < 1/n_k \qquad \text{and} \qquad \rho(\sigma_{n_k} - \sigma_{n_k}^2) < 1/n_k \tag{3.9}$$

and

$$d\big(u^{\sigma^1_{n_k}}, u^{\sigma^2_{n_k}}\big) > \varepsilon \tag{3.10}$$

Thus (3.9) and lemma 3.2 imply that $lim_{k\to\infty} d\big(u^{\sigma^1_{n_k}}, u^{\sigma^2_{n_k}}\big) = 0$. This contradicts (3.10). Assertion (ii) is proved. ∎

Proof of (iii), let $\sigma \in \mathcal{C}$. Since $\mathrm{Osc}(\sigma) = 0$, then there exists a decreasing sequence of strictly positive numbers δ_n ($\delta_n \downarrow 0$) such that

$$\sup\{d\big(u^{\sigma_1}), u^{\sigma_2}\big); \quad \sigma_i \in \mathcal{L} \quad \text{and} \quad \rho(\sigma - \sigma_i) < \delta_n \quad \text{for} \quad i = 1, 2\} < 1/n \tag{3.11}$$

But Lemma 3.1 implies that for each $n \in \mathbb{N}^*$, there exists a $\sigma_n \in \mathbf{Lip}$ such that $\rho(\sigma_n - \sigma) < \delta_n$. Since δ_n decreases, it follows from (3.11) that $d\left(u^{\sigma_n}, u^{\sigma_m}\right) < \max\left(\frac{1}{m}, \frac{1}{n}\right)$. Hence $(u^{\sigma_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(\mathcal{E}, \|.\|)$. Let u be its limit. We shall show that u satisfies equation $Eq(\sigma, 0)$. By the definition of the space $(\mathcal{E}, \|.\|)$ we have

$$\lim_{n\to\infty} \sup_{0\le s\le 1} \int_0^1 |u^{\sigma_n}(s,y) - u(s,y)|^2 dy = 0 \tag{3.12}$$

From (3.12), there exists a subsequence (n_k) such that

$$u^{\sigma_{n_k}}(t,x) \quad \text{converges to} \quad u(t,x) \qquad dP \times dt \times dx\text{-a.e} \tag{3.13}$$

It remains now to prove that for each $t \in [0,1]$

$$\lim_{n\to\infty} \int_0^t \int_0^1 G_{t-s}(x,y)\sigma_{n_k}(u^{\sigma_{n_k}})(s,y)W(ds,dy) = \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u)(s,y)W(ds,dy) \tag{3.14}$$

in Probability.

Without loss of generality we may assume that (3.14) holds without extracting subsequence. Let us set

$$I_n := \mathbb{E}\left|\int_0^t \int_0^1 G_{t-s}(x,y)\sigma_{n_k}(u^{\sigma_{n_k}})(s,y)W(ds,dy) - \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u)(s,y)W(ds,dy)\right|$$

$$I_n^{2p} \le \mathbb{E}\left|\int_0^t \int_0^1 G_{t-s}(x,y)^2 \left(\sigma_{n_k}(u^{\sigma_{n_k}}) - \sigma(u)(s,y)\right)^2 dyds\right|^p$$

Hence

$$\begin{aligned} I_n^{2p} \quad &\le \quad \frac{C_p}{N^{2p}} + C_p\rho_N(\sigma_n - \sigma)^{2p} \\ &\quad + C_p\mathbb{E}\int_0^t \int_0^1 \left|\sigma(u^{\sigma_{n_k}})(s,y) - \sigma(u)(s,y)\right|^{2p} dyds \end{aligned}$$

Lemma 3.2 shows thats $\lim_{n,N\to\infty} \frac{C_p}{N^{2p}} + C_p\rho_N(\sigma_n - \sigma)^{2p} = 0$. In another hand, since $\sigma \in \mathcal{C}$ then (3.14) implies that $\sigma(.,.,u^{\sigma_n}(.,.))$ converges to $\sigma(.,.,u(.,.))$ $d\mathbb{P} \times ds \times dx$–a.e. Hence the Lebesgue dominated convergence theorem allows us to deduce that

$$\lim_{n\to\infty} \mathbb{E}\int_0^t \int_0^1 \left|\sigma(u^{\sigma_{n_k}})(s,y) - \sigma(u)(s,y)\right|^{2p} dyds = 0.$$

This proves assertion (iii). ∎

Proof of Theorem 3.1. Lemma 3.2 and assertions (i) and (ii) of Lemma 3.3 imply that for each natural number n, the set $\mathcal{G}_n = \{\sigma \in \mathcal{C}; \quad \mathrm{Osc}(\sigma) < 1/n\}$ is a dense open subset of $(\mathcal{C}, \rho)$. Then by the Baire categories theorem the set $\mathcal{G} = \bigcap_{n \in \mathbb{N}^*} \mathcal{G}_n$ is a dense G_δ subset of the Baire space $(\mathcal{C}, \rho)$. Moreover, if $\sigma \in \mathcal{G}$ then lemma 3.3 (iii) implies that the corresponding equation $Eq(\sigma, 0)$ has one solution. Hence $\mathcal{G} \subset \mathcal{R}_e$. This implies that $\mathcal{R}_e$ is a residual subset in $(\mathcal{C}, \rho)$.

To prove that $\mathcal{R}_u$ is residual, we define the function $D_u : \mathcal{G} \longrightarrow \mathbb{R}_+$ as follows,

$$D_u(\sigma) = \sup\{d(u_1^\sigma, u_2^\sigma); u_i^\sigma \text{ is a solution to equation } Eq(\sigma, 0), i = 1, 2\}$$

and for each $n \in \mathbb{N}^*$ we put $\overline{\mathcal{G}}_n = \{\sigma \in \mathcal{G}; \quad D_u(\sigma) < 1/n\}$. By using Lemma 3.2 we see, as in the proof of Lemma 3.3 (ii), that the function D_u is upper semicontinuous on **Lip**. This implies that each $\overline{\mathcal{G}}_n$ contains the intersection of $\mathcal{G}$ and a dense open subset of $(\mathcal{C}, \rho)$. Thus the set $\overline{\mathcal{G}} = \bigcap_{n \in \mathbb{N}^*} \overline{\mathcal{G}}_n$ contains a dense G_δ subset of the Baire space $(\mathcal{C}, \rho)$. Hence it is residual in $(\mathcal{C}, \rho)$. Finally, if $\sigma \in \overline{\mathcal{G}}$ then the corresponding equation $Eq(\sigma, 0)$ has a unique solution. Thus $\overline{\mathcal{G}} \subset \mathcal{R}_u$. Theorem 3.1 follows. ■

4 Continuous dependence on the coefficient

For a given $\sigma \in \mathcal{C}$ we denote by $S(\sigma) = u^\sigma$ the solution of $Eq(\sigma, 0)$ when it exists.

Theorem 4.1 *There exists a second categoty set $\mathcal{R}_2$ such that the map $S : \mathcal{R}_2 \longrightarrow \mathcal{E}$ given by $S(\sigma) = u^\sigma$ is well defined and continuous at each point of $\mathcal{R}_2$.*

Proof. We shall show that S is continuous on $\overline{\mathcal{G}}$ (the dense G_δ set which has been defined in the proof of theorem 3.1). Suppose the contrary. Then there exist $\sigma \in \overline{\mathcal{G}}$, $\varepsilon > 0$ and a sequence $(\sigma_p) \subset \overline{\mathcal{G}}$ such that,

$$\lim_{p \to \infty} \rho(\sigma_p - \sigma) = 0 \qquad \text{and} \qquad d(S(\sigma)_p, S(\sigma)) \geq \varepsilon \quad \text{for each } p. \tag{4.15}$$

Fix $n \in \mathbb{N}$ such that $\varepsilon < 1/n$. Since $\overline{\mathcal{G}} \subset \mathcal{G}$ then there exists a decreasing sequence of strictly positive numbers δ_n ($\delta_n \downarrow 0$) and a sequence of functions $\sigma_n \in$ **Lip** such that,

$$\rho(g_n - \sigma) < \delta_n \qquad \text{and} \qquad d(S(g_n), S(\sigma)) < 1/n. \tag{4.16}$$

We choose p large enough as to have $\rho(\sigma_p - \sigma) < \delta_n - \rho(g_n - \sigma)$ then we use (4.16) to obtain $\rho(\sigma_p - g_n) < \delta_n$. Hence by lemma 3.2 we have $d(S(\sigma_p), S(g_n)) < 1/n$. Thus $d(S(\sigma_p), S(\sigma) \leq d(S(\sigma_p), S(g_n)) + d(S(g_n), S(\sigma)) < 1/n + 1/n < (2/3)\varepsilon$ which contradicts (4.15). Theorem 4.1 is proved. ■

5 The Picard successive approximations

For a given $f \in \mathcal{C}$ we denote by u_n^σ the sequence of propcesses defined by the following equation

$$\begin{cases} u_0^\sigma(t) &= \qquad G_t(x, u_0), \\ \\ u_{n+1}^\sigma(t,x) &= G_t(x, u_0) + \displaystyle\int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u_n^\sigma)(s,y)W(ds,dy) \end{cases} \tag{5.17}$$

Let $\mathcal{R}_3$ be the subset of $\mathcal{C}$ of all those $\sigma \in \mathcal{C}$ such that the corresponding sequence u_n^σ, defined by (5.17), converges in $(\mathcal{E}, \|.\|)$ to a solution u^σ of equation $Eq(\sigma, 0)$.

Theorem 5.1 *The set $\mathcal{R}_3$ is residual in $(\mathcal{C}, \rho)$.*

To prove this theorem we need the following lemma which is the analogue of the previous lemma 3.2.

Lemma 5.1 *Let σ be an element of* **Lip**. *Let $(\sigma_p)_{p\in\mathbb{N}}$ be a sequence in $\mathcal{R}_3$. We denote by u_n^σ [resp. $u_n^{\sigma_p}$] the sequence defined by equation (5.17). Assume that $\rho(\sigma_p - \sigma) \longrightarrow 0 \quad as \quad p \longrightarrow \infty$. Then $\lim_{p\to\infty} \sup_{n\in\mathbb{N}} \|u_n^{\sigma_p} - u_n^\sigma\| = 0$.*

Proof. Let u_n^σ (resp. $(u_n^{\sigma_p})$ be two solutions of equation (5.17) with coefficients σ and σ_p respectively. By Burkholder inequality we have for any $q > 6$

$$\mathbb{E}\int_0^1 |u_{n+1}^{\sigma_p}(t,x) - u_{n+1}^\sigma(t,x)|^q dx \leq C_q \mathbb{E}\int_0^t \int_0^1 \mid \sigma_p(u_n^{\sigma_p})(s,y) - \sigma(u_n^\sigma(s,y) \mid^q dyds$$

Let L be the Lipschitz constant of the function f. For a given positive number N let $A_{n,p}^N = \{(s,x,\omega); \quad |u_n^{\sigma_p}(s,x)|^2 + |u_n^\sigma(s,x)|^2 \geq N^2\}$ and $\overline{A}_{n,p}^N = \Omega \setminus A_{n,p}^N$ and denote by $\mathcal{X}_E$ the indicator function of the set E.
Put

$$\mathbb{E}\int_0^1 |u_{n+1}^{\sigma_p}(t,x) - u_{n+1}^\sigma(t,x)|^q dx = \varphi_{n+1}^p(t,q),$$

we argue as in the proof of lemma 3.2 to obtain the following inequality

$$\varphi_{n+1}^p(t,q) \leq \frac{C_q}{N^{2q}} + \rho_N^{2q}(\sigma_p - \sigma) + C_q L \int_0^t \varphi_n^p(s,q)ds.$$

Hence

$$\sup_{0\leq t\leq 1} \varphi_n^p(t,q) \leq \left(\frac{C_q}{N^{2q}} + \rho_N^{2q}(\sigma_p - \sigma)\right) e^{LC_q},$$

then we have

$$\sup_n \|u_n^{\sigma_p} - u_n^\sigma\|^q \leq \left(\frac{C_q}{N^{2q}} + \rho_N^{2q}(\sigma_p - \sigma)\right) e^{LC_q}.$$

We successively pass to the limit on p and N to get $\lim\limits_{p\to\infty} \sup\limits_n \|u_n^{\sigma_p} - u_n^\sigma\|^q = 0$ for each $t \in [0,1]$.
The Lemma 5.1 is proved. ∎

Proof of Theorem 5.1. Let $\tilde{\sigma} \in$ **Lip** and $k \in \mathbb{N}^*$. By lemma 5.1, there exists $\delta(\tilde{\sigma}, k) > 0$ such that for every $\sigma \in \mathcal{C}$ satisfying $\rho(\tilde{\sigma} - \sigma) < \delta(\tilde{\sigma}, k)$, the inequality $\|u_n^{\tilde{\sigma}} - u_n^\sigma\| < 1/k$ holds. By Lemma 3.1 and the Baire categories theorem the set $\mathcal{G}_1 = \bigcap_k \bigcup_{\tilde{\sigma}\in\mathbf{Lip}} \{\sigma \in \mathcal{C}; \rho(\tilde{\sigma} - \sigma) < \delta(\tilde{\sigma}, k)\}$ is a dense G_δ subset in the Baire space $(\mathcal{C}, \rho)$. We shall prove that for each $\sigma \in \mathcal{G}_1$ the sequence u_n^σ defined by (5.17) converges, in $(\mathcal{E}, \|.\|)$, to a solution of equation $Eq(\sigma, 0)$. Let $\sigma \in \mathcal{G}_1$ and $\varepsilon > 0$. We use lemma 5.1 and the fact that the sequence $u_n^{\tilde{\sigma}}$ converges for $\tilde{\sigma} \in$ **Lip** to show that a positive number N_0 exists such that for any $n, m \geq N_0$ the following holds

$$\|u_n^\sigma - u_m^\sigma\| \quad \leq \quad \|u_n^\sigma - u_n^{\tilde{\sigma}}\| + \|u_n^{\tilde{\sigma}} - u_m^{\tilde{\sigma}}\| + \|u_m^{\tilde{\sigma}} - u_m^\sigma\| < 3\varepsilon$$

Hence u_n^σ is a Cauchy sequence in the Banach space $(\mathcal{E}, \|.\|)$, and so its convergence follows. Let u be its limit. We shall show that u satisfies equation $Eq(\sigma, 0)$. Since u_n^σ converges to u in the space $(\mathcal{E}, \|.\|)$ we have immediatly

$$\lim_{n\to\infty} \mathbb{E} \sup_{0\le t\le 1} \sup_{0\le x\le 1} |u_{n+1}^\sigma(t,x) - u(t,x)|^2 = 0$$

Since f is bounded by M we have $E(|u_n^\sigma(t)|^2) \le M^2$ and then by Fatou's lemma we obtain $E(|u_n^\sigma(t) - u(t,x)|^2) \le 2M^2$. Thus the sequence (u_n^σ) converges to u in $L^2([0,1]\times\Omega)$. Since σ is bounded and continuous, then $\lim_{n\to\infty} \mathbb{E} \int_o^t \int_o^1 |\sigma(u_n^\sigma)(s,y) - \sigma(u)(s,y)|^2 dyds = 0$. Theorem 5.1 is proved. ∎

Remark. All the previous results remain valid when the coefficients grow at most linearly. This can be obtained by adapting the approximation lemma given in [1] and by using the result of [6].

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