Uniqueness of L^p solutions for multidimensional BSDEs and for systems of degenerate parabolic PDEs with superlinear growth generator.

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Abstract

We deal with the unique solvability of multidimensional backward stochastic differential equations (BSDEs) with a p-integrable terminal condition ($p > 1$) and a superlinear growth generator. We introduce a new *local condition*, on the generator (see Assumption (H4)), then we show that it ensures the existence and uniqueness, as well as the L^p-stability of solutions. Since the generator is of super linear growth, the uniform continuity is then not satisfied. Furthermore, the (local) monotony condition in the y-variable as well as the (local) Lipschitz condition in the z-variable are not needed. Since the assumptions we impose on the coefficient *are local in the three variables* y, z *and* ω, we then also cover the BSDEs with stochastic Lipschitz and/or stochastic monotone coefficient. Although we are focused on the multidimensional BSDEs, our uniqueness and stability results are new even in one-dimensional case. As application, we establish the existence and uniqueness of Sobolev solutions to systems of (possibly) degenerate semilinear parabolic partial differential equations (PDEs) having a super linear growth nonlinear term and a p-integrable terminal condition ($p > 1$). We cover certain systems of PDEs arising in physics, and in particular the logarithmic nonlinearity $u\log(|u|)$. The proofs we give are rather non-standard. And in particular, we introduce a new method which consists to show by using BSDEs that the uniqueness for a system of non-homogeneous semilinear PDEs can be derived from the uniqueness for the homogeneous PDE satisfied by its associated linear part.

1 Introduction

Let $(W_t)_{0\ \leq\ t\leq T}$ be a r-dimensional Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_t)_{0\ \leq\ t\leq T}$ denote the natural filtration of (W_t) such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, and ξ be an $\mathcal{F}_T$-measurable d-dimensional random variable. Let f be an $\mathbb{R}^d$-valued function defined on $[0,T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times r}$ such that for every $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{d\times r}$, the map $(t,\omega) \longmapsto f(t,\omega,y,z)$ is $\mathcal{F}_t$-progressively measurable. The BSDE under consideration is,

$$(E^{(\xi,f)}) \qquad Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dW_s \qquad 0 \leq t \leq T$$

The data ξ and f are respectively called the terminal condition and the coefficient or generator.

Since the paper [43], where the existence and uniqueness of solutions have been established for equation $(E^{(\xi,f)})$ under a uniformly Lipschitz generator f and a square integrable terminal data ξ, the theory of BSDEs has found further applications and has become a powerful tool in many fields such financial mathematics, optimal control, stochastic game, non-linear PDEs ... etc... The collected texts [24] give a useful introduction to the theory of BSDEs and some of their applications. See also [1, 2, 3, 4, 5, 6, 7, 9, 10, 16, 17, 25, 32, 37, 39, 40, 41, 42, 44, 45, 47] and the references therein.

Recently, the link between the solution of BSDEs and the "L_p-*theory of viscosity solution*" for PDEs with measurable coefficients ([18, 19]) has been established in [7].

It should be noted that, in contrast to the one dimensional case, a few results are known in *multi-dimensional BSDEs with local assumptions on the generator* f. This is partly due to the two following facts :

(i) The comparison methods which are the main tool in one dimensional BSDEs do not work in multidimensional case.

(ii) The usual localization procedure (by stopping times) is ineffective, especially for the z-variable. The first results which deal with the existence and uniqueness of solutions to multidimensional BSDEs *with local assumptions on the coefficient* f have been established in [1, 2, 3].

This paper is a detailed and completed version of [4]. The first part of the paper constitute a natural continuation and developments of our previous works [1, 2, 3]. It consists to establish the existence and uniqueness as well as the L^p-stability of strong solutions for BSDE $(E^{(\xi,f)})$ when *the terminal datum* ξ *is p-integrable* $(p>1)$ *[condition (H0)], the generator* f *is of superlinear growth in* (y,z) *[condition (H3)] and satisfies a new local assumption [condition (H4)].* The conditions we impose on the generator go beyond all existing ones in the literature of multidimensional BSDEs. For instance, we cover the nonlinearities $y\log(|y|)$ and $h(y)z\sqrt{|\log(|z|)|}$ which are, in our knowledge, not covered by the previous paper. Many other examples are listed in the second section of the paper. Actually, we allow to the generator to have the *strictly sub-quadratic growth*

$$| f(t,y,z) | \;\leq\; \eta_t + | y |^{\alpha} + | z |^{\alpha'},$$

for some $0<\alpha,\alpha'<2$ and some $q-$integrable process η with some $q>1$.

Due to the local assumptions on the generator, the usual techniques of BSDEs do not work in our situation. In the other hand, due to the superlinear growth of the generator, the techniques used in [1, 2] no longer work. Our approach, consists to establish a non standard a priori estimate between two solutions as follows : We consider two solutions (Y^1,Z^1) and (Y^2,Z^2) associated to suitable *parameters* (ξ_1,f_1) and (ξ_1,f_1), then we prove that for every $\varepsilon>0$ there exists $N_\varepsilon\in\mathbb{N}$ such that for every f satisfying our assumptions (see pp. 4 and 5 below for the assumptions),

$$\begin{aligned}&\mathbb{E}(\sup_{0\leq t\leq T}|Y_t^1-Y_t^2|^{\beta})+\mathbb{E}\big(\int_0^T|Z_s^1-Z_s^2|^2ds\big)^{\frac{\beta}{2}}\\&\qquad\leq\;\varepsilon\;+\;N_\varepsilon\big[\mathbb{E}(|\xi^1-\xi^2|^{\beta})+\mathbb{E}\int_0^T\big(\sup_{|y|,|z|\leq N_\varepsilon}(f_1-f)+\sup_{|y|,|z|\leq N_\varepsilon}(f_2-f)\big)(s,y,z)ds\big]\end{aligned}$$

where $\beta\in]1,2[$ is some constant.
The previous estimate allows to treat simultaneously the existence and uniqueness as well as, the L^p-stability of solutions. For instance, the existence (of solutions) is deduced by using a suitable approximation (ξ_n,f_n) of (ξ,f) and an appropriate localization procedure which is close to those given in [1, 2, 3]. However, in contrast to [3], we don't use the L^2-weak compactness of the approximating sequence (Y^n,Z^n). Here, we directly show that the sequence (Y^n,Z^n) strongly converges in some L^q space $(1<q<2)$ and, the limit satisfies the BSDE $(E^{(\xi,f)})$. Thus we can dispense both with weak convergence and the non-constructive choice of weakly convergent subsequences. We first establish the result for a small time duration, then for an arbitrary prescribed one.

To deal with the PDEs part, we start with the following example. Let $a(t,x)\geq0$ and consider the following semilinear Cauchy problem,

$$\begin{cases}\dfrac{\partial u}{\partial t}-a(t,x)\Delta u+u\log|u|=0 \;\text{ on }\; (0,\;\infty)\times\mathbb{R}^d\\ u(0^+)=\varphi>0\end{cases}\tag{1.1}$$

If we try to solve this PDE (1.1) by mathematical analysis methods, the nonlinear term $u\log|u|$ leads to difficulties that render ineffective the standard arguments, see e.g. [20, 31]. In the other hand, since the coefficient a can vanish, the solutions of PDE (1.1) will not be smooth enough, and therefore the uniqueness is rather hard to establish. The approach we present in this paper, to treat this kind of PDEs, is probabilistic and uses the BSDEs. The link between the strong solutions of BSDEs and the Sobolev solutions of semilinear PDEs was firstly established in [10] in the case where the nonlinear term F is at least uniformly Lipschitz and with sub-linear growth.

In this paper, we are concerned with the following system of (possibly degenerate) parabolic PDEs

$$(\mathcal{P}^{(g,F)}) \quad \begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \mathcal{L}u(t,x) + F(t,x,u(t,x),\sigma^*\nabla u(t,x)) = 0 & t \in]0,T[,\ x \in \mathbb{R}^k \\ u(T,x) = g(x) \quad x \in \mathbb{R}^k \end{cases}$$

where $\mathcal{L} := \dfrac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}\partial^2_{ij} + \sum_i b_i\partial_i$ And $\sigma : \mathbb{R}^k \longmapsto \mathbb{R}^{kr}$, $b : \mathbb{R}^k \longmapsto \mathbb{R}^k$, $g : \mathbb{R}^k \longmapsto \mathbb{R}^k$ and $F : [0,T] \times \mathbb{R}^k \times \mathbb{R}^d \times \mathbb{R}^{dr} \longmapsto \mathbb{R}^d$ are measurable functions.

Our main purpose consists to establish the existence and uniqueness of Sobolev solutions for the system of PDEs $(\mathcal{P}^{(g,F)})$, in the case where F is with super-linear growth in both u and ∇u, and satisfies the same conditions that f. This is done with g in some L^p-space, $p > 1$. Our result cover in particular the logarithmic nonlinearities $u\log(|u|)$ as well as $h(u)(\nabla u)\sqrt{|\log(|\nabla u|)|}$ where h is a suitable function. *The main feature consists to develop a method which allows us to prove that the uniqueness of the system of PDEs can be derived from the uniqueness of its associated BSDE. We first prove the existence and uniqueness in the class of solutions which are representable by BSDEs, and next we show that any solution is unique. To do this, we prove that 0 is the unique solution to the homogeneous linear PDE, then we use the BSDEs to derive the uniqueness for the non-homogeneous semilinear PDE.* More precisely, we prove that the system of semilinear PDEs

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \mathcal{L}u(t,x) + F(t,x,u(t,x),\nabla u(t,x)) = 0, & t \in]0,T[,\ x \in \mathbb{R}^k \\ u(T,x) = g(x), \quad x \in \mathbb{R}^k \end{cases}$$

has a unique solution *if and only if* 0 is the unique solution of the linear system

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \mathcal{L}u(t,x) = 0, & t \in]0,T[,\ x \in \mathbb{R}^k \\ u(T,x) = 0, \quad x \in \mathbb{R}^k \end{cases}$$

The paper is organized as follows. In section 2, we present the main result on BSDEs part and some illustrative examples. Section 3 is devoted to the proof of the result of section 2. The main result on PDEs part as well its proof, are treated in section 4.

We now give some motivations and short explanations for some topics which can be related to the present work.

• In terms of continuous-state branching processes, the logarithmic nonlinearity $u\log u$ corresponds to the Neveu branching mechanism. This process was introduced by Neveu in [38], and further studied in [11, 27, 28]. For instance, the super-process with Neveu's branching mechanism constructed in [27] is related to the Cauchy problem (1.1). Therefore, our result can be seen as an alternative approach to PDE (1.1), with possibly degenerate diffusion coefficient a.

• Since the degeneracy of the diffusion coefficient, our Proposition 4.2 (below) cover the PDE studied in [48] which arises in studying the motion of a particle acting under a force field perturbed by a noise (see e. g. Freidlin [30, 48] and Saintier [48]). Indeed, if $y(t) \in \mathbb{R}^d$ denotes such a motion acting under a force field $G(y, y')$ and W be the standard brownian motion, then $y(t)$ satisfies the SDE

$$y''(t) = G(y(t),\, y'(t)) + W'(t) \quad \text{with} \;\; y(0) = y_0,\; y'(0) = x_0$$

Setting $x(t) = y'(t)$ in the previous SDE, we obtain the system

$$\frac{d}{dt}\big(x(t),\, y(t)\big) = \big(G(x(t), y(t)),\, x(t)\big) + \big(W'(t),\, 0\big) \;\; \text{with} \;\big(x(0), y(0)\big) = \big(x_0, y_0\big)$$

The Kolmogorov operator associated to the previous system is degenerate and enter in our conditions.

• The logarithmic nonlinearities appear in some PDEs arising in physics, see e.g. [12, 13, 20, 21, 22, 31, 46, 49]. For instance, in [12] the construction of nonlinear wave quantum mechanics, based on Schrödinger-type equation, is with nonlinearity $-ku\log(|u|^2)$.

• The method we develop to study the systems of semilinear PDEs is based on BSDEs and the proofs are rather non-standard, especially for the uniqueness.

• Since the system of PDEs associated to the Markovian version of the BSDE $(E^{(\xi,f)})$ can be degenerate, our result also covers certain systems of first order PDEs.

• The BSDEs as well as the PDEs which we consider are interesting in themselves since the nonlinear part $f(t,y,z)$ can be neither locally Lipschitz in z nor locally monotone in y. Moreover, f is of a super linear growth than y and z, and hence it is also not uniformly continuous.

• It was proved recently (in [8]) that the BSDEs with logarithmic growth $|f(t,y,z)| \leq \eta_t + K|z|\sqrt{|\log|z||}$ (for some process η) appear in stochastic control problems.

2 First main result and some examples.

Throughout this paper, $p > 1$ is an arbitrary fixed real number and all the considered processes are $(\mathcal{F}_t)$-predictable.

2.1 Definition.

A solution of equation $(E^{(\xi,f)})$ is an $(\mathcal{F}_t)$-adapted and $\mathbb{R}^{d+dr}$-valued process (Y,Z) such that

$$\mathbb{E}\left(\sup_{t\leq T}|Y_t|^p + \Big(\int_0^T |Z_s|^2 ds\Big)^{\frac{p}{2}} + \int_0^T |f(s,Y_s,Z_s)|ds\right) < +\infty$$

and satisfies $(E^{(\xi,f)})$.

2.2 Assumptions

We consider the following assumptions on (ξ, f):

There exist $M \in \mathbb{L}^0(\Omega;\mathbb{L}^1([0,T];\mathbb{R}_+))$, $K \in \mathbb{L}^0(\Omega;\mathbb{L}^2([0,T];\mathbb{R}_+))$ and $\gamma \in]0, \dfrac{1\wedge(p-1)}{2}[$, such that (with $\lambda_s := 2M_s + \dfrac{K_s^2}{2\gamma}$) we have,

(H.0) $\quad \mathbb{E} \mid \xi \mid^p \; e^{\frac{p}{2}\int_0^T \lambda_s ds} < \infty,$

(H.1) f is continuous in (y,z) for almost all (t,ω)

(H.2) There exist η and $f^0 \in \mathbb{L}^0(\Omega\times[0,T];\mathbb{R}_+)$ satisfying

$$\mathbb{E}\Big(\int_0^T e^{\int_0^s \lambda_r dr}\eta_s ds\Big)^{\frac{p}{2}} < \infty\ , \qquad \mathbb{E}\Big(\int_0^T e^{\frac{1}{2}\int_0^s \lambda_r dr} f_s^0 ds\Big)^p < \infty$$

and such that :

$$\text{for every } t,y,z,\quad \langle y, f(t,y,z)\rangle \;\leq\; \eta_t + f_t^0|y| + M_t|y|^2 + K_t|y||z|$$

(H.3) There exist $\overline{\eta} \in \mathbb{L}^q(\Omega\times[0,T];\mathbb{R}_+))$ (for some $q > 1$) and $\alpha \in]1,p[, \alpha' \in]1, p\wedge 2[$ such that:

$$\text{for every } t,y,z,\quad \mid f(t,\omega,y,z)\mid \;\leq\; \overline{\eta}_t + \mid y\mid^{\alpha} + \mid z\mid^{\alpha'}.$$

(H.4) There exist $v \in \mathbb{L}^{q'}(\Omega\times[0,T];\mathbb{R}_+))$ (for some $q' > 0$) and $K' \in \mathbb{R}_+$ such that for every $N \in \mathbb{N}$ and every $y,\ y'\ z,\ z'$ satisfying $\mid y\mid, \mid y'\mid, \mid z\mid, \mid z'\mid \leq N$

$$\langle y-y', f(t,\omega,y,z) - f(t,\omega,y',z')\rangle \mathbb{1}_{\{v_t(\omega)\leq N\}} \leq K' \mid y-y'\mid^2 \, logA_N + \sqrt{K'\log A_N} \mid y-y'\mid\mid z-z'\mid + K'\frac{\log A_N}{A_N}$$

where A_N is a increasing sequence and satisfies $A_N > 1$, $\lim_{N\to\infty} A_N = \infty$ and $A_N \leq N^{\mu}$ for some $\mu > 0$.

2.3 The main result

Theorem 2.1. *Assume that **(H.0)-(H.4)** hold. Then, $(E^{(\xi,f)})$ has a unique solution (Y,Z) which satisfies,*

$$\mathbb{E}\sup_t \mid Y_t \mid^p e^{\frac{p}{2}\int_0^t \lambda_s ds} + \mathbb{E}\Big[\int_0^T e^{\int_0^s \lambda_r dr} \mid Z_s \mid^2 ds\Big]^{\frac{p}{2}}$$
$$\leq C\Big\{\mathbb{E} \mid \xi \mid^p e^{\frac{p}{2}\int_0^T \lambda_s ds} + \mathbb{E}\Big(\int_0^T e^{\int_0^s \lambda_r dr}\eta_s ds\Big)^{\frac{p}{2}} + \mathbb{E}\Big(\int_0^T e^{\frac{1}{2}\int_0^s \lambda_r dr} f_s^0 ds\Big)^p\Big\}$$

for some constant C depending only on p and γ.

We shall give some examples of BSDEs which satisfy the assumptions of Theorem 2.1. In our knowledge, these examples are not covered by the previous works in multidimensional BSDEs.

2.4 Examples.

Example 1. Let $f(y) := -y\log \mid y \mid$ then for all $\xi \in \mathbb{L}^p(\mathcal{F}_T)$ the following BSDE has a unique solution

$$Y_t = \xi - \int_t^T Y_s \log \mid Y_s \mid ds - \int_t^T Z_s dW_s.$$

Indeed, f satisfies **(H.1)-(H.3)** since $\langle y, f(y)\rangle \leq 1$ and $\mid f(y) \mid \leq 1 + \dfrac{1}{\varepsilon} \mid y \mid^{1+\varepsilon}$ for all $\varepsilon > 0$. In order to verify **(H.4)**, thanks to triangular inequality, it is sufficient to treat separately the two cases: $0 \leq\mid y \mid, \mid y' \mid\leq \dfrac{1}{N}$ and $\dfrac{1}{N} \leq\mid y \mid, \mid y' \mid\leq N$.
In the first case, since the map $x \mapsto -x\log x$ increases for $x \in]0, e^{-1}]$, we obtain for $N > e$

$$|f(y) - f(y')| \leq |f(y)| + |f(y')|$$
$$\leq 2\frac{\log N}{N}$$

In the second case, the finite increments theorem applied to f shows that

$$|f(y) - f(y')| \leq (1 + \log N) \mid y - y' \mid.$$

Hence **(H.4)** is satisfied for every $N > e$ with $v_s = 0$ and $A_N = N$.

Example 2. Let $g(y) := y\log \dfrac{\mid y \mid}{1+\mid y \mid}$ and $h \in \mathcal{C}(\mathbb{R}^{dr};\mathbb{R}_+)\bigcap\mathcal{C}^1(\mathbb{R}^{dr} - \{0\};\mathbb{R}_+)$ be such that

$$h(z) = \begin{cases} |z|\sqrt{-\log|z|} & \text{if } |z| < 1-\varepsilon_0 \\ |z|\sqrt{\log|z|} & \text{if } |z| > 1+\varepsilon_0 \end{cases}$$

where $\varepsilon_0 \in]0,1[$. Finally, we put $f(y,z) := g(y)h(z)$. Then for every $\xi \in \mathbb{L}^p(\mathcal{F}_T)$ the following BSDE has a unique solution

$$Y_t = \xi + \int_t^T f(Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

It is not difficult to see that f satisfies (H1). We shall prove that f satisfies (H2)-(H4).

(i) Since g is continuous, $g(0) = 0$ and $|g(y)|$ tends to 1 as $|y|$ tends to ∞, we deduce that g is bounded. Moreover, g satisfied $\langle y - y', g(y) - g(y')\rangle \leq 0$. Indeed, in one dimensional case it is not difficult to show that g is a decreasing function. Since, $-\langle y, y'\rangle \log \frac{|y|}{1+|y|} \leq -|y||y'|\log\frac{|y|}{1+|y|}$ (because

$\log \frac{|y|}{1+|y|} \leq 0$), we can reduce the multidimensional case to the one dimension case by developing the inner product as follows,

$$\begin{aligned}\langle y-y', g(y)-g(y')\rangle &\leq |y|^2 \log \frac{|y|}{1+|y|} + |y'|^2 \log \frac{|y'|}{1+|y'|} - |y||y'|(\log \frac{|y|}{1+|y|} + \log \frac{|y'|}{1+|y'|}) \\ &= (|y|-|y'|)(|y| \ \log \frac{|y|}{1+|y|} - |y'| \ \log \frac{|y'|}{1+|y'|}) \\ &= \langle |y|-|y'|, g(|y|)-g(|y'|)\rangle \\ &\leq 0\end{aligned}$$

(ii) The function $h(z)$ satisfies for all $\varepsilon > 0$

$$0 \leq h(z) \leq M + \frac{1}{\sqrt{2\varepsilon}} \mid z \mid^{1+\varepsilon}, \qquad \text{where} \quad M = \sup_{|z|\leq 1+\varepsilon_0} \mid h(z) \mid$$

The last inequality follows since $\sqrt{2\varepsilon \log|z|} = \sqrt{\log|z|^{2\varepsilon}} \leq |z|^{\varepsilon}$ for each $\varepsilon > 0$ and $|z| > 1$. (H3) follows now directly from the previous observations (i) and (ii). (H2) is satisfied since $\langle y, f(y,z)\rangle = \langle y, g(y)\rangle h(z) \leq 0$. To verify **(H.4)** it is enough to show that for every z, z' such that $\mid z \mid, \mid z' \mid \leq N$

$$\mid h(z) - h(z') \mid \leq c\left(\sqrt{\log N} \mid z - z' \mid + \frac{\log N}{N}\right)$$

for N large enough and some positive constant c. This can be proved by considering separately the following five cases, $0 \leq\mid z \mid, \mid z' \mid \leq \frac{1}{N}$, $\frac{1}{N} \leq\mid z \mid, \mid z' \mid \leq 1-\varepsilon_0$, $1-\varepsilon_0 \leq\mid z \mid, \mid z' \mid \leq 1+\varepsilon_0$ and $1+\varepsilon_0 \leq\mid z \mid, \mid z' \mid \leq N$.

In the first case (i.e. $0 \leq\mid z \mid, \mid z' \mid \leq \frac{1}{N}$), since the map $x \mapsto x\sqrt{-\log x}$ increases for $x \in [0, \frac{1}{\sqrt{e}}]$, we obtain $\quad |h(z)-h(z')| \leq |h(z)| + |h(z')| \leq 2\frac{1}{N}\sqrt{-\log \frac{1}{N}} \leq 2\frac{1}{N}\log N$ for $N > \sqrt{e}$.
The other cases can be proved by using the finite increments theorem.

Example 3. Let $(X_t)_{t\leq T}$ be an $(\mathcal{F}_t)-$adapted and $\mathbb{R}^k-$valued process satisfying the forward stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

where $X_0 \in \mathbb{R}^k$ and $\sigma, b : [0,T]\times\mathbb{R}^k \to \mathbb{R}^{kr}\times\mathbb{R}^k$ are measurable functions such that $\|\sigma(s,x)\| \leq c$ and $|b(s,x)| \leq c(1+|x|)$, for some constant c.

It is known from forward SDE's theory that there exist $\kappa > 0$ and $C > 0$ depending only on c, T, k such that

$$\mathbb{E}\exp(\kappa \sup_{t\leq T} \mid X_t \mid^2) \leq C\exp(C \mid X_0 \mid^2).$$

Consider the BSDE

$$Y_t = g(X_T) + \int_t^T \mid X_s \mid^{\overline{q}} Y_s - Y_s \log\mid Y_s \mid ds - \int_t^T Z_s dW_s.$$

where $\overline{q} \in]0,2[$ and g is a measurable function satisfying $\mid g(x) \mid \leq c\exp c \mid x \mid^{\overline{q}'}$, for some constants $c > 0$, $\overline{q}' \in [0,2[$.
The previous BSDE has a unique solution (Y,Z) which satisfies: for every $p > 1$ there exists a positive constant C such that

$$\mathbb{E}\sup_t \mid Y_t \mid^p + \mathbb{E}\left[\int_0^T \mid Z_s \mid^2 ds\right]^{\frac{p}{2}} \leq C\exp(C \mid X_0 \mid^2).$$

Indeed, one can show that
$i)$ $\langle y, f(t,y)\rangle \leq 1 + \mid X_t \mid^{\overline{q}} \mid y \mid^2$
$ii)$ Using Young inequality we obtain, for every $\epsilon > 0$ there is a constant $c_\epsilon > 0$ such, that

$$\mid f(t,y) \mid \leq c_\epsilon (1 + \mid X_t \mid^{\overline{q} c_\epsilon} + \mid y \mid^{1+\epsilon})$$

$iii)$ f satisfies assumption **(H.4)** with $v_s = \exp \mid X_s \mid^{\overline{q}}$ and $A_N = N$.

The following example shows that our assumptions enable to treat BSDEs with stochastic monotone coefficient

Example 4. Let (ξ, f) satisfying **(H.0)-(H.3)** and

$$(H'.4) \begin{cases} \text{There are a positive process } C \text{ satisfying } \mathbb{E}\displaystyle\int_0^T e^{q' C_s} ds < \infty \text{ (for some } q' > 0) \text{ and } K' \in \mathbb{R}_+ \\ \text{such that:} \\ \langle y - y', f(t,\omega,y,z) - f(t,\omega,y',z')\rangle \leq K' \mid y - y' \mid^2 \{C_t(\omega) + \mid \log \mid y - y' \mid \mid\} \\ \qquad\qquad + K' \mid y - y' \mid\mid z - z' \mid \sqrt{C_t(\omega) + \mid \log \mid z - z' \mid \mid}. \end{cases}$$

In particular we have for all z, z'

$$|f(t,\omega,y,z) - f(t,\omega,y,z')| \leq K' \mid z - z' \mid \sqrt{C_t(\omega) + \mid \log \mid z - z' \mid \mid}.$$

Therefore, the following BSDE has a unique solution

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.$$

To check **(H.4)**, it is enough to show that for some constant c we have
$\langle y - y', f(t,y,z) - f(t,y',z)\rangle \leq c \log N \left(\mid y - y' \mid^2 + \frac{1}{N} \right)$
$|f(t,y,z) - f(t,y,z')| \leq c\sqrt{\log N} \left(\mid z - z' \mid + \frac{1}{N} \right)$
whenever $v_s := e^{C_s} \leq N$ and $\mid y \mid, \mid y' \mid, \mid z \mid, \mid z' \mid \leq N$.
These two inequalities can be respectively proved by considering the following cases

$$\mid y - y' \mid \leq \frac{1}{2N}, \quad \frac{1}{2N} \leq \mid y - y' \mid \leq 2N.$$

and

$$\mid z - z' \mid \leq \frac{1}{2N}, \quad \frac{1}{2N} \leq \mid z - z' \mid \leq 2N.$$

Example 5. Let $(X_t)_{t \leq T}$ and ξ be as in example 3, let $F(t,x,y,z)$ be such that
$i)$ $F(t,x,.)$ is continuous
$ii)$ $|F(t,x,y,0)| \leq C \exp(C \mid x \mid^{\hat{q}}) + \mid y \mid^\alpha$, for some $\hat{q}, \alpha \in]0,2[$ and $C > 0$,
$iii)$ $\langle F(t,x,y,z) - F(t,x,y',z'), y - y'\rangle \leq K' \mid y - y' \mid^2 + K' \mid y - y' \mid\mid z - z' \mid$.
Let $\overline{q}, \overline{q}', \overline{q}" \geq 0$ such that $\overline{q} + \overline{q}" < 2$ and $\overline{q}' + \overline{q}" < 1$, the following BSDE has a unique solution

$$Y_t = \xi + \int_t^T |X_s|^{\overline{q}"} F(s, X_s, |X_s|^{\overline{q}} Y_s, |X_s|^{\overline{q}'} Z_s) ds - \int_t^T Z_s dW_s.$$

3 Proof of Theorem 2.1

We first give some a priori estimates from which we derive a stability result for BSDEs and next we use a suitable approximation of (ξ, f) to complete the proof. The difficulty comes from the fact that the generator f can be neither locally monotone in the variable y nor locally Lipschitz in the variable z and moreover, it also may have a superlinear growth in its two variables y and z.

3.1 Estimates for the solutions of equation $(E^{(\xi,f)})$.

In the first step, we give estimates for the processes Y and Z.

Proposition 3.1. *Let* $\Lambda_t := |Y_t|^2 e_t + 2\int_0^t e_s\eta_s ds + (\int_0^t e_s^{\frac{1}{2}} f_s^0 ds)^2$ *and* $e_t := \exp\int_0^t \lambda_s ds$.
Assume that ***(H.2)*** *hold and* $\mathbb{E}(\sup_{0\le\ s\le T} |Y_t|^p\, e_t^{\frac{p}{2}}) < \infty$.
Then, there exists a positive constant $C^{(p,\gamma)}$ *such that*

$$\mathbb{E}\sup_{0\le\ s\le T}\Lambda_s^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T e_s|Z_s|^2 ds\right)^{\frac{p}{2}} \le C^{(p,\gamma)}\mathbb{E}\Lambda_T^{\frac{p}{2}}.$$

To prove this proposition we need some lemmas.

Lemma 3.1. *For every* $\varepsilon > 0$, *every* $\beta > 1$ *and every positive functions* h *and* g *we have*

$$\int_t^T (h(s))^{\frac{\beta-1}{2}} g(s)ds \le \varepsilon \sup_{t\le s\le T} |\, h(s)\,|^{\frac{\beta}{2}} + \varepsilon^{1-\beta}\left(\int_t^T g(s)ds\right)^{\beta}.$$

Proof. Let $\varepsilon > 0$ and $\beta > 1$. Using Young's inequality we get for every δ and δ' such that $\frac{1}{\delta} + \frac{1}{\delta'} = 1$

$$\int_t^T (h(s))^{\frac{\beta-1}{2}} g(s)ds \le \frac{1}{\delta}\varepsilon^{\frac{(\beta-1)\delta}{\beta}} \sup_{t\le s\le T} |\, h(s)\,|^{\frac{(\beta-1)\delta}{2}} + \frac{\varepsilon^{\frac{(1-\beta)\delta'}{\beta}}}{\delta'}(\int_t^T g(s)ds)^{\delta'}$$

We now choose $\delta = \frac{\beta}{\beta-1}$ and use the fact that $\delta, \delta' > 1$. ∎

Lemma 3.2. *If* ***(H.2)*** *holds then for every* $\beta > 1+2\gamma$ *there exist positive constants* $C_1^{(\beta,\gamma)}, C_2^{(\beta,\gamma)}$ *such that for every* $\varepsilon > 0$, *every stopping time* $\tau \le T$ *and every* $t \le \tau$

$$\Lambda_t^{\frac{\beta}{2}} + \int_t^\tau \Lambda_s^{\frac{\beta-2}{2}} e_s|Z_s|^2 ds \ \le\ \varepsilon \sup_{t\le s\le\tau}\Lambda_s^{\frac{\beta}{2}} + \varepsilon^{(1-\beta)}C_1^{(\beta,\gamma)}\Lambda_\tau^{\frac{\beta}{2}} - C_2^{(\beta,\gamma)}\int_t^\tau \Lambda_s^{\frac{\beta}{2}-1} e_s\langle Y_s, Z_s dW_s\rangle.$$

Proof. Without loss of generality, we assume that η and f^0 are strictly positives.
It follows by using Itô's formula that for every $t \in [0, \tau]$,

$$|Y_t|^2 e_t + \int_t^\tau |Y_s|^2 \lambda_s e_s ds = e_\tau |Y_\tau|^2 + 2\int_t^\tau e_s\langle Y_s,\ f(s, Y_s, Z_s)\rangle ds - \int_t^\tau e_s \mid Z_s\mid^2 ds$$
$$- 2\int_t^\tau e_s\langle Y_s,\ Z_s dW_s\rangle.$$

Again Itô's formula, applied to the process Λ, shows that

$$\Lambda_t^{\frac{\beta}{2}} + \beta\int_t^\tau \Lambda_s^{\frac{\beta}{2}-1}\left(\frac{1}{2}|Y_s|^2\lambda_s e_s + e_s\eta_s + f_s^0 e_s^{\frac{1}{2}}\left[\int_0^s f_r^0 e_r^{\frac{1}{2}} dr\right]\right) ds$$
$$= \Lambda_\tau^{\frac{\beta}{2}} + \beta\int_t^\tau \Lambda_s^{\frac{\beta}{2}-1}\langle e_s Y_s, \quad f(s, Y_s, Z_s)\rangle ds - \frac{\beta}{2}\int_t^\tau \Lambda_s^{\frac{\beta}{2}-1}|Z_s|^2 e_s ds$$
$$- \beta\int_t^\tau e_s\Lambda_s^{\frac{\beta}{2}-1}\langle Y_s, \quad Z_s dW_s\rangle - \beta(\frac{\beta}{2}-1)\int_t^\tau e_s^2\Lambda_s^{\frac{\beta}{2}-2}\sum_{j=1}^r\left(\sum_{i=1}^d Y_s^i Z_s^{i,j}\right)^2 ds$$

Observe that $\sum_{j=1}^{r}\left(\sum_{i=1}^{d} Y_s^i Z_s^{i,j})\right)^2 \leq |Y_s|^2|Z_s|^2 \leq e_s^{-1}\Lambda_s\,|Z_s|^2$ then use the assumption **(H.2)** to get

$$\begin{aligned}
&\Lambda_t^{\frac{\beta}{2}} + \frac{\beta}{2}(1-2\gamma-(2-\beta)^+)\int_t^\tau \Lambda_s^{\frac{\beta}{2}-1} e_s\,|Z_s|^2\,ds \\
\leq\;& \Lambda_\tau^{\frac{\beta}{2}} + \beta\int_t^\tau \Lambda_s^{\frac{\beta}{2}-\frac{1}{2}} f_s^0 e_s^{\frac{1}{2}}\,ds - \beta\int_t^\tau \Lambda_s^{\frac{\beta}{2}-1}\langle e_s Y_s, \quad Z_s dW_s\rangle.
\end{aligned}$$

It follows from Lemma 3.1 with $g(s) = f_s^0 e_s^{\frac{1}{2}}$, since $\left(\int_t^\tau f_s^0 e_s^{\frac{1}{2}}\,ds\right)^\beta \leq \Lambda_\tau^{\frac{\beta}{2}}$, that for every $\varepsilon > 0$

$$\int_t^\tau \Lambda_s^{\frac{\beta}{2}-\frac{1}{2}} f_s^0 e_s^{\frac{1}{2}}\,ds \leq \;\varepsilon \sup_{t\leq s\leq\tau} \Lambda_s^{\frac{\beta}{2}} + \varepsilon^{1-\beta}\Lambda_\tau^{\frac{\beta}{2}}$$

Since $\beta > 1+2\gamma$ implies that $1-2\gamma-(2-\beta)^+ > 0$, Lemma 3.2 is proved. ■

Lemma 3.3. *Let* ***(H2)*** *be satisfied and assume that* $\mathbb{E}(\sup_{0\leq s\leq T}|Y_t|^p\, e_t^{\frac{p}{2}}) < \infty$.
Then,
1) *There exists a positive constant* $C^{(p,\gamma)}$ *such that for every* $\varepsilon > 0$*, we have*

$$\mathbb{E}\int_0^T \Lambda_s^{\frac{p-2}{2}} e_s|Z_s|^2 ds \;\;\leq\;\; \varepsilon\mathbb{E}(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}) \;+\; \varepsilon^{(1-p)}C_1^{(p,\gamma)}\mathbb{E}(\Lambda_T^{\frac{p}{2}}).$$

2) *There exists a positive constant* $C^{(p,\gamma)}$ *such that*

$$\mathbb{E}\Big(\int_0^T e_s|Z_s|^2 ds\Big)^{\frac{p}{2}} \;\;\leq\;\; C^{(p,\gamma)}\mathbb{E}(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}).$$

Proof. The first assertion follows by a standard martingale localization procedure. To prove the second assertion, we successively use Lemma 3.2 (with $\varepsilon = 1$ and $\beta = 2$), the Burkholder-Davis-Gundy inequality, the fact that $e_s|Y_s|^2 \leq \Lambda_s$ and Young's inequality to obtain

$$\begin{aligned}
\mathbb{E}\Big(\int_0^T e_s|Z_s|^2 ds\Big)^{\frac{p}{2}} &\leq C_1^{(p,\gamma)}\mathbb{E}\big(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}\big) + C_2^{(p,\gamma)}\mathbb{E}\big(|\int_t^T e_s\langle Y_s, Z_s dW_s\rangle|^{\frac{p}{2}}\big) \\
&\leq C_1^{(p,\gamma)}\mathbb{E}\big(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}\big) + C_2^{(p,\gamma)}\mathbb{E}\big(|\int_0^T e_s^2|Y_s|^2|Z_s|^2 ds|^{\frac{p}{4}}\big) \\
&\leq C_1^{(p,\gamma)}\mathbb{E}\big(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}\big) + C_2^{(p,\gamma)}\mathbb{E}\big(|\int_0^T \Lambda_s e_s|Z_s|^2 ds|^{\frac{p}{4}}\big) \\
&\leq C_1^{(p,\gamma)}\mathbb{E}\big(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}\big) + C_2^{(p,\gamma)}\mathbb{E}\big[\big(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{4}}\big)\big(\int_0^T e_s|Z_s|^2 ds\big)^{\frac{p}{4}}\big] \\
&\leq \big[C_1^{(p,\gamma)} + +2(C_2^{(p,\gamma)})^2\big]\mathbb{E}\big(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}\big) + \frac{1}{2}\mathbb{E}\big[\big(\int_0^T e_s|Z_s|^2 ds\big)^{\frac{p}{2}}\big] \\
&\leq [2C_1^{(p,\gamma)} + 4(C_2^{(p,\gamma)})^2] \quad \mathbb{E}\big(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}\big).
\end{aligned}$$

Lemma 3.3 is proved. ■

Lemma 3.4. *Let the assumptions of Lemma 3.3 be satisfied. Then, there exists a constant* $C^{(p,\gamma)}$ *such that*

$$\mathbb{E}(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}) \;\;\leq\;\; C^{(p,\gamma)}\mathbb{E}(\Lambda_T^{\frac{p}{2}}).$$

Proof. Lemma 3.2 and the Burkholder-Davis-Gundy inequality show that there exists a universal positive constant c such that for every $\varepsilon > 0$ and $t \leq T$

$$\begin{aligned}\mathbb{E} \sup_{0\leq s\leq T} \Lambda_s^{\frac{p}{2}} &\leq \varepsilon \mathbb{E}(\sup_{0\leq s\leq T} \Lambda_s^{\frac{p}{2}}) + \varepsilon^{(1-p)} C_1^{(p,\gamma)} \mathbb{E}(\Lambda_T^{\frac{p}{2}}) \\ &\quad + c C_2^{(p,\gamma)} \mathbb{E}\Big(\int_0^T \Lambda_s^{p-2}(|Y_s|^2 e_s) e_s |Z_s|^2 ds\Big)^{\frac{1}{2}}.\end{aligned}$$

Young's inequality gives, for every $\varepsilon' > 0$,

$$\begin{aligned}\mathbb{E}(\sup_{0\leq s\leq T} \Lambda_s^{\frac{p}{2}}) &\leq \varepsilon \mathbb{E}(\sup_{0\leq s\leq T} \Lambda_s^{\frac{p}{2}}) + \varepsilon^{(1-p)} C_1^{(p,\gamma)} \mathbb{E}(\Lambda_T^{\frac{p}{2}}) \\ &\quad + \varepsilon' \mathbb{E}(\sup_{0\leq s\leq T} \Lambda_t^{\frac{p}{2}}) + \frac{\Big[c C_2^{(p,\gamma)}\Big]^2}{\varepsilon'} \mathbb{E}\int_0^T \Lambda_s^{\frac{p-2}{2}} e_s |Z_s|^2 ds.\end{aligned}$$

Applying Lemma 3.3, we get for every $\varepsilon" > 0$

$$\begin{aligned}\mathbb{E}(\sup_{0\leq s\leq T} \Lambda_t^{\frac{p}{2}}) &\leq (\varepsilon + \varepsilon' + \frac{\Big[c C_2^{(p,\gamma)}\Big]^2 \varepsilon"}{\varepsilon'}) \mathbb{E}(\sup_{0\leq s\leq T} \Lambda_s^{\frac{p}{2}}) \\ &\quad + (\varepsilon^{(1-p)} C_1^{(p,\gamma)} + \frac{\Big[c C_2^{(p,\gamma)}\Big]^2 C_1^{(p,\gamma)} (\varepsilon")^{(1-p)}}{\varepsilon'}) \mathbb{E}(\Lambda_T^{\frac{p}{2}}).\end{aligned}$$

A suitable choice of $\varepsilon, \varepsilon', \varepsilon"$ allows to conclude the proof. ■

Proof of Proposition 3.1. It follows from Lemma 3.3 and Lemma 3.4. ■

Proposition 3.2. *If **(H.3)** holds then,*

$$\mathbb{E}\int_0^T |f(s, Y_s, Z_s)|^{\hat{\beta}} ds \leq 9^{p+q}(1+T)\big[1 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E} \sup_{0\leq s\leq T} |Y_s|^p + \mathbb{E}(\int_0^T |Z_s|^2 ds)^{\frac{p}{2}}\big]$$

where $\hat{\beta} := \dfrac{2}{\alpha'} \wedge \dfrac{p}{\alpha} \wedge \dfrac{p}{\alpha'} \wedge q$.

Proof. We successively use Assumption (H.3), Young's inequality and Hölder's inequality to show that

$$\begin{aligned}\mathbb{E}\int_0^T |f(s, Y_s, Z_s)|^{\hat{\beta}} ds &\leq \mathbb{E}\int_0^T (\overline{\eta}_s + |Y_s|^{\alpha} + |Z_s|^{\alpha'})^{\hat{\beta}} ds \\ &\leq 3^{\hat{\beta}} \mathbb{E}\int_0^T (\overline{\eta}_s^{\hat{\beta}} + |Y_s|^{\alpha\hat{\beta}} + |Z_s|^{\alpha'\hat{\beta}}) ds \\ &\leq 3^{\hat{\beta}} \mathbb{E}\int_0^T ((1+\overline{\eta}_s)^{\hat{\beta}} + (1+|Y_s|)^{\alpha\hat{\beta}} + (1+|Z_s|)^{\alpha'\hat{\beta}}) ds \\ &\leq 3^{\hat{\beta}} \mathbb{E}\int_0^T ((1+\overline{\eta}_s)^q + (1+|Y_s|)^p + (1+|Z_s|)^{p\wedge 2}) ds \\ &\leq 3^{\hat{\beta}} 3^{p+q} \mathbb{E}\int_0^T (1 + \overline{\eta}_s^q + |Y_s|^p + |Z_s|^{p\wedge 2}) ds \\ &\leq 3^{\hat{\beta}} 3^{p+q}\big[T + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + T\mathbb{E} \sup_{0\leq s\leq T} |Y_s|^p + T^{\frac{2-(p\wedge 2)}{2}} \mathbb{E}(\int_0^T |Z_s|^2 ds)^{\frac{p}{2}}\big] \\ &\leq 9^{p+q}(1+T)\big[1 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E} \sup_{0\leq s\leq T} |Y_s|^p + \mathbb{E}(\int_0^T |Z_s|^2 ds)^{\frac{p}{2}}\big].\end{aligned}$$

Proposition 3.2 is proved. ■

3.2 Estimate of the difference between two solutions.

The next proposition gives an estimate which is a key tool in the proofs.

Lemma 3.5. *Let $(\xi^i, f_i)_{i=1,2}$ satisfy **(H.3)** (with the same $\overline{\eta}, \alpha$ and α') and let (Y^i, Z^i) be a solution of $(E^{(\xi^i, f_i)})$. Then, there exist $\beta = \beta(p, q, \alpha, \alpha') \in]1, p \wedge 2[$, $r = r(p, q, \alpha, \alpha', K', \mu, q') > 0$ and $a = a(p, q, \alpha, \alpha', K', \mu, q') > 0$ such that for every $u \in [0, T], u' \in [u, T \wedge (u + r)]$, $N > 0$ and every function f satisfying **(H.4)***

$$\begin{aligned}
&\mathbb{E}(\sup_{u \le t \le u'} |Y_t^1 - Y_t^2|^\beta) \;+\; \mathbb{E}\int_u^{u'} \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}} ds \\
&\le\; N A_N^{1+\frac{\beta}{2}} \left[\mathbb{E}(|Y_{u'}^1 - Y_{u'}^2|^\beta) + \mathbb{E}\int_0^T \rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s ds \right] \\
&\quad+\; \frac{1}{A_N^a} \left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds \right].
\end{aligned}$$

where

$$\rho_N(f_i - f)(t, \omega) := \sup_{|y|, |z| \;\le\; N} |f(t, \omega, y, z) - f_i(t, \omega, y, z)|$$

and

$$\Theta_p^i \;:=\; \mathbb{E}(\sup_t |Y_t^i|^p) \;+\; \mathbb{E}\left(\int_0^T |Z_s^i|^2 ds\right)^{\frac{p}{2}}.$$

Proof. Let q be the number defined in assumption **(H3)** and q', K', μ those defined in assumption **(H4)**. Let $\overline{\gamma} > 0$ be such that $1 + 2\overline{\gamma} < \hat{\beta} := \frac{2}{\alpha'} \wedge \frac{p}{\alpha} \wedge \frac{p}{\alpha'} \wedge q$ and set $K" := K' + \frac{K'}{4\overline{\gamma}}$. Let $\beta \in]1 + 2\overline{\gamma}, \hat{\beta}[$ and $\nu \in]0, (1 - \frac{\beta}{\hat{\beta}})(1 \wedge q')[$. Let $r \in]0, \frac{\nu}{\mu\hat{\beta}K"} \wedge \frac{1}{2K"} \wedge 1[$.
For $N \in \mathbb{N}$, we set

$$\overline{e}_t := (A_N)^{2K"(t-u)} \quad \text{and} \quad \Delta_t := \{\left|Y_t^1 - Y_t^2\right|^2 + (A_N)^{-1}\}\overline{e}_t.$$

Using Itô's formula, we show that for every stopping time $\tau \in [u, u']$ and every $t \in [u, \tau]$

$$\begin{aligned}
\Delta_t^{\frac{\beta}{2}} \;&+\; 2\log(A_N)K" \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}} ds + \frac{\beta}{2}\int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \left|Z_s^1 - Z_s^2\right|^2 ds \\
&= \Delta_\tau^{\frac{\beta}{2}} - \beta \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \langle Y_s^1 - Y_s^2, (Z_s^1 - Z_s^2)\, dW_s \rangle \\
&\quad+\beta \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \langle Y_s^1 - Y_s^2, f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2) \rangle ds \\
&\quad-\beta(\tfrac{\beta}{2} - 1) \int_t^\tau \overline{e}_s^2 \Delta_s^{\frac{\beta}{2}-2} \sum_{j=1}^r \left(\sum_{i=1}^d (Y_{i,s}^1 - Y_{i,s}^2)(Z_{i,j,s}^1 - Z_{i,j,s}^2) \right)^2 ds \\
&= \Delta_\tau^{\frac{\beta}{2}} - \beta \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \langle Y_s^1 - Y_s^2, (Z_s^1 - Z_s^2)\, dW_s \rangle + \beta I_1 - \beta(\frac{\beta}{2} - 1) I_2,
\end{aligned} \tag{3.1}$$

where

$$I_1 := \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \langle Y_s^1 - Y_s^2, f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2) \rangle ds$$

and

$$I_2 := \int_t^\tau \overline{e}_s^2 \Delta_s^{\frac{\beta}{2}-2} \sum_{j=1}^r \left(\sum_{i=1}^d (Y_{i,s}^1 - Y_{i,s}^2)(Z_{i,j,s}^1 - Z_{i,j,s}^2) \right)^2 ds.$$

In order to complete the proof of Lemma 3.5 we need to estimate I_1 and I_2.

Estimate of I_1. Let $\Phi(s) := |Y_s^1| + |Y_s^2| + |Z_s^1| + |Z_s^2| + v_s$. Since $1\!\!1_{\{\Phi_s \leq N\}} \leq 1\!\!1_{\{v_s \leq N\}}$ and f satisfies **(H4)**, then a simple computation shows that

$$\begin{aligned} \langle Y_s^1 - Y_s^2, & f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)\rangle \\ & \leq \overline{e}_s^{\frac{-1}{2}} \Delta_s^{\frac{1}{2}} |f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)| 1\!\!1_{\{\Phi_s > N\}} \\ & \quad + 2N[\rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s] 1\!\!1_{\{v_s \leq N\}} \\ & \quad + [K\text{''} \log(A_N) \overline{e}_s^{-1} \Delta_s + \overline{\gamma} \left|Z_s^1 - Z_s^2\right|^2] 1\!\!1_{\{\Phi_s \leq N\}} \end{aligned}$$

Therefore, using Lemma 3.1 with $h_s = \Delta_s$, we get

$$\begin{aligned} I_1 \leq & \int_t^\tau \overline{e}_s^{\frac{1}{2}} \Delta_s^{\frac{\beta-1}{2}} |f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)| 1\!\!1_{\{\Phi_s > N\}} ds \\ & + 2N \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} [\rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s] 1\!\!1_{\{v_s \leq N\}} ds \\ & + \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} [K\text{''} \log(A_N) \overline{e}_s^{-1} \Delta_s + \overline{\gamma} \left|Z_s^1 - Z_s^2\right|^2] 1\!\!1_{\{\Phi_s \leq N\}} ds \\ \leq & \, \varepsilon \sup_{s \in [u,u']} \Delta_s^{\frac{\beta}{2}} \\ & + \varepsilon^{(1-\beta)} \overline{e}_{u'}^{\frac{\beta}{2}} \int_u^{u'} |f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)|^\beta 1\!\!1_{\{\Phi_s > N\}} ds \\ & + 2N \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} [\rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s] 1\!\!1_{\{v_s \leq N\}} ds \\ & + \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} [K\text{''} \log(A_N) \overline{e}_s^{-1} \Delta_s + \overline{\gamma} \left|Z_s^1 - Z_s^2\right|^2] 1\!\!1_{\{\Phi_s \leq N\}} ds \end{aligned}$$

Estimate of I_2. Since

$$\sum_{j=1}^r \left(\sum_{i=1}^d (Y_{i,s}^1 - Y_{i,s}^2)(Z_{i,j,s}^1 - Z_{i,j,s}^2) \right)^2 \leq \left|Y_s^1 - Y_s^2\right|^2 \left|Z_s^1 - Z_s^2\right|^2 \leq \overline{e}_s^{-1} \Delta_s \left|Z_s^1 - Z_s^2\right|^2$$

then

$$I_2 \leq \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \left|Z_s^1 - Z_s^2\right|^2 ds.$$

Now, coming back to equation (3.1) and taking into account the above estimates we get for every $\varepsilon > 0$,

$$
\begin{aligned}
&\Delta_t^{\frac{\beta}{2}} + \frac{\beta}{2}(\beta - 1 - 2\overline{\gamma}) \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \left|Z_s^1 - Z_s^2\right|^2 ds \\
\leq\ & \overline{e}_\tau^{\frac{\beta}{2}} |Y_\tau^1 - Y_\tau^2|^\beta + \frac{\overline{e}_{u'}^{\frac{\beta}{2}}}{A_N^{\frac{\beta}{2}}} + \beta\varepsilon \sup_{s\in[u,u']} \Delta_s^{\frac{\beta}{2}} \\
&+\beta\varepsilon^{(1-\beta)}\overline{e}_{u'}^{\frac{\beta}{2}} \int_u^{u'} |f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)|^\beta \mathbb{1}_{\{\Phi_s > N\}} ds \\
&+2N\beta\overline{e}_\tau^{\frac{\beta}{2}} A_N^{1-\frac{\beta}{2}} \int_u^\tau \rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s \mathbb{1}_{\{v_s \ \leq \ N\}} ds \\
&-\beta \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right) dW_s\rangle.
\end{aligned}
\tag{3.2}
$$

For a given $\hbar > 1$, let $\tau_\hbar$ be the stopping time defined by

$$
\tau_\hbar := \inf\{s \geq u, \quad |Y_s^1 - Y_s^2|^2 + \int_u^s |Z_r^1 - Z_r^2|^2 dr \geq \hbar\} \wedge u',
$$

Choose $\tau = \tau_\hbar$, $t = u$, then pass to the expectation in equation (3.2) to obtain, when $\hbar \to \infty$,

$$
\begin{aligned}
&\frac{\beta}{2}(\beta - 1 - 2\overline{\gamma})\mathbb{E} \int_u^{u'} \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \left|Z_s^1 - Z_s^2\right|^2 ds \\
\leq\ & \overline{e}_{u'}^{\frac{\beta}{2}} \mathbb{E}(|Y_{u'}^1 - Y_{u'}^2|^\beta) + \frac{\overline{e}_{u'}^{\frac{\beta}{2}}}{A_N^{\frac{\beta}{2}}} + \beta\varepsilon\mathbb{E}(\sup_{s\in[u,u']} \Delta_s^{\frac{\beta}{2}}) \\
&+\beta\varepsilon^{(1-\beta)}\overline{e}_{u'}^{\frac{\beta}{2}}\mathbb{E} \int_u^{u'} |f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)|^\beta \mathbb{1}_{\{\Phi_s > N\}} ds \\
&+2N\beta\overline{e}_{u'}^{\frac{\beta}{2}} A_N^{1-\frac{\beta}{2}} \mathbb{E} \int_u^{u'} \rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s \mathbb{1}_{\{v_s \ \leq \ N\}} ds.
\end{aligned}
\tag{3.3}
$$

Return back to (3.2) and use the Burkholder-Davis-Gundy inequality to show that there exists a universal constant c such that

$$
\begin{aligned}
\mathbb{E}(\sup_{u\leq t\leq T} \Delta_t^{\frac{\beta}{2}}) \leq\ & \overline{e}_{u'}^{\frac{\beta}{2}} \mathbb{E}(|Y_{u'}^1 - Y_{u'}^2|^\beta) + \frac{\overline{e}_{u'}^{\frac{\beta}{2}}}{A_N^{\frac{\beta}{2}}} + \beta\varepsilon\mathbb{E}(\sup_{s\in[u,u']} \Delta_s^{\frac{\beta}{2}}) \\
&+ \beta\varepsilon^{(1-\beta)}\overline{e}_{u'}^{\frac{\beta}{2}}\mathbb{E} \int_u^{u'} |f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)|^\beta \mathbb{1}_{\{\Phi_s > N\}} ds \\
&+ 2N\beta\overline{e}_{u'}^{\frac{\beta}{2}} A_N^{1-\frac{\beta}{2}} \mathbb{E} \int_u^{u'} \rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s \mathbb{1}_{\{v_s \ \leq \ N\}} ds \\
&+ c\beta\mathbb{E}(\int_u^T \overline{e}_s^2 \Delta_s^{\beta-2} \sum_{j=1}^r [\sum_{i=1}^d (Y_{i,s}^1 - Y_{i,s}^2)(Z_{ij,s}^1 - Z_{ij,s}^2)]^2 ds)^{\frac{1}{2}}.
\end{aligned}
$$

But, there exists a positive constant C_β depending only on β such that

$$c\beta\mathbb{E}(\int_u^{u'} \overline{e}_s^2\Delta_s^{\beta-2}\sum_{j=1}^r[\sum_{i=1}^d(Y_{i,s}^1-Y_{i,s}^2)(Z_{ij,s}^1-Z_{ij,s}^2)]^2ds)^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\mathbb{E}(\sup_{u\leq t\leq u'}\Delta_t^{\frac{\beta}{2}})+C_\beta\mathbb{E}\int_u^{u'}\overline{e}_s\Delta_s^{\frac{\beta}{2}-1}|Z_s^1-Z_s^2|^2ds.$$

Use (3.3) and take ε small enough to obtain the existence of a positive constant $C=C(\beta,\overline{\gamma})$ such that

$$\mathbb{E}(\sup_{u\leq t\leq u'}\Delta_t^{\frac{\beta}{2}})+\mathbb{E}\int_u^{u'}\overline{e}_s\Delta_s^{\frac{\beta}{2}-1}|Z_s^1-Z_s^2|^2ds$$
$$\leq C\left[\overline{e}_{u'}^{\frac{\beta}{2}}\mathbb{E}|Y_{u'}^1-Y_{u'}^2|^\beta+\frac{\overline{e}_{u'}^{\frac{\beta}{2}}}{A_N^{\frac{\beta}{2}}}+\overline{e}_{u'}^{\frac{\beta}{2}}\sup_i\mathbb{E}\int_u^{u'}|f_i(s,Y_s^i,Z_s^i)|^\beta 1\!\!1_{\{\Phi_s>N\}}ds\right.$$
$$\left.+N\overline{e}_{u'}^{\frac{\beta}{2}}A_N^{1-\frac{\beta}{2}}\mathbb{E}\int_u^{u'}\rho_N(f_1-f)_s+\rho_N(f_2-f)_s 1\!\!1_{\{v_s\ \leq\ N\}}ds\right].$$

We shall estimate $J:=\sup_i\mathbb{E}\int_u^{u'}|f_i(s,Y_s^i,Z_s^i)|^\beta 1\!\!1_{\{\Phi_s>N\}}ds$, $i=1,2$.
Using the fact that $1\!\!1_{\{\Phi_s>N\}}\leq 1\!\!1_{\{v_s>5^{-1}N\}}+1\!\!1_{\{|Y_s^1|>5^{-1}N\}}+1\!\!1_{\{|Y_s^2|>5^{-1}N\}}+1\!\!1_{\{|Z_s^1|>5^{-1}N\}}+1\!\!1_{\{|Z_s^2|>5^{-1}N\}}$ and $1\!\!1_{\{a>b\}}\leq\frac{a^\nu}{b^\nu}$ for every $a,b,\nu>0$, we show that for every $N>1$

$$J\leq\left(\frac{5}{N}\right)^\nu\sup_i\mathbb{E}\int_u^{u'}|f_i(s,Y_s^i,Z_s^i)|^\beta v_s^\nu ds$$
$$+\left(\frac{5}{N}\right)^\nu\sup_i\mathbb{E}\int_u^{u'}|f_i(s,Y_s^i,Z_s^i)|^\beta|Y_s^1|^\nu ds$$
$$+\left(\frac{5}{N}\right)^\nu\sup_i\mathbb{E}\int_u^{u'}|f_i(s,Y_s^i,Z_s^i)|^\beta|Y_s^2|^\nu ds$$
$$+\left(\frac{5}{N}\right)^\nu\sup_i\mathbb{E}\int_u^{u'}|f_i(s,Y_s^i,Z_s^i)|^\beta|Z_s^1|^\nu ds.$$
$$+\left(\frac{5}{N}\right)^\nu\sup_i\mathbb{E}\int_u^{u'}|f_i(s,Y_s^i,Z_s^i)|^\beta|Z_s^2|^\nu ds.$$

using Young's inequality, one can prove that there exists a positive constant C such that for every $N>1$

$$J\leq\frac{C}{N^\nu}\left\{1+\Theta_p^1+\Theta_p^2+\sup_i\mathbb{E}\int_u^{u'}|f_i(s,Y_s^i,Z_s^i)|^{\beta(\frac{q'}{q'-\nu}\vee\frac{2}{2-\nu}\vee\frac{p}{p-\nu})}ds+\mathbb{E}\int_u^{u'}v_s^{q'}ds\right\}.$$

where $\Theta_p^i:=\mathbb{E}(\sup_t|Y_t^i|^p)+\mathbb{E}\left(\int_0^T|Z_s^i|^2ds\right)^{\frac{p}{2}}$.
Using Proposition 3.2, we get (since $\beta(\frac{q'}{q'-\nu}\vee\frac{2}{2-\nu}\vee\frac{p}{p-\nu})\leq\hat{\beta}$)

$$J\leq\frac{C}{N^\nu}\left\{1+\Theta_p^1+\Theta_p^2+\mathbb{E}\int_0^T|\overline{\eta}_s|^q ds+\mathbb{E}\int_u^{u'}v_s^{q'}ds\right\}.$$

Hence, for $a := (\frac{\nu}{\mu} \wedge \frac{\beta}{2}) - \beta r K"$ and N large enough we get (since $A_N \leq N^{\mu}$ by assumption {bf(H.4)),

$$\mathbb{E} \sup_{u\leq t\leq u'} \Delta_t^{\frac{\beta}{2}} + \mathbb{E}\int_u^{u'} \overline{e}_s \Delta_s^{\frac{\beta}{2}-1}|Z_s^1 - Z_s^2|^2 ds$$
$$\leq \; N A_N^{1+\frac{\beta}{2}} \left[\mathbb{E}|Y_{u'}^1 - Y_{u'}^2|^{\beta} + \mathbb{E}\int_0^T \rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s 1\!\!1_{\{v_s \;\leq\; N\}} ds \right]$$
$$+ \frac{1}{A_N^a} \left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds \right].$$

Lemma 3.5 is proved. ■

As a consequence of lemma 3.5, we have

Lemma 3.6. *Let $(\xi^i, f_i)_{i=1,2}$ satisfies **(H.3)** (with the same $\overline{\eta}, \alpha$ and α') and let (Y^i, Z^i) be a solution of $(E^{(\xi^i, f_i)})$. Then, there exists $\beta = \beta(p, q, \alpha, \alpha') \in]1, p \wedge 2[$ such that for every $\varepsilon > 0$ there is an integer $N_{\varepsilon} = N_{\varepsilon}(p, q, \alpha, \alpha', K', \mu, q', \varepsilon, (A_N)_N)$ such that for every function f satisfying **(H.4)***

$$\mathbb{E}(\sup_{0\leq t\leq T} |Y_t^1 - Y_t^2|^{\beta}) + \mathbb{E}\int_0^T \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}} ds$$
$$\leq \; N_{\varepsilon} \left[\mathbb{E}|\xi^1 - \xi^2|^{\beta} + \mathbb{E}\int_0^T \rho_{N_{\varepsilon}}(f_1 - f)_s + \rho_{N_{\varepsilon}}(f_2 - f)_s ds \right]$$
$$+ \varepsilon \left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds \right].$$

Proof. Let $(u_0 = 0 < ... < u_{\ell+1} = T)$ be a subdivision of $[0, T]$ such that for every $i \in \{0, .., \ell\}$

$$u_{i+1} - u_i \leq r$$

From lemma 3.5 we have : for all $\varepsilon > 0$ there is an integer N_{ε} such that for every function f satisfying **(H.4)**

$$\mathbb{E}(\sup_{u_\ell\leq t\leq T} |Y_t^1 - Y_t^2|^{\beta}) + \mathbb{E}\int_{u_\ell}^T \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}} ds$$
$$\leq \; N_{\varepsilon} \left[\mathbb{E}(|\xi^1 - \xi^2|^{\beta}) + \mathbb{E}\int_0^T \rho_{N_{\varepsilon}}(f_1 - f)_s + \rho_{N_{\varepsilon}}(f_2 - f)_s ds \right]$$
$$+ \varepsilon \left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds \right].$$

Assume that for some $i \in \{0, .., \ell\}$ we have for all $\varepsilon > 0$ there is an integer N_{ε} such that for every function f satisfying **(H.4)**

$$\mathbb{E}(\sup_{u_{i+1}\leq t\leq T} |Y_t^1 - Y_t^2|^{\beta}) + \mathbb{E}\int_{u_{i+1}}^T \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}} ds$$
$$\leq \; N_{\varepsilon} \left[\mathbb{E}(|\xi^1 - \xi^2|^{\beta}) + \mathbb{E}\int_0^T \rho_{N_{\varepsilon}}(f_1 - f)_s + \rho_{N_{\varepsilon}}(f_2 - f)_s ds \right]$$
$$+ \varepsilon \left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds \right].$$

Then, for every $\varepsilon' > 0$ there is an integer $N_{\varepsilon'}$ such that for every function f satisfying **(H.4)**

$$\begin{aligned}
&\mathbb{E}(\sup_{u_i\le t\le T}|Y_t^1-Y_t^2|^\beta)+\mathbb{E}\int_{u_i}^T\frac{|Z_s^1-Z_s^2|^2}{(1+|Y_s^1-Y_s^2|^2)^{1-\frac{\beta}{2}}}ds\\
&\le \mathbb{E}(\sup_{u_i\le t\le u_{i+1}}|Y_t^1-Y_t^2|^\beta)+\mathbb{E}\int_{u_i}^{u_{i+1}}\frac{|Z_s^1-Z_s^2|^2}{(1+|Y_s^1-Y_s^2|^2)^{1-\frac{\beta}{2}}}ds\\
&+N_{\varepsilon'}\left[\mathbb{E}(|\xi^1-\xi^2|^\beta)+\mathbb{E}\int_0^T\rho_{N_{\varepsilon'}}(f_1-f)_s+\rho_{N_{\varepsilon'}}(f_2-f)_s ds\right]\\
&+\varepsilon'\left[1+\Theta_p^1+\Theta_p^2+\mathbb{E}\int_0^T\overline{\eta}_s^q ds+\mathbb{E}\int_0^T v_s^{q'}ds\right].
\end{aligned}$$

Using Lemma 3.5 we obtain; for every $\varepsilon', \varepsilon" > 0$ there exist $N_{\varepsilon'} > 0$ and $N_{\varepsilon"} > 0$ such that for every function f satisfying **(H.4)**

$$\begin{aligned}
&\mathbb{E}(\sup_{u_i\le t\le T}|Y_t^1-Y_t^2|^\beta)+\mathbb{E}\int_{u_i}^T\frac{|Z_s^1-Z_s^2|^2}{(1+|Y_s^1-Y_s^2|^2)^{1-\frac{\beta}{2}}}ds\\
&\le N_{\varepsilon"}\left[\mathbb{E}(|Y_{u_{i+1}}^1-Y_{u_{i+1}}^2|^\beta)+\mathbb{E}\int_0^T\rho_{N_{\varepsilon"}}(f_1-f)_s+\rho_{N_{\varepsilon"}}(f_2-f)_s ds\right]\\
&+N_{\varepsilon'}\left[\mathbb{E}(|\xi^1-\xi^2|^\beta)+\mathbb{E}\int_0^T\rho_{N_{\varepsilon'}}(f_1-f)_s+\rho_{N_{\varepsilon'}}(f_2-f)_s ds\right]\\
&+2\varepsilon'\left[1+\Theta_p^1+\Theta_p^2+\mathbb{E}\int_0^T\overline{\eta}^q ds+\mathbb{E}\int_0^T v_s^{q'}ds\right]\\
&\le N_{\varepsilon'}N_{\varepsilon"}\ \mathbb{E}(|\xi^1-\xi^2|^\beta)\\
&+(N_{\varepsilon'}N_{\varepsilon"}+2N_{\varepsilon'})\ \mathbb{E}\int_0^T\rho_{(N_{\varepsilon'}N_{\varepsilon"})}(f_1-f)_s+\rho_{(N_{\varepsilon'}N_{\varepsilon"})}(f_2-f)_s ds\\
&+(2\varepsilon'+\varepsilon"N_{\varepsilon'})\left[1+\Theta_p^1+\Theta_p^2+\mathbb{E}\int_0^T\overline{\eta}_s^q ds+\mathbb{E}\int_0^T v_s^{q'}ds\right].
\end{aligned}$$

For $\varepsilon > 0$, let $\varepsilon' := \dfrac{\varepsilon}{4}$ and $\varepsilon" := \dfrac{\varepsilon}{2N_{(\frac{\varepsilon}{4})}}$, we then deduce that there exists an integer N_ε such that for every function f satisfying **(H.4)**

$$\begin{aligned}
&\mathbb{E}(\sup_{u_i\le t\le T}|Y_t^1-Y_t^2|^\beta)+\mathbb{E}\int_{u_i}^T\frac{|Z_s^1-Z_s^2|^2}{(1+|Y_s^1-Y_s^2|^2)^{1-\frac{\beta}{2}}}ds\\
&\le\ N_\varepsilon\left[\mathbb{E}(|\xi^1-\xi^2|^\beta)+\mathbb{E}\int_0^T\rho_{N_\varepsilon}(f_1-f)_s+\rho_{N_\varepsilon}(f_2-f)_s ds\right]\\
&+\varepsilon\left[1+\Theta_p^1+\Theta_p^2+\mathbb{E}\int_0^T\overline{\eta}_s^q ds+\mathbb{E}\int_0^T v_s^{q'}ds\right].
\end{aligned}$$

We complete the proof by induction ■

Proposition 3.3. *Let $(\xi^i, f_i)_{i=1,2}$ satisfies **(H.3)** (with the same $\overline{\eta}, \alpha$ and α') and let (Y^i, Z^i) be a solution of $(E^{(\xi^i,f_i)})$. Then, there exists $\beta = \beta(p, q, \alpha, \alpha') \in]1, p\wedge 2[$ such that for every $\varepsilon > 0$ there is* *an integer $N_\varepsilon = N_\varepsilon(p, q, \alpha, \alpha', K', \mu, q', \varepsilon, (A_N)_N)$ such that for every function f satisfying* ***(H.4)***

$$\begin{aligned}
&\mathbb{E}(\sup_{0\le t\le T}|Y_t^1 - Y_t^2|^\beta) + \mathbb{E}\left(\int_0^T |Z_s^1 - Z_s^2|^2 ds\right)^{\frac{\beta}{2}}\\
&\le N_\varepsilon\left[\mathbb{E}(|\xi^1 - \xi^2|^\beta) + \mathbb{E}\int_0^T \rho_{N_\varepsilon}(f_1 - f)_s + \rho_{N_\varepsilon}(f_2 - f)_s ds\right]\\
&\quad + \varepsilon\left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds\right],
\end{aligned}$$

where $\Theta_p^i := \mathbb{E}(\sup_t |Y_t^i|^p) + \mathbb{E}\left(\int_0^T |Z_s^i|^2 ds\right)^{\frac{p}{2}}$.

Proof. Using Hölder's inequality, Young's inequality and the fact that $\dfrac{\beta}{2} < 1$, we obtain for all $\varepsilon' > 0$

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T |Z_s^1 - Z_s^2|^2 ds\right)^{\frac{\beta}{2}}\\
&\le \mathbb{E}\left\{\left[\int_0^T \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}} ds\right]^{\frac{\beta}{2}} \sup_{s\le T}\left(1 + |Y_s^1 - Y_s^2|^2\right)^{(1-\frac{\beta}{2})\frac{\beta}{2}}\right\}\\
&\le \left[\mathbb{E}\int_0^T \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}} ds\right]^{\frac{\beta}{2}} \left(1 + \mathbb{E}(\sup_{s\le T}|Y_s^1 - Y_s^2|^\beta)\right)^{\frac{2-\beta}{2}}\\
&\le \left[\mathbb{E}(\sup_{s\le T}|Y_s^1 - Y_s^2|^\beta) + \mathbb{E}\int_0^T \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}} ds\right]^{\frac{\beta}{2}}\\
&\quad + \left[\mathbb{E}(\sup_{0\le t\le T}|Y_t^1 - Y_t^2|^\beta) + \mathbb{E}\int_0^T \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}} ds\right]\\
&\le \varepsilon' + (1 + \varepsilon'^{\frac{\beta-2}{\beta}})\left[\mathbb{E}(\sup_{0\le t\le T}|Y_t^1 - Y_t^2|^\beta) + \mathbb{E}\int_0^T \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}} ds\right].
\end{aligned}$$

Use lemma 3.5 to conclude that for every $\varepsilon', \varepsilon" > 0$

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T |Z_s^1 - Z_s^2|^2 ds\right)^{\frac{\beta}{2}}\\
&\le \varepsilon' + (1 + \varepsilon'^{\frac{\beta-2}{\beta}})N_{\varepsilon"}\left[\mathbb{E}(|\xi^1 - \xi^2|^\beta) + \mathbb{E}\int_0^T \rho_{N_{\varepsilon"}}(f_1 - f)_s + \rho_{N_{\varepsilon"}}(f_2 - f)_s ds\right]\\
&\quad + \varepsilon"(1 + \varepsilon'^{\frac{\beta-2}{\beta}})\left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds\right].
\end{aligned}$$

Letting $\varepsilon' = \dfrac{\varepsilon}{2}$ and $\varepsilon" = \dfrac{\varepsilon}{2(1 + (\frac{\varepsilon}{2})^{\frac{\beta-2}{2}})}$, we finish this proof of proposition 3.3. ■

Remark 3.1. *The uniqueness of equation $(E^{(\xi,f)})$ follows by letting $f_1 = f_2 = f$ and $\xi_1 = \xi_2 = \xi$ in Proposition 3.3.*

The following stability result follows from propositions (3.3), (3.2) and (3.1)

Proposition 3.4. *Let (ξ, f) satisfies* **(H.0)-(H.4)** *and $(\xi^n, f_n)_n$ satisfies* **(H.0)-(H.3)** *uniformly on n. Assume moreover that*

(a) $\xi^n \to \xi$ *a.s.* *and* $\sup_n \mathbb{E}\big(|\xi_n|^p \exp(\frac{p}{2}\int_0^T \lambda_s ds)\big) < \infty$

(b) For every $N \in \mathbb{N}^*$, $\lim_n \rho_N(f_n - f) = 0$ *a.e.*

(c) for every $n \in \mathbb{N}^*$, the BSDE $(E^{(\xi^n, f_n)})$ has a solution (Y^n, Z^n) which satisfies,

$$\sup_n \mathbb{E}(\sup_{t\le T} |Y_t^n|^p e^{\frac{p}{2}\int_0^T \lambda_s ds}) < \infty.$$

Then, there exists $(Y, Z) \in \mathbb{L}^p(\Omega; \mathcal{C}([0,T];\mathbb{R}^d)) \times \mathbb{L}^p(\Omega; \mathbb{L}^2([0,T];\mathbb{R}^{dr}))$ *such that*

i) $$\mathbb{E}(\sup_t |Y_t|^p e^{\frac{p}{2}\int_0^t \lambda_s ds}) + \mathbb{E}\left[\int_0^T e^{\int_0^s \lambda_r dr} |Z_s|^2 ds\right]^{\frac{p}{2}} \le C^{p,\gamma}\left\{\mathbb{E}(|\xi|^p e^{\frac{p}{2}\int_0^T \lambda_s ds}) + \mathbb{E}\left(\int_0^T e^{\int_0^s \lambda_r dr}\eta_s ds\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T e^{\frac{1}{2}\int_0^s \lambda_r dr} f_s^0 ds\right)^p\right\}$$

ii) for every $p' < p$, $(Y^n, Z^n) \longrightarrow (Y, Z)$ strongly in $\mathbb{L}^{p'}(\Omega; \mathcal{C}([0,T];\mathbb{R}^d)) \times \mathbb{L}^{p'}(\Omega; \mathbb{L}^2([0,T];\mathbb{R}^{dr}))$.

iii) for every $\hat\beta < \frac{2}{\alpha'} \wedge \frac{p}{\alpha} \wedge \frac{p}{\alpha'} \wedge q$, $\lim_{n\to\infty} \mathbb{E}\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|^{\hat\beta} ds = 0$

Moreover, (Y, Z) is the unique solution of $(E^{(\xi, f)})$.

Proof. From Proposition 3.1, Proposition 3.2 and Proposition 3.3, we have

$a')$ $$\left[\mathbb{E}(\sup_t |Y_t^n|^p e^{\frac{p}{2}\int_0^t \lambda_s ds}) + \mathbb{E}\left(\int_0^T e^{\int_0^t \lambda_s ds}|Z_s^n|^2 ds\right)^{\frac{p}{2}}\right] \le C^{p,\gamma}\sup_n\left\{\mathbb{E}(|\xi^n|^p e^{\frac{p}{2}\int_0^T \lambda_s ds}) + \mathbb{E}\left(\int_0^T e^{\int_0^s \lambda_r dr}\eta_s ds\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T e^{\frac{1}{2}\int_0^s \lambda_r dr} f_s^0 ds\right)^p\right\} := D.$$

$b')$ $$\mathbb{E}\int_0^T |f_n(s, Y_s^n, Z_s^n)|^{\hat\beta} ds \le C(1 + D + \int \bar\eta_s^q ds).$$

$c')$ There exists $\beta > 1$ such that for every $\varepsilon > 0$ there exists $N_\varepsilon > 0$:

$$\mathbb{E}(\sup_t |Y_t^n - Y_t^m|^\beta) + \mathbb{E}\left(\int_0^T |Z_s^n - Z_s^m|^2 ds\right)^{\frac{\beta}{2}} \le N_\varepsilon \mathbb{E}\left[|\xi^n - \xi^m|^\beta + \int_0^T \rho_{N_\varepsilon}(f_n - f)_s + \rho_{N_\varepsilon}(f_m - f)_s ds\right] + \varepsilon\left[1 + 2D + \mathbb{E}\int_0^T \bar\eta_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds\right].$$

We deduce the existence of $(Y, Z) \in \mathbb{L}^p(\Omega; \mathcal{C}([0,T];\mathbb{R}^d)) \times \mathbb{L}^p(\Omega; \mathbb{L}^2([0,T];\mathbb{R}^{dr}))$ such that

i) $$\mathbb{E}(\sup_t |Y_t|^p e^{\frac{p}{2}\int_0^t \lambda_s ds}) + \mathbb{E}\left[\int_0^T e^{\int_0^s \lambda_r dr} |Z_s|^2 ds\right]^{\frac{p}{2}} \le C^{p,\gamma}\left\{\mathbb{E}(|\xi|^p e^{\frac{p}{2}\int_0^T \lambda_s ds}) + \mathbb{E}\left(\int_0^T e^{\int_0^s \lambda_r dr}\eta_s ds\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T e^{\frac{1}{2}\int_0^s \lambda_r dr} f_s^0 ds\right)^p\right\}$$

ii) for all $p' < p$, $(Y^n, Z^n) \longrightarrow (Y, Z)$ strongly in $\mathbb{L}^{p'}(\Omega; \mathcal{C}([0,T];\mathbb{R}^d)) \times \mathbb{L}^{p'}(\Omega; \mathbb{L}^2([0,T];\mathbb{R}^{dr}))$.

Let us prove *iii)*. Set $a := \limsup_{n\to\infty} \mathbb{E}\int_0^T |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|^{\hat\beta} ds$. Consider a subsequence n' of n such that $a := \lim_{n'\to\infty} \mathbb{E}\int_0^T |f(s, Y_s^{n'}, Z_s^{n'}) - f(s, Y_s, Z_s)|^{\hat\beta} ds$ and, $(Y^{n'}, Z^{n'}) \to (Y, Z)$ a.e. Assumption **(H.3)** and the continuity of f ensure that $a = 0$. It remains to prove that

$$\limsup_{n\to\infty}\mathbb{E}\int_0^T |f_n(s,Y_s^n,Z_s^n)-f(s,Y_s^n,Z_s^n)|^{\hat\beta}ds=0$$

We use Hölder's inequality, the previous claim b'), Proposition 3.2 and Chebychev's inequality to get

$$\begin{aligned}
&\mathbb{E}\int_0^T |f_n(s,Y_s^n,Z_s^n)-f(s,Y_s^n,Z_s^n)|^{\hat\beta}ds\\
&\le \mathbb{E}\int_0^T \rho_N(f_n-f)_s^{\hat\beta}ds+(\mathbb{E}\int_0^T |f_n(s,Y_s^n,Z_s^n)-f(s,Y_s^n,Z_s^n)|^{r\hat\beta}ds)^{\frac{1}{r}}(\mathbb{E}\int_0^T 1_{|Y_s^n|+|Z_s^n|\ge N}ds)^{\frac{r-1}{r}}\\
&\le \mathbb{E}\int_0^T \rho_N(f_n-f)_s^{\hat\beta}ds+\frac{C(r)}{N^{\frac{(r-1)(p\wedge 2)}{r}}},
\end{aligned}$$

for some reel $r>1$ such that $r\hat\beta<\dfrac{2}{\alpha'}\wedge\dfrac{p}{\alpha}\wedge\dfrac{p}{\alpha'}\wedge q$.
We successively let $n\longrightarrow\infty$ and $N\longrightarrow\infty$ to derive assertion $iii)$. Proposition 3.4 is proved ■

3.3 Approximation

We shall construct a sequence (ξ^n,f_n) which converges in a suitable sense to (ξ,f) and which has good properties. With the help of this approximation, we can construct a solution (Y,Z) to the BSDE $(E^{(\xi,f)})$ by using Proposition 3.4.

Let h_t is a predictable process such that $0<h_t\le 1$ and set $\overline{\Lambda}_t:=\eta_t+\overline{\eta}_t+f_t^0+M_t+K_t+\frac{1}{h_t}$

Proposition 3.5. *Assume that (ξ,f) satisfies **(H.0)–(H.3)**. Then there exists a sequence (ξ^n,f_n) such that*
(a) For each n, ξ^n is bounded, $|\xi^n|\le|\xi|$ and ξ^n converges to ξ a.s.
(b) For each n, f_n is uniformly Lipschitz in (y,z).
(c) $|f_n(t,\omega,y,z)|\ \le\ 1\!\!1_{\{\overline{\Lambda}_t\ \le\ n,\ |y|\le n,\ |z|\le n\}}\{\overline{\eta}_t+|\,y\,|^{\alpha}+|\,z\,|^{\alpha'}+2ph_t\}\le 2p+3n^p$.
(d) $<y,f_n(t,\omega,y,z)>\ \le\ 1\!\!1_{\{\overline{\Lambda}_t\ \le\ n\}}\{\eta_t+f_t^0|y|+M_t|y|^2+K_t|y||z|+10h_t\}$.
(e) For every N, $\rho_N(f_n-f)(t,\omega)\longrightarrow 0$ as $n\longrightarrow\infty$ a.e (t,ω).
(f) For every N, $\rho_N(f_n-f)(t,\omega)\ \le\ 2\{\overline{\eta}_t+N^{\alpha}+N^{\alpha'}+2ph_t\}$.

Proof. Let $\psi:\mathbb{R}\longrightarrow[0,\dfrac{\exp(-1)}{c_1}]$ defined by:

$$\psi(x):=\begin{cases} c_1^{-1}\exp(-\dfrac{1}{1-x^2}) & \text{if } |x|<1\\ 0 & \text{else}\end{cases}$$

where $c_1=\displaystyle\int_{-1}^1\exp(-\frac{1}{1-x^2})dx$.

Let $m:=\dfrac{n^{2p}}{h_t}$. the sequence (ξ^n,f_n) defined by : $\xi^n:=\xi 1_{\{|\xi|\le n\}}$ and

$$\begin{aligned}
f_n(t,y,z)=&(c_1e)^2 1\!\!1_{\{\overline{\Lambda}_t\ \le\ n\}}\psi(n^{-2}|y|^2)\psi(n^{-2}|z|^2)\times\\
&m^{(d+dr)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{dr}}f(t,y-u,z-v)\Pi_{i=1}^d\psi(mu_i)\Pi_{i=1}^d\Pi_{j=1}^r\psi(mv_{ij})dudv,
\end{aligned}$$

satisfies the required properties. Indeed, (a) is obvious. (e) follows from the definition of f_n. (f) follows from assumption **(H.3)** and assertion (c). We shall prove assertions (b), (c) and (d).

(b) For a fixed t and ω, $f_n(t,\omega,.,.)$ is smooth and with compact support in $[-n,n]^{d+dr}$. Moreover

$$|\ \nabla_{y,z}f_n(t,\omega,y,z)\ |\le Cn^{2p+2},$$

where ∇ denotes the gradient and C is a positive constant.

(c) For all (t,ω,y,z) such that $\overline{\Lambda}_t \leq n$, $|y|\leq n$ and $|z|\leq n$ we obtain, by using assumption **(H.3)**, that

$$
\begin{aligned}
| f_n(t,y,z) | \leq & m^{(d+dr)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{dr}} | f(t,y-u,z-v) | \Pi_{i=1}^d \psi(mu_i)\Pi_{i=1}^d\Pi_{j=1}^r\psi(mv_{ij})dudv \\
& \leq \overline{\eta}_t + | y |^\alpha + | z |^{\alpha'} + m^d\int_{\mathbb{R}^d}\Big(| y-u |^\alpha - | y |^\alpha \Big)\Pi_{i=1}^d\psi(mu_i)du \\
& + m^{dr}\int_{\mathbb{R}^{dr}}\Big(| z-v |^{\alpha'} - | z |^{\alpha'} \Big)\Pi_{i=1}^d\Pi_{j=1}^r\psi(mv_{ij})dv \\
& \leq \overline{\eta}_t + | y |^\alpha + | z |^{\alpha'} + \alpha(n+\frac{h_t}{n^{2p}})^{\alpha-1}\frac{h_t}{n^{2p}} + \alpha'(n+\frac{h_t}{n^{2p}})^{\alpha'-1}\frac{h_t}{n^{2p}} \\
& \leq \overline{\eta}_t + | y |^\alpha + | z |^{\alpha'} + 2ph_t
\end{aligned}
$$

(d) For all (t,ω,y,z) such that $\overline{\Lambda}_t \leq n$, $|y|\leq n$ and $|z|\leq n$ we obtain, by using assumptions **(H.2)** – **(H.3)**, that

$$
\begin{aligned}
\langle y, f_n(t,y,z)\rangle \leq & (c_1e)^2\psi(n^{-2}|y|^2)\psi(n^{-2}|z|^2)\times \\
& m^{(d+dr)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{dr}}\langle f(t,y-u,z-v), y-u\rangle\Pi_{i=1}^d\psi(mu_i)\Pi_{i=1}^d\Pi_{j=1}^r\psi(mv_{ij})dudv \\
& + m^{(d+dr)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{dr}} | f(t,y-u,z-v) || u | \Pi_{i=1}^d\psi(mu_i)\Pi_{i=1}^d\Pi_{j=1}^r\psi(mv_{ij})dudv \\
& \leq \eta_t + f_t^0|y| + M_t|y|^2 + K_t|y||z| + 10h_t
\end{aligned}
$$

■

Remark 3.2. *Theorem 2.1 follows now from Proposition 3.4 and Proposition 3.5.*

4 The second main result : Systems of degenerate PDEs

In this section, we consider the system of semilinear PDEs associated to the Markovian version of the BSDE ($E^{\xi,f}$), for which we establish the existence and uniqueness of a weak (Sobolev) solution. In particular, we introduce a new method of proving, by means of BSDE, that uniqueness to a system of inhomogeneous semilinear PDE results from the uniqueness of its associated homogeneous system of linear PDE.

4.1 Formulation of the problem.

Let $\sigma:\mathbb{R}^k \longmapsto \mathbb{R}^{kr}$, $b:\mathbb{R}^k \longmapsto \mathbb{R}^k$, $g:\mathbb{R}^k \longmapsto \mathbb{R}^k$, and $F:[0,T]\times\mathbb{R}^k\times\mathbb{R}^d\times\mathbb{R}^{dr}\longmapsto\mathbb{R}^d$ be measurable functions. Consider the system of semilinear PDEs

$$
(\mathcal{P}^{(g,F)}) \quad \begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \mathcal{L}u(t,x) + F(t,x,u(t,x),\sigma^*\nabla u(t,x)) = 0 & t\in]0,T[,\ x\in\mathbb{R}^k \\ u(T,x) = g(x) & x\in\mathbb{R}^k \end{cases}
$$

where $\mathcal{L} := \dfrac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}\partial_{ij}^2 + \sum_i b_i\partial_i$.

The diffusion process associated to the operator $\mathcal{L}$ satisfies,

$$
X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r, \qquad t\leq s\leq T
$$

We assume throughout this section that $\sigma\in\mathcal{C}_b^3(\mathbb{R}^k,\mathbb{R}^{kr})$, and $b\in\mathcal{C}_b^2(\mathbb{R}^k,\mathbb{R}^k)$.

We define,

$$
\mathcal{H}^{1+} := \bigcup_{\delta\geq 0,\beta>1}\left\{ v\in\mathcal{C}([0,T];\mathbb{L}^\beta(\mathbb{R}^k,e^{-\delta|x|}dx;\mathbb{R}^d)) : \int_0^T\int_{\mathbb{R}^k}|\sigma^*\nabla v(s,x)|^\beta e^{-\delta|x|}dxds < \infty \right\}
$$

Definition 4.1. *A (weak) solution of* $(\mathcal{P}^{(g,F)})$ *is a function* $u \in \mathcal{H}^{1+}$ *such that for every* $t \in [0,T]$ *and* $\varphi \in C_c^1([0,T] \times \mathbb{R}^d)$

$$\int_t^T < u(s), \frac{\partial \varphi(s)}{\partial s} > ds + < u(t), \varphi(t) >= < g, \varphi(T) > + \int_t^T < F(s,.,u(s),\sigma^* \nabla u(s)), \varphi(s) > ds$$
$$+ \int_t^T < Lu(s), \varphi(s) > ds$$

where $< f(s), h(s) >= \int_{\mathbb{R}^k} f(s,x)h(s,x)dx$.

Observe that an integrating by part shows that,

$$< Lu(s), \varphi(s) > = \; - \int_{\mathbb{R}^k} \frac{1}{2} \langle \sigma^* \nabla u(s,x); \sigma^* \nabla \varphi(s,x) \rangle dx \; - < u(s), div(\tilde{b}\varphi)(s) >$$

where $\tilde{b}_i := b_i - \frac{1}{2} \sum_j \partial_j (\sigma\sigma^*)_{ij}$

4.2 Assumptions

Consider the following assumptions:

There exist $\delta \geq 0$ and $\overline{p} > 1$ such that

(A.0) $g(x) \in \mathbb{L}^{\overline{p}}(\mathbb{R}^k, e^{-\delta|x|}dx; \mathbb{R}^d)$

(A.1) $F(t,x,.,.)$ is continuous for a.e. (t,x)

(A.2) $\begin{cases} \text{There are } \eta' \in \mathbb{L}^{\frac{\overline{p}}{2} \vee 1}([0,T] \times \mathbb{R}^k, e^{-\delta|x|}dtdx; \mathbb{R}_+)), \\ f^{0'} \in \mathbb{L}^{\overline{p}}([0,T] \times \mathbb{R}^k, e^{-\delta|x|}dtdx; \mathbb{R}_+)), \text{ and } M, M' \in \mathbb{R}_+ \text{ such that} \\ \\ \langle y, F(t,x,y,z) \rangle \leq \eta'(t,x) + f^{0'}(t,x)|y| + (M + M'|x|)|y|^2 + \sqrt{M + M'|x|}|y||z| \end{cases}$

(A.3) $\begin{cases} \text{There are } \overline{\eta}' \in \mathbb{L}^q([0,T] \times \mathbb{R}^k, e^{-\delta|x|}dtdx; \mathbb{R}_+)) \text{ (for some } q > 1), \; \alpha \in]1, \overline{p}[\\ \text{and } \alpha' \in]1, \overline{p} \wedge 2[\text{ such that} \\ \\ |F(t,x,y,z)| \leq \overline{\eta}'(t,x) + |y|^\alpha + |z|^{\alpha'} \end{cases}$

(A.4) $\begin{cases} \text{There are } K, r \in \mathbb{R}_+ \text{ such that for every } N \in \mathbb{N} \text{ and every } x, y, \; y', \; z, \; z' \\ \text{satisfying : } \; e^{r|x|}, \; | \, y \, |, \; | \, y' \, |, \; | \, z \, |, \; | \, z' \, | \leq N, \\ \\ \langle y - y'; F(t,x,y,z) - F(t,x,y',z') \rangle \leq K \log N \left(\frac{1}{N} + |y - y'|^2 \right) + \sqrt{K \log N} |y - y'||z - z'|. \end{cases}$

4.3 Existence and uniqueness for $(\mathcal{P}^{(g,F)})$

Theorem 4.1. *Let* $p \in]\alpha \vee \alpha', \overline{p}[$ *if* $M' > 0$ *and* $p = \overline{p}$ *if* $M' = 0$. *Under assumption* ***(A.0)-(A.4)*** *we have*

1) The PDE $(\mathcal{P}^{(g,F)})$ *has a unique (weak) solution* u *on* $[0,T]$

2) For every $t \in [0,T]$ *there exists* $D_t \subset \mathbb{R}^k$ *such that*

i) $\int_{\mathbb{R}^k} 1\!\!1_{D_t^c} \, dx = 0$, *where* $D_t^c := \mathbb{R}^k \setminus D_t$.

ii) for every $t \in [0,T]$ *and every* $x \in D_t$, *the BSDE* $(E^{(\xi^{t,x}, f^{t,x})})$ *has a unique solution* $(Y^{t,x}, Z^{t,x})$ *on* $[t,T]$

where $\xi^{t,x} := g(X_T^{t,x})$ *and* $f^{t,x}(s,y,z) := 1\!\!1_{\{s>t\}} F(s, X_s^{t,x}, y, z)$

3) For every $t \in [0,T]$

$$\big(u(s,X_s^{t,x}),\sigma^*\nabla u(s,X_s^{t,x})\big) = \big(Y_s^{t,x},Z_s^{t,x}\big) \qquad a.e.(s,x,\omega)$$

4) There exists a positive constant C depending only on $\delta, M, M', p, \bar{p}$, $|\sigma|_\infty, |b|_\infty$ and T such that

$$\sup_{0\le t\le T}\int_{\mathbb{R}^k} \mid u(t,x)\mid^p e^{-\delta'|x|}dx + \int_0^T\int_{\mathbb{R}^k} \mid \sigma^*\nabla u(t,x)\mid^{p\wedge 2} e^{-\delta'|x|}dtdx$$
$$\le C\left(1\!\!1_{[M'\neq 0]} + \int_{\mathbb{R}^k} \mid g(x)\mid^{\overline{p}} dx + \int_{\mathbb{R}^k}\int_0^T \eta'(s,x)^{\frac{\overline{p}}{2}\vee 1}dsdx + \int_{\mathbb{R}^k}\int_0^T f^{0'}(s,x)^{\overline{p}}dsdx\right)$$

where $\delta' = \delta + \kappa' + 1\!\!1_{[M'\neq 0]}$ and $\kappa' := \dfrac{p\overline{p}M'T}{(\overline{p}-p)}\sup(4,\frac{2p}{p-1})$.

4.4 Proof of Theorem 4.1.

A) **Existence.**

Lemma 4.1. 1) *There exists $\kappa > 0$ depending only on $|\sigma|_\infty, |b|_\infty$ and T such that*

$$\sup_{t,x}\mathbb{E}[\exp(\kappa \sup_{t\le s\le T} \mid X_s^{t,x} - x\mid^2)] < \infty. \tag{4.0}$$

In particular, for every $r > 0$ there is a constant $C(r,\kappa)$ such that for each (t,x)

$$\mathbb{E}[\exp(r\sup_{t\le s\le T}\mid X_s^{t,x}\mid)] \le C(r,\kappa)\exp(r\mid x\mid)$$

2) *For every $\delta \ge 0$ there exists a constant $C_{\delta,T} > 1$ such that for every $\varphi \in \mathbb{L}^0(\mathbb{R}^k)$, $t \in [0,T]$ and $s \in [t,T]$*

$$C_{\delta,T}^{-1}\int_{\mathbb{R}^k}|\varphi(x)|e^{-\delta|x|}dx \le \mathbb{E}\int_{\mathbb{R}^k}|\varphi(X_s^{t,x})|e^{-\delta|x|}dx \le C_{\delta,T}\int_{\mathbb{R}^k}|\varphi(x)|e^{-\delta|x|}dx. \tag{4.2}$$

Moreover for every $\delta \ge 0$ there exists a constant $C_{\delta,T} > 1$ such that for every $\psi \in \mathbb{L}^0([0,T]\times\mathbb{R}^k)$, $t \in [0,T]$ and $s \in [t,T]$

$$C_{\delta,T}^{-1}\int_{\mathbb{R}^k}\int_t^T|\psi(s,x)|dse^{-\delta|x|}dx \le \mathbb{E}\int_{\mathbb{R}^k}\int_t^T|\psi(s,X_s^{t,x})|dse^{-\delta|x|}dx \le C_{\delta,T}\int_{\mathbb{R}^k}\int_t^T|\psi(s,x)|dse^{-\delta|x|}dx.$$

Proof. The first assertion is well known. Its particular case follows by using triangular and Young's inequalities. Indeed

$$\begin{aligned}\mathbb{E}[\exp(r\sup_{t\le s\le T}\mid X_s^{t,x}\mid)] &\le \exp(r\mid x\mid)\mathbb{E}[\exp(r\sup_{t\le s\le T}\mid X_s^{t,x}-x\mid)]\\ &\le \exp(r\mid x\mid)\mathbb{E}[\exp(\frac{r}{\sqrt{\kappa}}\sqrt{\kappa}\sup_{t\le s\le T}\mid X_s^{t,x}-x\mid)]\\ &\le \exp(\frac{r^2}{\kappa})\exp(r\mid x\mid)\mathbb{E}[\exp(\kappa\sup_{t\le s\le T}\mid X_s^{t,x}-x\mid^2)].\end{aligned}$$

For the second assertion, see [40]. ∎

Lemma 4.2. *Let $p \in]\alpha \vee \alpha', \bar{p}[$ if $M' > 0$ and $p = \bar{p}$ if $M' = 0$. Let $t \in [0,T]$. There exists $D_t \subset \mathbb{R}^k$ such that*
i) $\int_{D_t^c} 1\, dx = 0$
ii) for every $x \in D_t$

$$\mathbb{E}(|\, g(X_T^{t,x})\,|^p\, e^{\frac{p}{2}\int_t^T \lambda_s^{t,x} ds}) + \mathbb{E}\left(\int_t^T \eta'(s, X_s^{t,x}) e^{\int_t^s \lambda_r^{t,x} dr} ds\right)^{\frac{p}{2}}$$
$$+ \mathbb{E}\left(\int_t^T f^{0'}(s, X_s^{t,x}) e^{\frac{1}{2}\int_t^s \lambda_r^{t,x} dr} ds\right)^p + \mathbb{E}\int_t^T \overline{\eta}'(s, X_s^{t,x})^q ds \; < +\infty,$$

where $\lambda_s^{t,x} := (M + M'|X_s^{t,x}|)\sup(4, \frac{2p}{p-1})$.

Proof . Using Hölder's inequality, Young's inequality and Lemma 4.1 we get

$$\mathbb{E}(|\, g(X_T^{t,x})\,|^p\, e^{\frac{p}{2}\int_t^T \lambda_s^{t,x} ds}) + \mathbb{E}\left(\int_t^T \eta'(s, X_s^{t,x}) e^{\int_t^s \lambda_r^{t,x} dr} ds\right)^{\frac{p}{2}}$$
$$+\mathbb{E}\left(\int_t^T f^{0'}(s, X_s^{t,x}) e^{\frac{1}{2}\int_t^s \lambda_r^{t,x} dr} ds\right)^p + \mathbb{E}\int_t^T \overline{\eta}'(s, X_s^{t,x})^q ds$$
$$\leq C\left(\mathbb{E}(|\, g(X_T^{t,x})\,|^{\overline{p}}) + \mathbb{E}\int_t^T \eta'(s, X_s^{t,x})^{\frac{\overline{p}}{2}\vee 1} + \mathbb{E}\int_t^T f^{0'}(s, X_s^{t,x})^{\overline{p}} + \mathbb{E}\int_t^T \overline{\eta}'(s, X_s^{t,x})^q ds + 1\!\!1_{[M'\neq 0]} e^{\kappa'|x|}\right)$$

for some constant C depending only on $M, M', p, \bar{p}$, $|\sigma|_\infty, |b|_\infty$ and T.
We put,

$$\Gamma^{t,x} := C\big(\mathbb{E}(|\, g(X_T^{t,x})\,|^{\overline{p}}) + \mathbb{E}\int_t^T \eta'(s, X_s^{t,x})^{\frac{\overline{p}}{2}\vee 1} + \mathbb{E}\int_t^T f^{0'}(s, X_s^{t,x})^{\overline{p}} + \mathbb{E}\int_t^T \overline{\eta}'(s, X_s^{t,x})^q ds + 1\!\!1_{[M'\neq 0]} e^{\kappa'|x|}\big).$$

Using Lemma 4.1-2) and assumptions **(A.0)-(A.3)**, one can show that

$$\int_{\mathbb{R}^k} \Gamma^{t,x} e^{-\delta'|x|} dx \; < \; \infty$$

where $\delta' = \delta + \kappa' + 1$. The set $D_t := \{x; \quad \Gamma^{t,x} < \infty\}$. Lemma 4.2 is proved. ■

Lemma 4.3. *Assume **(A.0)-(A.4)**. Let $p \in]\alpha \vee \alpha', \bar{p}[$ if $M' > 0$ and $p = \bar{p}$ if $M' = 0$. Then, for every $t \in [0,T]$ and every $x \in D_t$, the BSDE $(E^{(\xi^{t,x}, f^{t,x})})$ has a unique solution $(Y^{t,x}, Z^{t,x})$ which satisfies, for every $t \in [0,T]$ and every $x \in D_t$,*

$$\begin{aligned}&\mathbb{E}\big(\sup_{t\leq s\leq T} |\, Y_s^{t,x}\,|^p\,\big) + \mathbb{E}\big(\int_t^T |\, Z_s^{t,x}\,|^2\, ds\big)^{\frac{p}{2}} \\ &\leq C\big[\mathbb{E}(|\, g(X_T^{t,x})\,|^{\overline{p}}) + \mathbb{E}\int_t^T \eta'(s, X_s^{t,x})^{\frac{\overline{p}}{2}\vee 1} ds + \mathbb{E}\int_t^T f^{0'}(s, X_s^{t,x})^{\overline{p}} ds + 1\!\!1_{[M'\neq 0]} e^{\kappa'|x|}\big]\end{aligned} \tag{4.3}$$

for some constant C depending only on $M, M', p, \bar{p}$, $|\sigma|_\infty, |b|_\infty$ and T .

Proof. For every $t \in [0,T]$ and $x \in D_t$, $(\xi^{t,x}, f^{t,x})$ satisfies **(H.0)-(H.4)** with $\gamma = \inf\{\frac{1}{4}, \frac{p-1}{4}\}$, $M_s = M + M'|X_s^{t,x}|$, $K_s = \sqrt{M + M'|X_s^{t,x}|}$, $\eta_s = \eta'(s, X_s^{t,x})$, $f_s^0 = f^{0'}(s, X_s^{t,x})$, $\overline{\eta}_s = \overline{\eta}'(s, X_s^{t,x})$, $v_s = \exp(r|X_s^{t,x}|)$ and $A_N = N$. Hence, Lemma 4.3 follows from Theorem 2.1 and Lemma 4.2. ■

Set,

$$g_n(x) := g(x) 1\!\!1_{\{|g(x)|\leq n\}},$$

$$F_n(t,x,y,z) := (n^{2p} e^{|x|})^{(d+dr)} (c_1 e)^2 1\!\!1_{\{\eta'(t,x) + \overline{\eta}'(t,x) + f^{0'}(t,x) + |x| \leq n\}} \psi(n^{-2}|y|^2) \psi(n^{-2}|z|^2) \times$$
$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^{dr}} F(t,x,y-u,z-v) \Pi_{i=1}^d \psi(n^{2p} e^{|x|} u_i) \Pi_{i=1}^d \Pi_{j=1}^r \psi(n^{2p} e^{|x|} v_{ij}) du dv,$$

$\xi_n^{t,x} := g_n(X_T^{t,x})$
and
$f_n^{t,x}(s,y,z) := 1\!\!1_{\{s>t\}} F_n(s, X_s^{t,x}, y, z).$

It is not difficult to see that the sequence (g_n, F_n) satisfies **(A.0)-(A.3)** uniformly in n. Hence $(\xi_n^{t,x}, f_n^{t,x})$ satisfies **(H.0)-(H.3)** uniformly in n. Moreover, for every $n \in \mathbb{N}^*$, $(\xi_n^{t,x}, f_n^{t,x})$ is bounded and $f_n^{t,x}$ is globally Lipschitz.
Let $(Y^{t,x,n}, Z^{t,x,n})$ be the unique solution of BSDE $(E^{(\xi_n^{t,x}, f_n^{t,x})})$. Let $p \in]\alpha \vee \alpha', \bar{p}[$ if $M' > 0$ and $p = \bar{p}$ if $M' = 0$. Arguing as in Lemma 4.3, we show that for every t, $x \in D_t$ and every $n \in \mathbb{N}^*$

$$\mathbb{E}(\sup_{t\le s\le T} \mid Y_s^{t,x,n} \mid^p) + \mathbb{E}\Big(\int_t^T \mid Z_s^{t,x,n} \mid^2 ds\Big)^{\frac{p}{2}} \le C\bigg(\mathbb{E}\int_t^T e^{-(\frac{\bar{p}}{2}\vee 1)|X_s^{t,x}|} ds + \mathbb{E}(\mid g(X_T^{t,x}) \mid^{\bar{p}}) + \\ + \mathbb{E}\int_t^T \eta'(s, X_s^{t,x})^{\frac{\bar{p}}{2}\vee 1} ds + \mathbb{E}\int_t^T f^{0'}(s, X_s^{t,x})^{\bar{p}} ds + 1\!\!1_{[M'\neq 0]} e^{\kappa'|x|}\bigg) \tag{4.4}$$

for some constant $C = C(\bar{p})$ not depending on (t,x,n). To see this, use proposition 3.5 (with $h_s := e^{-|X_s^{t,x}|}$), Proposition 3.1 and the proof of proposition 3.4-a).

According to [10], we have

Lemma 4.4. *There exists a unique solution u^n to the problem,*

$$(\mathcal{P}^{(g_n,F_n)}) \begin{cases} \dfrac{\partial u^n(t,x)}{\partial t} + \mathcal{L}u^n(t,x) + F_n(t,x,u^n(t,x), \sigma^*\nabla u^n(t,x)) = 0, & t \in]0,T[,\ x \in \mathbb{R}^k \\ u^n(T,x) = g_n(x), \quad x \in \mathbb{R}^k \end{cases}$$

such that for every t

$$u^n(s, X_s^{t,x}) = Y_s^{t,x,n} \quad \textit{and} \quad \sigma^*\nabla u^n(s, X_s^{t,x}) = Z_s^{t,x,n} \qquad a.e\ (s,\omega,x).$$

From Proposition 3.4-(ii) we have

Lemma 4.5. *(Stability) For every $t \in [0,T]$, $x \in D_t$ and $p' < \bar{p}$,*

$$\lim_n \left[\mathbb{E}(\sup_{0\le s\le T} \mid Y_s^{t,x,n} - Y_s^{t,x} \mid^{p'}) + \mathbb{E}\left(\int_t^T \mid Z_s^{t,x,n} - Z_s^{t,x} \mid^2 ds\right)^{\frac{p'}{2}}\right] = 0.$$

Using Lemma 4.1−2), inequality (4.4), Lemma 4.4, Lemma 4.5 and the Lebesgue dominated convergence theorem, we obtain

Lemma 4.6. *(Covergence of PDE) For every $p' < \bar{p}$,*

$$\lim_{n,m} \sup_{0\le t\le T} \int_{\mathbb{R}^k} \mid u^n(t,x) - u^m(t,x) \mid^{p'} e^{-\delta'|x|} dx = 0$$
$$\lim_{n,m} \int_0^T \int_{\mathbb{R}^k} \mid \sigma^*\nabla u^n(t,x) - \sigma^*\nabla u^m(t,x) \mid^{p'\wedge 2} e^{-\delta'|x|} dtdx = 0.$$

Using Lemma 4.1, Lemma 4.6 and the fact that $\mathcal{H}^{1+}$ is complete, we prove that exists $u \in \mathcal{H}^{1+}$ such that for every $p' < \bar{p}$,

i) $\sup_{0\le t\le T} \int_{\mathbb{R}^k} \mid u(t,x) \mid^{p'} e^{-\delta'|x|} dx + \int_0^T \int_{\mathbb{R}^k} \mid \sigma^*\nabla u(t,x) \mid^{p'\wedge 2} e^{-\delta'|x|} dtdx < \infty$

ii) $\lim_n \sup_{0\le t\le T} \int_{\mathbb{R}^k} \mid u^n(t,x) - u(t,x) \mid^{p'} e^{-\delta'|x|} dx = 0$

$iii)$ $\lim_n \mathbb{E} \int_{\mathbb{R}^k} \left(\int_t^T \mid \sigma^* \nabla u^n(s, X_s^{t,x}) - \sigma^* \nabla u(s, X_s^{t,x}) \mid^2 e^{-\delta'|x|} ds \right)^{\frac{p'}{2}} dx = 0 \qquad \forall t \in [0, T]$

$iv)$ $(u(s, X_s^{t,x}), \sigma^* \nabla u(s, X_s^{t,x})) = (Y_s^{t,x}, Z_s^{t,x})$ a.e.

In another hand, from Proposition 3.2 and Proposition 3.4 we respectively have for every $t \in [0, T]$ and $x \in D_t$

$$\mathbb{E} \int_t^T |F_n(s, X_s^{t,x}, u^n(s, X_s^{t,x}), \sigma^* \nabla u^n(s, X_s^{t,x})|^{\hat{\beta}} ds \leq C \left(1 + \Theta_p^{t,x,n} + \mathbb{E} \int_t^T |\overline{\eta}'(s, X_s^{t,x})|^q ds \right)$$

and

$$\lim_n \mathbb{E} \int_t^T |F_n(s, X_s^{t,x}, u^n(s, X_s^{t,x}), \sigma^* \nabla u^n(s, X_s^{t,x})) - F(s, X_s^{t,x}, u(s, X_s^{t,x}), \sigma^* \nabla u(s, X_s^{t,x}))|^{\hat{\beta}} ds = 0$$

where $\hat{\beta}$ is some real in $]1, \infty[$, C is some constant not depending on (t, x, n) and $\Theta_p^{t,x,n} = \mathbb{E} \sup_s |Y_s^{t,x,n}|^p + \mathbb{E}\big(\int_t^T |Z_s^{t,x,n}|^2 ds \big)^{\frac{p}{2}}$.
We deduce from Lemma 4.1, the Lebesgue dominated convergence theorem and inequality (4.4) that

$$\lim_n \int_0^T \int_{\mathbb{R}^d} |F_n(s, x, u^n(s, x), \sigma^* \nabla u^n(s, x)) - F(s, x, u(s, x), \sigma^* \nabla u(s, x))|^{\hat{\beta}} e^{-(1+\delta')|x|} dx ds = 0.$$

As a consequence of Lemma 4.3 and the proof of Proposition 3.4, we get the following existence result for the problem $(\mathcal{P}^{(g,F)})$.

Proposition 4.1. *Under assumptions **(A.0)-(A.4)**, the PDE $(\mathcal{P}^{(g,F)})$ has a unique solution u such that $u(s, X_s^{t,x}) = Y_s^{t,x}$ and $\sigma^* \nabla u(s, X_s^{t,x}) = Z_s^{t,x}$. Moreover, letting $p \in]\alpha \vee \alpha', \overline{p}[$ if $M' > 0$ and $p = \overline{p}$ if $M' = 0$, then there is a constant C depending only on $\delta', M, M', p, \overline{p}$, $|\sigma|_\infty, |b|_\infty$ and T such that*

$$\begin{aligned} &\sup_{0 \leq t \leq T} \int_{\mathbb{R}^k} \mid u(t, x) \mid^p e^{-\delta'|x|} dx + \int_0^T \int_{\mathbb{R}^k} \mid \sigma^* \nabla u(t, x) \mid^{p \wedge 2} e^{-\delta'|x|} dt dx \\ &\leq C \left(1 + \int_{\mathbb{R}^k} \mid g(x) \mid^{\overline{p}} dx + \int_{\mathbb{R}^k} \int_0^T \eta'(s, x)^{\frac{\overline{p}}{2} \vee 1} ds dx + \int_{\mathbb{R}^k} \int_0^T f^{0'}(s, x)^{\overline{p}} ds dx \right) \end{aligned}$$

where $\delta' = \delta + \kappa' + 1$ and $\kappa' := \dfrac{p\overline{p}M'T}{(\overline{p} - p)} \sup(4, \frac{2p}{p-1})$.

B) **Uniqueness.**

Due to the degeneracy of the diffusion coefficient, the solution of the homogeneous linear PDEs is not sufficiently smooth and hence we can not use it as a test function. In order to construct a suitable test function, we need the following lemma. This lemma is interesting in itself since it gives a uniform estimate for a regularized degenerate PDE.

Let $\mathcal{W}_q^{1,2}([0,\ T] \times \mathbb{R}^d)$ denotes the Sobolev space of all funcions $u(t, x)$ defined on $\mathbb{R}_+ \times \mathbb{R}^d$ such that both u and all the generalized derivatives $D_t u$, $D_x u$, and $D_{xx}^2 u$ belong to $L^q([0,\ T] \times \mathbb{R}^d)$.

Lemma 4.7. *Let $\varepsilon \in]0, 1[$, $g \in \mathcal{C}_c^\infty([0, T] \times \mathbb{R}^k; \mathbb{R})$. Then, the PDE*

$$(\mathcal{P}_\varepsilon(g)) \begin{cases} \dfrac{\partial \phi^\varepsilon(t, x)}{\partial t} - \dfrac{1}{2} div(\sigma\sigma^* \nabla \phi^\varepsilon) - \varepsilon \bigtriangleup \phi^\varepsilon(t, x) + \langle \tilde{b}(x); \nabla \phi^\varepsilon(t, x) \rangle = g(t, x) \\ \phi^\varepsilon(0, x) = 0 \quad x \in \mathbb{R}^k \end{cases}$$

has a unique solution ϕ^ε which satisfies :

(i) $\phi^\varepsilon \in \bigcap_{q > \frac{3}{2}} \mathcal{W}_q^{1,2}([0, T] \times \mathbb{R}^k; \mathbb{R}) \cap \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^k; \mathbb{R})$

(ii) $\sup_{(\varepsilon, t, x)} \left\{ |\dfrac{\partial \phi^\varepsilon}{\partial t}(t, x)| + |\nabla \phi^\varepsilon(t, x)| + |\phi^\varepsilon(t, x)| \right\} < \infty.$

Proof. The existence and uniqueness, of the solution ϕ^ε, follow from [34] (p. 318 and pp. 341 − 342). We shall prove an uniform estimates for ϕ^ε and for their first derivatives. These estimates can be established by adapting the proofs given in Krylov [33] pp. 330 − 344. However, we give here a probabilistic proof which is very simple. We assume that the dimension k is 1. Let $X_t^\varepsilon(x)$ denotes the diffusion process associated to the problem $(\mathcal{P}_\varepsilon(g))$. For simplicity, we assume that g does not depend from t and the drift coefficient of $X_t^\varepsilon(x)$ is zero. The process $X_t^\varepsilon(x)$ is then the unique (strong) solution of the following SDE

$$X_t^\varepsilon(x) = x + \int_0^t \sigma_\varepsilon(X_s^\varepsilon(x))dW_s, \qquad 0 \le t \le T$$

Let $M := \sup_{(\varepsilon,t,x)}(|g'(X_t^\varepsilon(x))| + |\sigma(t,x)| + |\sigma'(t,x)|)$. Since the coefficients σ_ε is smooth and uniformly elliptic, then the solution ϕ^ε belongs to $\mathcal{C}^{1,2}$. Hence, Itô's formula shows that,

$$\phi^\varepsilon(t,x) = -\mathbb{E}\int_t^T g(X_s^\varepsilon(x))ds.$$

Since $g \in \mathcal{C}_c^\infty$, we immediately get

$$\sup_{(\varepsilon,t,x)} \left\{ |\frac{\partial \phi^\varepsilon}{\partial t}(t,x)| + |\phi^\varepsilon(t,x)| \right\} < \infty.$$

Since $\sigma_\varepsilon \in \mathcal{C}_b^3$, we can show that

$$|\frac{\partial \phi^\varepsilon(t,x)}{\partial x}| \le M\mathbb{E}\int_t^T |\frac{\partial X_s^\varepsilon(x)}{\partial x}|ds$$

It remains to show that $\sup_{(\varepsilon,t,x)} \mathbb{E}(|\frac{\partial X_t^\varepsilon(x)}{\partial x}|) < \infty$.
Since $|\sigma'_\varepsilon(t,x)| \le |\sigma'(t,x)| \le \sup_{(t,x)} |\sigma'(t,x)| \le M$, we have

$$\begin{aligned}\mathbb{E}(|\frac{\partial X_t^\varepsilon(x)}{\partial x}|^2) &\le 1 + \mathbb{E}\int_0^t |\sigma'_\varepsilon(X_s^\varepsilon(x))|^2 |\frac{\partial X_s^\varepsilon(x)}{\partial x}|^2 ds \\ &\le 1 + M^2\mathbb{E}\int_0^t |\frac{\partial X_s^\varepsilon(x)}{\partial x}|^2 ds\end{aligned}$$

The Gronwall Lemma gives now the desired result.
In multidimensional case, the proof can be performed similarly since it is based on the fact that the first derivative of σ_ε is bounded uniformly in ε, which is valid in multidimensional case also, see Freidlin [29], III § 3.2, pp. 188-193. Lemma 4.7 is proved. ■

Remark 4.1. (i) *According to Krylov estimate (because σ_ε is uniformly elliptic), the previous proof (in dimension one) remains valid also when the coefficients σ and b are Lipschitz only.*
(ii) *Since in our situation $\sigma \in \mathcal{C}_b^3(\mathbb{R}^k, \mathbb{R}^{kr})$ and $b \in \mathcal{C}_b^2(\mathbb{R}^k, \mathbb{R}^k)$, we can estimate also the second derivative of ϕ^ε. More precisely we have*

$$\sup_{(\varepsilon,t,x)} \left\{ \phi^\varepsilon(t,x)| + |\frac{\partial \phi^\varepsilon}{\partial t}(t,x)| + |\nabla\phi^\varepsilon(t,x)| + |D^2\phi^\varepsilon(t,x)| \right\} < \infty.$$

Proof of Remark 4.1 . Let B_t be a d-dimensional Wiener process stochastically independent of W_t and consider the SDE :

$$X_s^{t,x}(\varepsilon) = x + \int_t^s \bar{b}(X_r^{t,x}(\varepsilon))dr + \int_t^s \sigma(X_r^{t,x}(\varepsilon))dW_r + \sqrt{2\varepsilon}(B_s - B_t), \qquad t \le s \le T$$

where $\bar{b}(x) := \tilde{b}(x) - \frac{1}{2}\sum_j \partial_j(\sigma\sigma^*)_{.j}(x) = b(x) - \sum_j \partial_j(\sigma\sigma^*)_{.j}(x)$
Itô's formula shows that,

$$\phi^\varepsilon(T-t,x) = \mathbb{E}\int_t^T g(r, X_r^{t,x}(\varepsilon))dr$$

Then

$$\partial_i\phi^\varepsilon(T-t,x) = \mathbb{E}\int_t^T \langle \nabla g(r, X_r^{t,x}(\varepsilon)); \partial_i X_r^{t,x}(\varepsilon)\rangle dr$$

and

$$\partial^2_{ij}\phi^\varepsilon(T-t,x) = \mathbb{E}\int_t^T \langle \nabla g(r, X_r^{t,x}(\varepsilon)); \partial^2_{ij} X_r^{t,x}(\varepsilon)\rangle + \langle D^2 g(r, X_r^{t,x}(\varepsilon))\, \partial_i X_r^{t,x}(\varepsilon); \partial_j X_r^{t,x}(\varepsilon)\rangle dr$$

On other hand,

$$\partial_i(X_s^{t,x})_k(\varepsilon) = \delta_{ik} + \int_t^s \langle \nabla \bar{b}_k(X_r^{t,x}(\varepsilon)); \partial_i X_r^{t,x}(\varepsilon)\rangle dr + \sum_n \int_t^s \langle \nabla\sigma_{kn}(X_r^{t,x}(\varepsilon)); \partial_i X_r^{t,x}(\varepsilon)\rangle dW_r^n$$

and

$$\begin{aligned}\partial^2_{ij}(X_s^{t,x})_k(\varepsilon) = \quad & \int_t^s \langle \nabla \bar{b}_k(X_r^{t,x}(\varepsilon)); \partial^2_{ij} X_r^{t,x}(\varepsilon)\rangle dr + \sum_n \int_t^s \langle \nabla\sigma_{kn}(X_r^{t,x}(\varepsilon)); \partial^2_{ij} X_r^{t,x}(\varepsilon)\rangle dW_r^n \\ & + \int_t^s \langle D^2\bar{b}_k(X_r^{t,x}(\varepsilon))\partial_j X_r^{t,x}(\varepsilon); \partial_i X_r^{t,x}(\varepsilon)\rangle dr \\ & + \sum_n \int_t^s \langle D^2\sigma_{kn}(X_r^{t,x}(\varepsilon))\partial_j X_r^{t,x}(\varepsilon); \partial_i X_r^{t,x}(\varepsilon)\rangle dW_r^n\end{aligned}$$

Itô's formula gives

$$\begin{aligned}\mathbb{E}|\partial_i(X_s^{t,x})_k(\varepsilon)|^4 \quad &= \quad \delta_{ik} + 4\mathbb{E}\int_t^s \langle \nabla\bar{b}_k(X_r^{t,x}(\varepsilon))\ ;\ \partial_i X_r^{t,x}(\varepsilon)\rangle\ (\partial_i(X_r^{t,x})_k(\varepsilon))^3 dr \\ &\quad +6\sum_n \mathbb{E}\int_t^s |\langle \nabla\sigma_{kn}(X_r^{t,x}(\varepsilon))\ ;\ \partial_i X_r^{t,x}(\varepsilon)\rangle|^2\ (\partial_i(X_r^{t,x})_k(\varepsilon))^2 dr \\ &\leq \quad \delta_{ik} + \sup_x (2|\nabla\bar{b}_k(x)| + \sum_n |\nabla\sigma_{kn}(x)|^2) \int_t^s \mathbb{E}|\partial_i X_r^{t,x}(\varepsilon)|^4 dr\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}|\partial^2_{ij}(X_s^{t,x})_k(\varepsilon)|^2 \quad &= \quad 2\mathbb{E}\int_t^s \langle \nabla\bar{b}_k(X_r^{t,x}(\varepsilon)); \partial^2_{ij} X_r^{t,x}(\varepsilon)\rangle\ \partial^2_{ij}(X_r^{t,x})_k(\varepsilon) dr \\ &\quad + \sum_n \mathbb{E}\int_t^s |\langle \nabla\sigma_{kn}(X_r^{t,x}(\varepsilon))\ ;\ \partial^2_{ij} X_r^{t,x}(\varepsilon)\rangle|^2 dr \\ &\quad +2\mathbb{E}\int_t^s \langle D^2\bar{b}_k(X_r^{t,x}(\varepsilon))\ \partial_j X_r^{t,x}(\varepsilon\ ;\ \partial_i X_r^{t,x}(\varepsilon)\rangle\ \partial^2_{ij}(X_r^{t,x})_k(\varepsilon) dr \\ &\quad + \sum_n \mathbb{E}\int_t^s |\langle D^2\sigma_{kn}(X_r^{t,x}(\varepsilon))\ \partial_j X_r^{t,x}(\varepsilon)\ ;\ \partial_i X_r^{t,x}(\varepsilon)\rangle|^2 dr \\ &\leq \quad \sup_x (2|\nabla\bar{b}_k(x)| + 2|D^2\bar{b}_k(x)| + \sum_n |\nabla\sigma_{kn}(x)|^2)\mathbb{E}\int_t^s |\partial^2_{ij} X_r^{t,x}(\varepsilon)|^2 dr \\ &\quad + \sup_x (|D^2\bar{b}_k(x)| + \sum_n |D^2\sigma_{kn}(x)|^2) \int_t^s \mathbb{E}|\partial_j X_r^{t,x}(\varepsilon)|^4 + \mathbb{E}|\partial_i X_r^{t,x}(\varepsilon)|^4 dr\end{aligned}$$

We deduce that

$$\begin{aligned}
\mathbb{E}|\partial_i(X_s^{t,x})(\varepsilon)|^4 &\le k^2 + k^2 \textstyle\sum_j \sup_x \left(2|\nabla \bar{b}_n(x)| + \sum\limits_n |\nabla \sigma_{jn}(x)|^2\right) \displaystyle\int_t^s \mathbb{E}|\partial_i X_r^{t,x}(\varepsilon)|^4 dr \\
&\le k^2 e^{k^2 T \sum_j \sup_x (2|\nabla \bar{b}_n(x)| + \sum_n |\nabla \sigma_{jn}(x)|^2)} \qquad \text{(Gronwall's Lemma)}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}|\partial^2_{ij}(X_s^{t,x})(\varepsilon)|^2 &\le k \sup_x \left(2|\nabla \bar{b}_k(x)| + 2|D^2 \bar{b}_k(x)| + \sum_n |\nabla \sigma_{kn}(x)|^2\right) \mathbb{E} \int_t^s |\partial^2_{ij} X_r^{t,x}(\varepsilon)|^2 dr \\
&\quad + k^3 T \sup_x \left(|D^2 \bar{b}_k(x)| + \sum_n |D^2 \sigma_{kn}(x)|^2\right) k^2 e^{k^2 T \sum_j \sup_x (2|\nabla \bar{b}_n(x)| + \sum_n |\nabla \sigma_{jn}(x)|^2)} \\
&\le k^3 T \sup_x \left(|D^2 \bar{b}_k(x)| + \sum_n |D^2 \sigma_{kn}(x)|^2\right) k^2 e^{k^2 T \sum_j \sup_x (2|\nabla \bar{b}_n(x)| + \sum_n |\nabla \sigma_{jn}(x)|^2)} \\
&\quad \times e^{kT \sup_x (2|\nabla \bar{b}_k(x)| + 2|D^2 \bar{b}_k(x)| + \sum_n |\nabla \sigma_{kn}(x)|^2)} \qquad \text{(Gronwall's Lemma)}
\end{aligned}$$

Since $g \in \mathcal{C}_c^\infty$, $\sigma \in \mathcal{C}_b^3(\mathbb{R}^k, \mathbb{R}^{kr})$ and $b \in \mathcal{C}_b^2(\mathbb{R}^k, \mathbb{R}^k)$ we get

$$\sup_{(\varepsilon,t,x)} \left\{ \phi^\varepsilon(t,x)| + |\frac{\partial \phi^\varepsilon}{\partial t}(t,x)| + |\nabla \phi^\varepsilon(t,x)| + |D^2 \phi^\varepsilon(t,x)| \right\} < \infty.$$

Lemma 4.7 is proved. ■

Lemma 4.8. 0 *is the unique solution of the PDE*

$$(\mathcal{P}^{(0,-div(\tilde{b})(x)y)}) \begin{cases} \dfrac{\partial w(t,x)}{\partial t} + \mathcal{L} w(t,x) + div(\tilde{b})(x) w(t,x) = 0 \quad t \in]0,T[, \ x \in \mathbb{R}^k \\ w(T,x) = 0 \quad x \in \mathbb{R}^k \end{cases}$$

satisfying for some $\beta > 1$

$$\sup_{0 \le t \le T} \int_{\mathbb{R}^k} | w(t,x) |^\beta + | w(t,x) | \, dx + \int_0^T \int_{\mathbb{R}^k} | \sigma^* \nabla w(t,x) |^\beta + | \sigma^* \nabla w(t,x) | \, dt dx < \infty. \qquad (4.1)$$

Proof. Let w be a solution of $(\mathcal{P}(0, -div(\tilde{b})(x)y))$ satisfying (4.1) and consider $w_n \in \mathcal{C}_c^\infty(R^k)$ such that

$$\int_0^T \int_{\mathbb{R}^k} |w(s,x) - w_n(s,x)| dx ds + \int_0^T \int_{\mathbb{R}^k} |\sigma^* \nabla (w(s,x) - w_n(s,x))| dx ds \to 0.$$

Let $\varepsilon \in]0,1[$, $g \in \mathcal{C}_c^\infty([0,T] \times \mathbb{R}^k; \mathbb{R})$ and consider the unique solution $\phi^\varepsilon \in \cap_{q > \frac{3}{2}} \mathcal{W}_q^{1,2}([0,T] \times \mathbb{R}^k; \mathbb{R}) \cap \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^k; \mathbb{R})$ of the following problem

$$(\mathcal{P}_\varepsilon(g)) \begin{cases} \dfrac{\partial \phi^\varepsilon(t,x)}{\partial t} - \dfrac{1}{2} div(\sigma \sigma^* \nabla \phi^\varepsilon) - \varepsilon \triangle \phi^\varepsilon(t,x) + \langle \tilde{b}(x); \nabla \phi^\varepsilon(t,x) \rangle = g(t,x) \rangle \\ \phi^\varepsilon(0,x) = 0 \quad x \in \mathbb{R}^k \end{cases}$$

The existence and uniqueness of ϕ^ε follows from Lemma 4.7.

Let $(\psi_i)_{i \in \mathbb{N}} \subset \mathcal{C}_c^\infty(R^k)$ be such that $\psi_i \in [0,1]$, $\psi_i \to 1$ uniformly on every compact set and $\nabla \psi_i \to 0$ uniformly on $\mathbb{R}^k$. By considering $\phi^\varepsilon \psi_i$ as a test function, we have

$$\int_0^T \int_{\mathbb{R}^k} \left[w \frac{\partial \phi^\varepsilon}{\partial t} + \frac{1}{2} \langle \sigma^* \nabla w; \sigma^* \nabla \phi^\varepsilon \rangle + w \langle \tilde{b}; \nabla \phi^\varepsilon \rangle \right] \psi_i dx dt + \\ \int_0^T \int_{\mathbb{R}^k} \left[\frac{1}{2} \langle \sigma^* \nabla w; \sigma^* \nabla \psi_i \rangle + w \langle \tilde{b}; \nabla \psi_i \rangle \right] \phi^\varepsilon dx dt = 0.$$

Introducing w_n and integrating by part we obtain

$$\int_0^T \int_{\mathbb{R}^k} w_n \psi_i \left[\frac{\partial \phi^\varepsilon}{\partial t} - \frac{1}{2} div(\sigma\sigma^* \nabla \phi^\varepsilon) + \langle \tilde{b}; \nabla \phi^\varepsilon \rangle \right] dtdx = \chi_1^{\varepsilon,i}(n) + \chi_2^{\varepsilon,n}(i),$$

where

$$\chi_1^{\varepsilon,i}(n) := -\int_0^T \int_{\mathbb{R}^k} \left[(w - w_n) \frac{\partial \phi^\varepsilon}{\partial t} + \frac{1}{2} \langle \sigma^* \nabla (w - w_n); \sigma^* \nabla \phi^\varepsilon \rangle + (w - w_n) \langle \tilde{b}; \nabla \phi^\varepsilon \rangle \right] \psi_i dxdt$$

and

$$\chi_2^{\varepsilon,n}(i) := -\int_0^T \int_{\mathbb{R}^k} \langle \frac{1}{2} \phi^\varepsilon \sigma\sigma^* \nabla w + \phi^\varepsilon w \tilde{b} - \frac{1}{2} w_n \sigma\sigma^* \nabla \phi^\varepsilon \ ; \ \nabla \psi_i \rangle dxdt.$$

From Lemma 4.7, we have

$$\sup_\varepsilon \sup_{(t,x)} \left\{ | \frac{\partial \phi^\varepsilon}{\partial t}(t,x)| + |\nabla \phi^\varepsilon(t,x)| + |\phi^\varepsilon(t,x)| \right\} < \infty.$$

Hence

$$\sup_{\varepsilon,i} |\chi_1^{\varepsilon,i}(n)| \longrightarrow_{n\to\infty} 0$$

and

$$\sup_{\varepsilon,n} |\chi_2^{\varepsilon,n}(i)| \longrightarrow_{i\to\infty} 0.$$

Observe that an integrating by part shows that $\int_0^T \int_{\mathbb{R}^k} w_n \psi_i \bigtriangleup \phi^\varepsilon dxdt = -\int_0^T \int_{\mathbb{R}^k} \nabla(w_n \psi_i) \nabla \phi^\varepsilon dxdt$, then use the Lebesgue dominated convergence theorem to deduce that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^k} wg(t,x)dxdt &= \lim_n \lim_i \lim_\varepsilon \int_0^T \int_{\mathbb{R}^k} w_n \psi_i (g(t,x) + \varepsilon \bigtriangleup \phi^\varepsilon) dxdt \\ &= \lim_n \lim_i \lim_\varepsilon (\chi_1^{\varepsilon,i}(n) + \chi_2^{\varepsilon,n}(i)) \\ &= 0. \end{aligned}$$

Lemma 4.8 is proved. ■

Proof of uniqueness for $(\mathcal{P}^{(g,F)})$**.** The proof is divided into three steps.

Step1. 0 *is the unique solution of* $(\mathcal{P}^{(0,0)})$ *satisfying the inequality (4.1)* Lemma 4.8.

Let w_1 be a solution of $(\mathcal{P}^{(0,0)})$ satisfying the inequality (4.1) Lemma 4.8. Then, by Lemma 4.8 it is also the unique solution of $(\mathcal{P}^{(0, div\tilde{b}(x)y - div\tilde{b}(x)w_1(t,x))})$ satisfying the inequality (4.1) Lemma 4.8. Indeed, if u is a solution of $(\mathcal{P}^{(0, div\tilde{b}(x)y - div\tilde{b}(x)w_1(t,x))})$, then $u - w_1$ is a solution of $(\mathcal{P}^{(0, div\tilde{b}(x)y)})$ and hence $u - w_1 = 0$ by Lemma 4.8.
From Proposition 4.1, the process $(w_1(s, X_s^{t,x}), \sigma^* \nabla w_1(s, X_s^{t,x}))$ is the unique solution of BSDE $(E^{(0, div\tilde{b}(X_s^{t,x})y - div\tilde{b}(X_s^{t,x})w_1(s, X_s^{t,x}))})$. Thanks to the uniqueness of this BSDE and Lemma 4.1-2), we get $w_1 = 0$.

Step2. 0 *is the unique solution of* $(\mathcal{P}^{(0,0)})$.

Let w_1 be a solution of $(\mathcal{P}^{(0,0)})$. Since $w_1 \in \mathcal{H}^{1+}$, then there exist $\beta' > 1, \delta' \geq 0$ such that,

$$\sup_{0\le t\le T} \int_{\mathbb{R}^k} | \ w_1(t,x) \ |^{\beta'} \ e^{-\delta'|x|} dx + \int_0^T \int_{\mathbb{R}^k} | \ \sigma^* \nabla w_1(t,x) \ |^{\beta'} \ e^{-\delta'|x|} dxdt < \infty.$$

Let $\delta > \delta'$ and set $\tilde{w}_1 := w_1 f(x)$ where $f \in \mathcal{C}^2(\mathbb{R}^k; \mathbb{R}_+^*)$ such that $f(x) = e^{-\delta|x|}$ if $| \ x \ | > 1$.
By Lemma 4.8, $\tilde{w}_1$ is the unique solution to the PDE

$$(\mathcal{P}_1^{(0,0)}) \qquad \begin{cases} \dfrac{\partial w(t,x)}{\partial t} + \mathcal{L}w(t,x) + div(\tilde{b})(x)w(t,x) + H(x)\tilde{w}_1(t,x) + \langle \overline{H}(x), \sigma^*\nabla\tilde{w}_1(t,x)\rangle = 0 \\ w(T,x) = 0 \end{cases}$$

satisfying the inequality (4.1) Lemma 4.8, where H and $\overline{H}$ are some bounded and continuous functions. Proposition 4.1 implies that $(\tilde{w}_1(s, X_s^{t,x}), \sigma^*\nabla\tilde{w}_1(s, X_s^{t,x}))$ is the unique solution of the BSDE $(E^{(0, div(\tilde{b})(X_s^{t,x})y + H(X_s^{t,x})\tilde{w}_1(s,X_s^{t,x}) + \langle \overline{H}(X_s^{t,x}), \sigma^*\nabla\tilde{w}_1(s,X_s^{t,x})\rangle)})$. Hence $\tilde{w}_1 = 0$, which implies that $w_1 = 0$.

Step 3. $(\mathcal{P}^{(g,F)})$ *has a unique solution if and only if* 0 *is the unique solution of* $(\mathcal{P}^{(0,0)})$.

By Proposition 4.1, there exists a unique solution u of the problem $(\mathcal{P}^{(g,F)})$ such that, $u(s, X_s^{t,x}) = Y_s^{t,x}$ and $\sigma^*\nabla u(s, X_s^{t,x}) = Z_s^{t,x}$.
Let u' be another solution of $(\mathcal{P}^{(g,F)})$ and set

$$\hat{F}(t,x) = F(s,x,u(s,x),\sigma^*\nabla u(s,x)) - F(s,x,u'(s,x),\sigma^*\nabla u'(s,x)).$$

The function $w := u - u'$ is then a solution of the problem

$$(\mathcal{P}^{(0,\hat{F})}) \qquad \begin{cases} \dfrac{\partial w(t,x)}{\partial t} + \mathcal{L}w(t,x) + \hat{F}(t,x) = 0 \quad t \in]0,T[,\ x \in \mathbb{R}^k \\ w(T,x) = 0 \quad x \in \mathbb{R}^k \end{cases}$$

In other hand, since $(0,\hat{F})$ satisfies assumptions **(A.0)-(A.4)**, then Proposition 4.1 ensures the existence of a unique solution $\hat{w}$ to the problem $(\mathcal{P}^{(0,\hat{F})})$ such that, $\hat{w}(s, X_s^{t,x}) = \hat{Y}_s^{t,x}$ and $\sigma^*\nabla\hat{w}(s, X_s^{t,x}) = \hat{Z}_s^{t,x}$, where $(\hat{Y}_s^{t,x}, \hat{Z}_s^{t,x})$ is the unique solution of

$$\hat{Y}_s^{t,x} = \int_s^T \hat{F}(r, X_r^{t,x})dr - \int_s^T \hat{Z}_r^{t,x} dW_r$$

The uniqueness of $(\mathcal{P}^{(0,\hat{F})})$ (which follows from step 2) allows us to deduce that

$$u'(s, X_s^{t,x}) = Y_s^{t,x} - \hat{Y}_s^{t,x} \qquad \text{and} \qquad \sigma^*\nabla u'(s, X_s^{t,x}) = Z_s^{t,x} - \hat{Z}_s^{t,x}.$$

This implies that $u'(t, X_s^{t,x})$ is a solution to BSDE $(E^{(g,F)})$. The uniqueness of this BSDE shows that $u'(t, X_s^{t,x}) = u(t, X_s^{t,x})$. We get that $u(t,x) = u'(t,x)$ *a.e.* by using Lemma 4.1-2). Theorem 4.1 is proved. ∎

As consequence, we have : Let $g \in \mathbb{L}^p([0,T] \times \mathbb{R}^k, e^{-\delta|x|}dx; \mathbb{R}^d)$ for some $p > 1$ and $\delta \geq 0$. Let $A : [0,T] \times \mathbb{R}^k \longrightarrow \mathbb{R}^{d\times d}$, $B : [0,T] \times \mathbb{R}^k \longrightarrow (\mathbb{R}^d)^{dr}$ and $C : [0,T] \times \mathbb{R}^k \longrightarrow \mathbb{R}^{d\times d}$ be measurable functions which satisfy :

There exists a positive constant K such that for all (t,x)

$$\|A(t,x)\| + \|B(t,x)\|^2 \leq K(1+|x|), \ \ \|C(t,x)\| \leq K \ \text{ and } \ C(t,x) \geq 0.$$

We then have

Proposition 4.2. *Let* $g \in \mathbb{L}^p([0,T]\times\mathbb{R}^k, e^{-\delta|x|}dx; \mathbb{R}^d)$ *for some* $p > 1$ *and* $\delta \geq 0$. *Let* $A : [0,T]\times\mathbb{R}^k \longmapsto \mathbb{R}^{d\times d}$, $B : [0,T] \times \mathbb{R}^k \longmapsto (\mathbb{R}^d)^{dr}$ *and* $C : [0,T] \times \mathbb{R}^k \longmapsto \mathbb{R}^{d\times d}$ *be measurable functions. Assume that there exists a positive constant* $K > 0$ *such that for every* (t,x), $0 \leq C(t,x) \leq K$ *and,*
$\|A(t,x)\| + \|B(t,x)\|^2 \leq K(1+|x|)$, $\|C(t,x)\| \leq K$
Then, the PDE

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} + \mathcal{L}w(t,x) + A(t,x)w(t,x) + \langle\langle\, B(t,x);\, \sigma^*\nabla w(t,x)\,\rangle\rangle - C(t,x)w(t,x)\log|w(t,x)| = 0, \\ w(T,x) = g(x) \quad x \in \mathbb{R}^k \end{cases}$$

has a unique solution w *and* $(w(s, X_s^{t,x}), \sigma^*\nabla w(s, X_s^{t,x}))$ *is the unique solution of*

$$E^{\left(g(X_T^{t,x}),\ A(s,X_s^{t,x})y + \langle\langle B(s,X_s^{t,x});z\rangle\rangle - C(s,X_s^{t,x})y\log|y|\right)},$$

where $\langle\langle B; z\rangle\rangle := \sum_{i=1}^{d}\sum_{j=1}^{r} B_{ij}Z_{ij}$.

Set $F(t,x,y,z) := A(t,x)y + \langle\langle B(t,x);z\rangle\rangle - C(t,x)y\log|y|$.
Arguing as in the introductory examples, we show the following claims 1)–3). The claim 2) follows by using Young's inequality.
1) $\langle y, F(t,x,y,z)\rangle \leq K + (K+K|x|)|y|^2 + \sqrt{K+K|x|}|y||z|$

2) for all $\varepsilon > 0$ there is a constant C_ε such that

$|F(t,x,y,z)| \leq C_\varepsilon(1+|x|^{C_\varepsilon} + |y|^{1+\varepsilon} + |z|^{1+\varepsilon})$

3) for every $N > 3$ and every $x, y,\ y'\ z,\ z'$ satisfying $e^{|x|},\ |\ y\ |,\ |\ y'\ |,\ |\ z\ |,\ |\ z'\ | \leq N$:

$$\langle y-y'; F(t,x,y,z) - F(t,x,y',z')\rangle \leq K'\log N\left(\frac{1}{N} + |y-y'|^2\right) + \sqrt{K'\log N}|y-y'||z-z'|,$$

where $K' := 1 + 4Kd + K^2$.
So assumptions **(A.0)-(A.4)** are satisfied for (g, F). ■

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