APPENDIX A. FINITE ELEMENT THEORY.

Write the potential energy to obtain a variational for F.E.M. use. The system's potential energy is:

$$\Pi = U - W_p \qquad \textbf{(1)}$$

where U is the strain energy and W_p is the virtual work done by the system in deformation. The strain energy is computed as a volume integral of the strain density U_0.

$$U_0 = \frac{1}{2}[\epsilon]^T [S][\epsilon] \qquad \textbf{(2)}$$

(Notice the similarities between the strain density and Hooke's law for springs: ½ $k\,x^2$).

$$U = \int\int_v\int U_0 \, dV \qquad \textbf{(3)}$$

To obtain the strain density, we need the strain vector and a stiffness matrix. The strain vector and stiffness matrix from Ques. 1, Appendix A for the isometric, small deformation assumption is:

$$\epsilon = \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{yz} \\ \epsilon_{zx} \end{pmatrix}; \qquad \textbf{(4)} \qquad S = \begin{pmatrix} \frac{2\mu(-1+\nu)}{-1+2\nu} & \frac{2\mu\nu}{1-2\nu} & \frac{2\mu\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{2\mu\nu}{1-2\nu} & \frac{2\mu(-1+\nu)}{-1+2\nu} & \frac{2\mu\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{2\mu\nu}{1-2\nu} & \frac{2\mu\nu}{1-2\nu} & \frac{2\mu(-1+\nu)}{-1+2\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}; \qquad \textbf{(5)}$$

resulting in:

$$U_0 = 2A\Big\{(-1+\nu)\,\epsilon_{xx}^2 + \frac{1}{2}(-1+2\nu)\,\epsilon_{xy}^2 - \epsilon_{yy}^2 + \nu\,\epsilon_{yy}^2 - \frac{1}{2}\epsilon_{yz}^2 + \nu \\ \frac{1}{2}\epsilon_{zx}^2 + \nu\,\epsilon_{zx}^2 - 2\nu\,\epsilon_{yy}\,\epsilon_{zz} - \epsilon_{zz}^2 + \nu\,\epsilon_{zz}^2 - 2\nu\,\epsilon_{xx}\,(\epsilon_{yy} + \epsilon_z \qquad \textbf{(6)}$$

where $A = \dfrac{\mu}{-2 + 4\nu}$:

From the above equations and the small deformation strains, the following derivatives are needed and can be written as an array:

$$\nabla \underset{\sim}{u} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} & \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \qquad \textbf{(7)}$$

where for small deformations, the strain vector can be defined in terms of equation 7.

$$\epsilon = \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{yz} \\ \epsilon_{zx} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2\frac{\partial u}{\partial x} \\ 2\frac{\partial v}{\partial y} \\ 2\frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial} \\ \frac{\partial}{\partial} \\ \frac{\partial}{\partial} \\ \frac{\partial}{\partial} \\ \frac{\partial}{\partial} \\ \frac{\partial}{\partial} \\ \frac{\partial}{\partial} \\ \frac{\partial}{\partial} \\ \frac{\partial}{\partial} \end{pmatrix} \qquad \textbf{(8)}$$

For an eight node (3-D) Hexahedral (Brick) element, assume a trial solution u(x,y,z), v(x,y,z), w(x,y,z) as a function of interpolation function and the degrees of freedom:

$$\underline{\boldsymbol{u}} = \boldsymbol{N}\underline{\boldsymbol{d}} \qquad \textbf{(9)}$$

where ***u*** is the trial solution (u, v, and w), ***N*** is the shape functions as defined by the parent element, and ***d*** is the degree of freedom vector for the nodes.

The shape functions are generalized as

$$N_i = \frac{1}{8}(1 + r_i r)(1 + s_i s)(1 + t_i t) \qquad i = 1, \ldots, 8 \qquad \textbf{(10)}$$

where r, s, t is the parent element coordinate system that the element will be mapped too. The coordinates of r, s, and t for nodal coordinates are (for a square element)...

<u>Table</u>

Node #	r_i	s_i	t_i
1	-1	-1	-1
2	1	-1	-1
3	1	1	-1
4	-1	1	-1
5	-1	-1	1
6	1	-1	1
7	1	1	1
8	-1	1	1

$$\begin{pmatrix} \frac{1}{8}(1-r)(1-s)(1-t) \\ \frac{1}{8}(1+r)(1-s)(1-t) \\ \frac{1}{8}(1+r)(1+s)(1-t) \\ \frac{1}{8}(1-r)(1+s)(1-t) \\ \frac{1}{8}(1-r)(1-s)(1+t) \\ \frac{1}{8}(1+r)(1-s)(1+t) \\ \frac{1}{8}(1+r)(1+s)(1+t) \\ \frac{1}{8}(1-r)(1+s)(1+t) \end{pmatrix}^T \qquad \textbf{(11)}$$

$\boldsymbol{N_i}$ =

Nodal coordinates before deformation are in the x, y, z coordinate system. In order to transform to the u, v, w coordinate system, we must define nodal coordinates in terms of *r*, *s* and *t*. To do this we need transformation functions or isoparametric mapping functions. So, given an actual element with coordinates (xi, yi, zi) for each node (for eight node element)

$$\begin{aligned} xi &= (x1\ \ x2\ \ x3\ \ x4\ \ x5\ \ x6\ \ x7\ \ x8)^T; \\ yi &= (y1\ \ y2\ \ y3\ \ y4\ \ y5\ \ y6\ \ y7\ \ y8)^T; \\ zi &= (z1\ \ z2\ \ z3\ \ z4\ \ z5\ \ z6\ \ z7\ \ z8)^T; \\ Ni &= (N1\ \ N2\ \ N3\ \ N4\ \ N5\ \ N6\ \ N7\ \ N8)^T; \end{aligned} \tag{12}$$

then the interpolation functions for the actual element represented in the parent element coordinates (r, s, and t) is

$$\begin{aligned} x &= Ni^T \cdot xi \\ y &= Ni^T \cdot yi \\ z &= Ni^T \cdot zi \end{aligned} \tag{13}$$

with the displacement vector **d** defined as:

$$\mathbf{d} = (u1\ \ v1\ \ w1\ \ u2\ \ v2\ \ w2\ \ \ldots\ \ u8\ \ v8\ \ w8)^T; \tag{14}$$

and

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} N1 & 0 & 0 & N2 & 0 & 0 & \ldots & N8 & 0 & 0 \\ 0 & N1 & 0 & 0 & N2 & 0 & 0 & \ldots & N8 & 0 \\ 0 & 0 & N1 & 0 & 0 & N2 & 0 & 0 & \ldots & N8 \end{pmatrix} \mathbf{d} = \begin{pmatrix} Nu^T \\ Nv^T \\ Nw^T \end{pmatrix} \mathbf{d} =] \tag{15}$$

Now because the interpolation functions are in terms of the parent element and eventually strains are needed in the original coordinate system, a transformation is needed to switch back and forth. For the derivatives of **u**...

$$\begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \\ \frac{\partial u}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} \equiv \mathbf{J} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} \qquad \mathbf{(16)}$$

where **J** is the Jacobian matrix. The derivatives of u (with respect to x, y, and z) can be computed by inverting the Jacobian matrix **J**, resulting in (with substitution from the approximate solution above),

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \\ \frac{\partial u}{\partial t} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \frac{\partial (\mathbf{Nu}^T \mathbf{d})}{\partial r} \\ \frac{\partial (\mathbf{Nu}^T \mathbf{d})}{\partial s} \\ \frac{\partial (\mathbf{Nu}^T \mathbf{d})}{\partial t} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \frac{\partial \mathbf{Nu}^T}{\partial r} \\ \frac{\partial \mathbf{Nu}^T}{\partial s} \\ \frac{\partial \mathbf{Nu}^T}{\partial t} \end{pmatrix} \begin{pmatrix} u1 \\ v1 \\ w1 \\ \vdots \\ w8 \end{pmatrix} = \begin{pmatrix} \mathbf{B_u^T} \\ \mathbf{B_u^T} \\ \mathbf{B_u^T} \end{pmatrix} \qquad \mathbf{(17)}$$

$$\begin{pmatrix} \frac{\partial \mathbf{Nu}^T}{\partial r} \\ \frac{\partial \mathbf{Nu}^T}{\partial s} \\ \frac{\partial \mathbf{Nu}^T}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial N1}{\partial r} & 0 & 0 & \frac{\partial N2}{\partial r} & 0 & 0 & & \frac{\partial N8}{\partial r} & 0 & 0 \\ \frac{\partial N1}{\partial s} & 0 & 0 & \frac{\partial N2}{\partial s} & 0 & 0 & \ldots & \frac{\partial N8}{\partial s} & 0 & 0 \\ \frac{\partial N1}{\partial t} & 0 & 0 & \frac{\partial N2}{\partial t} & 0 & 0 & & \frac{\partial N8}{\partial t} & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{pmatrix} \cdot \mathbf{Nu}^T \qquad \mathbf{(18)}$$

Likewise for v and w,

$$\begin{pmatrix} \frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial s} \\ \frac{\partial v}{\partial t} \end{pmatrix} = \mathbf{J} \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial z} \end{pmatrix} \qquad \mathbf{(19)}$$

$$\begin{pmatrix} \frac{\partial w}{\partial r} \\ \frac{\partial w}{\partial s} \\ \frac{\partial w}{\partial t} \end{pmatrix} = \mathbf{J} \begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{pmatrix} \qquad \mathbf{(20)}$$

and with the reverse as

$$\begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial z} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial s} \\ \frac{\partial v}{\partial t} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \frac{\partial \mathbf{Nv}^T}{\partial r} \\ \frac{\partial \mathbf{Nv}^T}{\partial s} \\ \frac{\partial \mathbf{Nv}^T}{\partial t} \end{pmatrix} \begin{pmatrix} u1 \\ v1 \\ w1 \\ \vdots \\ w8 \end{pmatrix} = \begin{pmatrix} \mathbf{B_{vx}^T} \\ \mathbf{B_{vy}^T} \\ \mathbf{B_{vz}^T} \end{pmatrix} \mathbf{d} \qquad \mathbf{(21)}$$

and also

$$\begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \frac{\partial w}{\partial r} \\ \frac{\partial w}{\partial s} \\ \frac{\partial w}{\partial t} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{wx}^{T} \\ \mathbf{B}_{wy}^{T} \\ \mathbf{B}_{wz}^{T} \end{pmatrix} \mathbf{d} \tag{22}$$

Resulting in:

$$\epsilon = \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{yz} \\ \epsilon_{zx} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{B}_{ux}^{T} \\ \mathrm{B}_{uy}^{T} \\ \mathrm{B}_{uz}^{T} \\ \mathrm{B}_{vx}^{T} \\ \mathrm{B}_{vy}^{T} \\ \mathrm{B}_{vz}^{T} \\ \mathrm{B}_{wx}^{T} \\ \mathrm{B}_{wy}^{T} \\ \mathrm{B}_{wz}^{T} \end{pmatrix} \mathbf{d} \equiv \mathbf{B}^{T}\,\mathbf{d} \tag{23}$$

Now to get all of these derivatives for the Jacobian and to check the determinant over the interval to see if the mapping was good. Ideally, one would check the mapping by making sure that the determinant of the Jacobian is not zero anywhere over the element. But for the 3-D case, this proves to be very difficult so common convention is to check whether the determinant of the Jacobian at all of the gauss points (of integration) are the same sign and if it is, then assuming it is not zero anywhere.

Remember the strain energy in equations 2 and 3. Substituting with the definition gained above, results in

$$U_0 = \frac{1}{2} \boldsymbol{B}^T \boldsymbol{d}\, \boldsymbol{S} \boldsymbol{B}^T \boldsymbol{d} = \frac{1}{2} \boldsymbol{d}^T \boldsymbol{B}\, \boldsymbol{S} \boldsymbol{B}^T \boldsymbol{d} \tag{24}$$

and

$$U = \int\!\!\int\!\!\int_{v} \frac{1}{2} \boldsymbol{d}^T \boldsymbol{B}\, \boldsymbol{S} \boldsymbol{B}^T \boldsymbol{d}\, \mathrm{dV} = \frac{1}{2} \boldsymbol{d}^T \int\!\!\int\!\!\int_{v} \boldsymbol{B}\, \boldsymbol{S} \boldsymbol{B}^T\, \mathrm{dV}\, \boldsymbol{d} = \frac{1}{2} \boldsymbol{d}^T \boldsymbol{K} \boldsymbol{d} \tag{25}$$

where ***K*** is the element stiffness matrix

$$\boldsymbol{K} = \int\!\!\int\!\!\int_{v} \boldsymbol{B}\, \boldsymbol{S} \boldsymbol{B}^T\, \mathrm{dV} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \boldsymbol{B}\, \boldsymbol{S} \boldsymbol{B}^T \det(\boldsymbol{J})\, dr\, ds\, dt \tag{26}$$

The virtual work done by the system against the forces can be written as:

$$W_p = \iiint_v (Fx\,u + Fy\,v + Fz\,w)\,dV + \iint_A (Px\,u + Py\,v + Pz\,w)\,dA \qquad \textbf{(27)}$$

or

$$W_p = \iiint_v (u \;\; v \;\; w)\begin{pmatrix} Fx \\ Fy \\ Fz \end{pmatrix} dV + \iint_A (u \;\; v \;\; w)\begin{pmatrix} Px \\ Py \\ Pz \end{pmatrix} dA \qquad \textbf{(28)}$$

rewritten as,

$$W_p = -\boldsymbol{d}^T \rho \iiint_v N\,(N^T)\,dV\,\ddot{\boldsymbol{d}} + \boldsymbol{d}^T \iint_A N\,(\mathbf{P})\,dA = \boldsymbol{d}^T\,\mathbf{W_F} + \boldsymbol{d}^T\,\mathbf{W_P} \qquad \textbf{(29)}$$

where Fx, Fy and Fz are the body forces per unit volume and Px, Py, and Pz are surface or boundary forces per unit area. Let the body forces per unit volume be the inertial forces:

$$Fx = -\rho\frac{\partial^2 u}{\partial t^2}, \qquad Fy = -\rho\frac{\partial^2 v}{\partial t^2}, \qquad Fz = -\rho\frac{\partial^2 w}{\partial t^2}. \qquad \textbf{(30)}$$

Note that if there is no time dependence (static case) then there are no body forces per unit volume. The Bondry forces per unit area only happen at the boundary and will be primarily from the air flow and air pressure. ***F*** is the body force vector and ***P*** is the equivalent nodal load vector.

$$W_F = -\rho\iiint_v (N\,N)^T\,dV; \quad \text{and} \quad W_P = \iint_A \mathbf{N}\;\mathbf{P}\;dA. \qquad \textbf{(31)}$$

The surface or boundary forces per area are external to the element and for the element here would be of the form...

$$\mathbf{P} = (\,pu1 \;\; pv1 \;\; pw1 \;\; pu2 \;\; pv2 \;\; pw2 \;\; \ldots \;\; pu8 \;\; pv8 \;\; pw8\,)^T; \qquad \textbf{(32)}$$

These components are literally the stresses in each direction imposed by external forces. So, the potential energy is

$$\begin{aligned} \Pi &= \iiint_v U_0\,dv - \iiint_v (Fx\,u + Fy\,v + Fz\,w)\,dV \\ &\quad - \iint_A (Px\,u + Py\,v + Pz\,w)\,dA \qquad \textbf{(33)} \\ &= \frac{1}{2}\boldsymbol{d}^T \boldsymbol{K}\,\boldsymbol{d} + \boldsymbol{d}^T\,\mathbf{W_F}\,\ddot{\boldsymbol{d}} - \boldsymbol{d}^T\,\mathbf{W_P} \end{aligned}$$

For F.E.M. purposes, we want the variation of the potential energy, where each term is each element is zero with respect to its modal coordinates. Let each integral represented by $\boldsymbol{I}_k$.

$$\frac{\partial \Pi^e}{\partial U_i} = \sum_{k=1}^{K} \frac{\partial I_k}{\partial U_i} = 0; \quad i = 1, 2, 3; \qquad \frac{\partial \Pi^e}{\partial V_i} = \sum_{k=1}^{K} \frac{\partial I_k}{\partial V_i} = 0; \quad i = 1, 2, 3; \qquad \frac{\partial \Pi^e}{\partial W_i} = \sum_{k=1}^{K} \frac{\partial I_k}{\partial W_i} = 0; \quad i = 1, 2, 3; \tag{34}$$

The derivatives with respect to individual nodal points in u, v, and w will then be taken (minimization) and set equal to zero as stated in the equations above. If $\boldsymbol{K}$ is symmetric, then the minimization is easy with one large matrix equation:

$$\boldsymbol{K}\boldsymbol{d} = -\boldsymbol{W_F}\,\ddot{\boldsymbol{d}} + \boldsymbol{W_P} \tag{35}$$

Viscous damping effects can be included by replacing μ in the stiffness matrix $\boldsymbol{K}$ and replacing it with $\mu + \varsigma\, \partial/\partial t$. Let this new addition be the damping matrix $\boldsymbol{D}$, where $\boldsymbol{D} = \frac{\eta}{\mu}\boldsymbol{K}$. Putting this all together results in our dynamic equation.

$$\boldsymbol{K}\boldsymbol{d} + \boldsymbol{D}\dot{\boldsymbol{d}} + \boldsymbol{W_F}\,\ddot{\boldsymbol{d}} - \boldsymbol{W_P} = 0 \tag{36}$$

For static problems, anything with a time dependence is nullified. Leaving our static equation to be:

$$\boldsymbol{K}\boldsymbol{d} - \boldsymbol{W_P} = 0 \tag{37}$$