Appendix B: Isotropic Materials in Continuum Mechanics

Isotropic materials have the property of having uniform material properties in every direction. The particular stiffness matrix **S** for isotropic materials, where the shears are symmetric, can be found in many engineering books (Cook *et al*, 1989).

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{pmatrix} = \begin{pmatrix} [{}^3 s] & [{}^3 0] \\ [{}^3 0] & [{}^3 s'] \end{pmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{yz} \\ \epsilon_{zx} \end{pmatrix} \quad \textbf{(1)}$$

Where

$$[{}^3 s] = \frac{2\,\mu}{1 - 2\,\nu} \begin{pmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{pmatrix} \quad \textbf{(2)}$$

$$[{}^3 s'] = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \quad \textbf{(3)}$$

and μ is the shear modulus, and ν is the Poisson's ratio. The constitutive equation can be expanded in six individual relations.

$$\tau_{xy} = \mu\,\epsilon_{xy} \qquad \sigma_x = \frac{2\,\mu}{1-2\,\nu}\left[(1-\nu)\,\epsilon_{xx} + \nu\,\epsilon_{yy} + \nu\,\epsilon_{zz}\right]$$

$$\tau_{yz} = \mu\,\epsilon_{yz} \qquad \sigma_y = \frac{2\,\mu}{1-2\,\nu}\left[\nu\,\epsilon_{xx} + (1-\nu)\,\epsilon_{yy} + \nu\,\epsilon_{zz}\right] \quad \textbf{(4)}$$

$$\tau_{zx} = \mu\,\epsilon_{zx} \qquad \sigma_z = \frac{2\,\mu}{1-2\,\nu}\left[\nu\,\epsilon_{xx} + \nu\,\epsilon_{yy} + (1-\nu)\,\epsilon_{zz}\right]$$

Remember that the strains were define in Appendix A; equation 19. The full version of equation

A19 was labeled equations 5.

$$
\begin{aligned}
\epsilon_{xx} &= \frac{1}{2}\Big(\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial \zeta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial x}\frac{\partial \psi}{\partial x} + \\
&\quad \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \zeta}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \xi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial x}\Big) \\
\epsilon_{xy} &= \frac{1}{2}\Big(\frac{\partial \xi}{\partial y} + \frac{\partial \psi}{\partial x} + \frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial \psi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial \zeta}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial \xi}{\partial x}\frac{\partial \psi}{\partial y} + \\
&\quad \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} + \frac{\partial \zeta}{\partial x}\frac{\partial \psi}{\partial y} + \frac{\partial \xi}{\partial x}\frac{\partial \zeta}{\partial y} + \frac{\partial \psi}{\partial x}\frac{\partial \zeta}{\partial y} + \frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial y}\Big) \\
\epsilon_{yy} &= \frac{1}{2}\Big(\frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \xi}{\partial y} + \frac{\partial \psi}{\partial y}\frac{\partial \xi}{\partial y} + \frac{\partial \zeta}{\partial y}\frac{\partial \xi}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \psi}{\partial y} + \\
&\quad \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial y} + \frac{\partial \zeta}{\partial y}\frac{\partial \psi}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial \zeta}{\partial y}\frac{\partial \zeta}{\partial y}\Big) \\
\epsilon_{yz} &= \frac{1}{2}\Big(\frac{\partial \psi}{\partial z} + \frac{\partial \zeta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \xi}{\partial z} + \frac{\partial \psi}{\partial y}\frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial y}\frac{\partial \xi}{\partial z} + \frac{\partial \xi}{\partial y}\frac{\partial \psi}{\partial z} + \\
&\quad \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial z} + \frac{\partial \zeta}{\partial y}\frac{\partial \psi}{\partial z} + \frac{\partial \xi}{\partial y}\frac{\partial \zeta}{\partial z} + \frac{\partial \psi}{\partial y}\frac{\partial \zeta}{\partial z} + \frac{\partial \zeta}{\partial y}\frac{\partial \zeta}{\partial z}\Big) \\
\epsilon_{zz} &= \frac{1}{2}\Big(\frac{\partial \zeta}{\partial z} + \frac{\partial \zeta}{\partial z} + \frac{\partial \xi}{\partial z}\frac{\partial \xi}{\partial z} + \frac{\partial \psi}{\partial z}\frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial z}\frac{\partial \xi}{\partial z} + \frac{\partial \xi}{\partial z}\frac{\partial \psi}{\partial z} + \\
&\quad \frac{\partial \psi}{\partial z}\frac{\partial \psi}{\partial z} + \frac{\partial \zeta}{\partial z}\frac{\partial \psi}{\partial z} + \frac{\partial \xi}{\partial z}\frac{\partial \zeta}{\partial z} + \frac{\partial \psi}{\partial z}\frac{\partial \zeta}{\partial z} + \frac{\partial \zeta}{\partial z}\frac{\partial \zeta}{\partial z}\Big) \\
\epsilon_{zx} &= \frac{1}{2}\Big(\frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z} + \frac{\partial \xi}{\partial z}\frac{\partial \xi}{\partial x} + \frac{\partial \psi}{\partial z}\frac{\partial \xi}{\partial x} + \frac{\partial \zeta}{\partial z}\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial z}\frac{\partial \psi}{\partial x} + \\
&\quad \frac{\partial \psi}{\partial z}\frac{\partial \psi}{\partial x} + \frac{\partial \zeta}{\partial z}\frac{\partial \psi}{\partial x} + \frac{\partial \xi}{\partial z}\frac{\partial \zeta}{\partial x} + \frac{\partial \psi}{\partial z}\frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial z}\frac{\partial \zeta}{\partial x}\Big)
\end{aligned}
\tag{5}
$$

For small deformations, the product terms can be assumed to be negligible, resulting in equations 6.

$$
\begin{aligned}
\epsilon_{xx} &= \frac{\partial \xi}{\partial x} & \qquad \epsilon_{xy} &= \frac{1}{2}\left(\frac{\partial \xi}{\partial y} + \frac{\partial \psi}{\partial x}\right) \\
\epsilon_{yy} &= \frac{\partial \psi}{\partial y} & \qquad \epsilon_{yz} &= \frac{1}{2}\left(\frac{\partial \psi}{\partial z} + \frac{\partial \zeta}{\partial y}\right)
\end{aligned}
\tag{6}
$$

$$\epsilon_{zz} = \frac{\partial \zeta}{\partial z} \qquad \epsilon_{zx} = \frac{1}{2}\left(\frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z}\right)$$

The equations of motion for a isotropic material assuming small deformation without body forces would be obtained by getting the stresses from equation A15 using the strains from equation 6, resulting in the stresses in equation 7, and substituting them into the equations of motion, Manuscript; equation 6, resulting in equation 8.

$$\sigma_x = \frac{2\mu}{1-2\nu}\left[(1-\nu)\frac{\partial \xi}{\partial x} + \nu\frac{\partial \psi}{\partial y} + \nu\frac{\partial \zeta}{\partial z}\right]$$

$$\sigma_y = \frac{2\mu}{1-2\nu}\left[\nu\frac{\partial \xi}{\partial x} + (1-\nu)\frac{\partial \psi}{\partial y} + \nu\frac{\partial \zeta}{\partial z}\right]$$

$$\sigma_z = \frac{2\mu}{1-2\nu}\left[\nu\frac{\partial \xi}{\partial x} + \nu\frac{\partial \psi}{\partial y} + (1-\nu)\frac{\partial \zeta}{\partial z}\right] \qquad \textbf{(7)}$$

$$\tau_{xy} = \frac{1}{2}\mu\left(\frac{\partial \psi}{\partial z} + \frac{\partial \zeta}{\partial y}\right)$$

$$\tau_{yz} = \frac{1}{2}\mu\left(\frac{\partial \psi}{\partial z} + \frac{\partial \zeta}{\partial y}\right)$$

$$\tau_{zx} = \frac{1}{2}\mu\left(\frac{\partial \zeta}{\partial x} + \frac{\partial \xi}{\partial z}\right)$$

$$\frac{\partial}{\partial x}\left\{\frac{2\mu}{1-2\nu}\left[(1-\nu)\frac{\partial \xi}{\partial x}+\nu\frac{\partial \psi}{\partial y}+\nu\frac{\partial \zeta}{\partial z}\right]\right\}+\frac{\partial}{\partial y}\left\{\frac{1}{2}\mu\left[\frac{\partial \psi}{\partial z}+\frac{\partial \zeta}{\partial y}\right]\right\}$$
$$+\frac{\partial}{\partial z}\left\{\frac{1}{2}\mu\left[\frac{\partial \zeta}{\partial x}+\frac{\partial \xi}{\partial z}\right]\right\}=\rho\frac{\partial^2 \xi}{\partial t^2}$$

$$\frac{\partial}{\partial x}\left\{\frac{1}{2}\mu\left[\frac{\partial \psi}{\partial z}+\frac{\partial \zeta}{\partial y}\right]\right\}+\frac{\partial}{\partial y}\left\{\frac{2\mu}{1-2\nu}\left[\nu\frac{\partial \xi}{\partial x}+(1-\nu)\frac{\partial \psi}{\partial y}+\nu\frac{\partial \zeta}{\partial z}\right]\right\}$$
$$+\frac{\partial}{\partial z}\left\{\frac{1}{2}\mu\left[\frac{\partial \psi}{\partial z}+\frac{\partial \zeta}{\partial y}\right]\right\}=\rho\frac{\partial^2 \psi}{\partial t^2} \qquad \textbf{(8)}$$

$$\frac{\partial}{\partial x}\left\{\frac{1}{2}\mu\left[\frac{\partial \zeta}{\partial x}+\frac{\partial \xi}{\partial z}\right]\right\}+\frac{\partial}{\partial y}\left\{\frac{1}{2}\mu\left[\frac{\partial \psi}{\partial z}+\frac{\partial \zeta}{\partial y}\right]\right\}$$
$$+\frac{\partial}{\partial z}\left\{\frac{2\mu}{1-2\nu}\left[\nu\frac{\partial \xi}{\partial x}+\nu\frac{\partial \psi}{\partial y}+(1-\nu)\frac{\partial \zeta}{\partial z}\right]\right\}=\rho\frac{\partial^2 \zeta}{\partial t^2}$$

The equations of motion labeled 8 can be analytically solved for specific cases. Note that if the assumption of small deformations was not made, the process for obtaining the equations of motion would be the same except that the full strains in equation 5 would be used instead the ones in 6. It is easy to see how the equations of motion for this case would have more terms than could fit on this page. For this reason, they were not shown.