Appendix A: General Continuum Mechanics.

CONTINUUM MECHANICS

Continuum mechanics, or the mechanics of deformable bodies, is the basic theory underlying non-point mass Newtonian mechanics. Continuum mechanics is just a general Newtonian mechanics description of the movement and deformation of bodies or fields. An important part of continuum mechanics is a constitutive equation. Zupkas and Fung (1985) said, “the first step in the analysis of the biomechanics of any organ is to obtain its constitutive equation.” A general constitutive equation for isotopic and transversely isotropic material with descriptions of small- and large-deformations will be discussed.

Given a spacial body, define its *resting* (or *undeformed* or *material*) *coordinates* for any arbitrary point as x_0, y_0, z_0. After deformation or applied force, the new configuration or coordinates, sometimes called *spacial coordinates,* for any arbitrary point is x, y, z. Define a displacement vector ***U*** to describe a strained configuration between the resting and spacial coordinates.

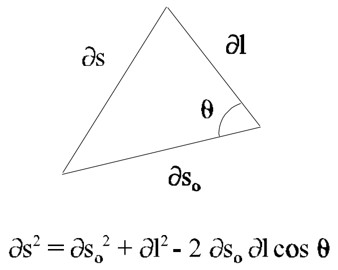
$$\begin{aligned} \boldsymbol{U} &= (x - x_0)\,\boldsymbol{i} + (y - y_0)\,\boldsymbol{j} + (z - z_0)\,\boldsymbol{k} \\ &= \xi\,\boldsymbol{i} + \psi\,\boldsymbol{j} + \zeta\,\boldsymbol{k} \end{aligned} \tag{1}$$

The *components of displacement* (or the *displacement field*) are defined as ξ, ψ, and ζ.

The rate of change of the deformation vector is called the velocity field. Assume ds_0 to be and infinitesimal length of an element in the resting or reference configuration and ds to be the corresponding length of the element in the deformed state.

$$\begin{aligned} ds_0{}^2 &= dx_0{}^2 + dy_0{}^2 + dz_0{}^2 = dp_i{}^2 \\ ds^2 &= dx^2 + dy^2 + dz^2 = dq_i{}^2 \end{aligned} \tag{2}$$

From the law of cosines (figure 1), for a mapping of points when there is only an angular and a linear change in *ds* relative to a in a single plane, the difference of the definitions in (2) is equation (3).



$$\partial s^2 = \partial s_o^2 + \partial l^2 - 2\,\partial s_o\,\partial l\cos\theta$$

Figure 1. The law of Cosines.

$$ds^2 - ds_0{}^2 = dl^2 - 2\,ds_0\,dl\cos\theta \tag{3}$$

Equation 3 takes into account the linear deformation (*dl*) and the angular deformation (cosΘ). This is only for a single plane so other direction cosines are needed for deformation in all three dimensions. For a general definition of deformation in any arbitrary direction, equation 4 was defined.

$$D^2 = ds^2 - ds_0{}^2 \tag{4}$$

Differential lengths for component length in the *x* direction is shown by equation 5.

$$dx_0 = \frac{\partial x_0}{\partial x}dx + \frac{\partial x_0}{\partial y}dy + \frac{\partial x_0}{\partial z}dz = \frac{\partial x_0}{\partial q_i}dq_i \tag{5}$$

The other components can be similarly written as shown in equation 6.

$$dy_0 = \frac{\partial y_0}{\partial q_i}dq_i; \qquad dz_0 = \frac{\partial z_0}{\partial q_i}dq_i \tag{6}$$

Substituting the component lengths into equation 4 results in equation 7, making the definition of the *Almani stress tensor* (equation 8).

$$\begin{aligned} \boldsymbol{D}^2 &= ds^2 - {ds_0}^2 \\ &= {dq_j}^2 - \left[\frac{\partial x_0}{\partial q_i}\, dq_i\right]^2 - \left[\frac{\partial y_0}{\partial q_i}\, dq_i\right]^2 - \left[\frac{\partial z_0}{\partial q_i}\, dq_i\right]^2 \\ &= {dq_j}^2 - \left[\frac{\partial p_j}{\partial q_i}\, dq_i\right]^2 \\ &= 2\,\epsilon_{ij}\, dq_i\, dq_j \qquad i,\ j = 1, 2, 3 \end{aligned} \tag{7}$$

$$\epsilon_{ij} = \frac{1}{2}\left(\delta_{ij} - \delta_{kl}\,\frac{\partial p_k}{\partial q_i}\,\frac{\partial p_l}{\partial q_j}\right) \tag{8}$$

The partial derivatives above are called deformation gradients. Reversing these gradients by solving for the other variable results in equation 9 for the *dx* component.

$$dx = \frac{\partial x}{\partial p_i}\, dp_i \tag{9}$$

Doing this for all three directions, the deformation is redefined as equation 10 with the *Green-St. Venant strain tensor* being defined (eqn. 11).

$$\boldsymbol{D}^2 = 2\, e_{ij}\, dp_i\, dp_j \tag{10}$$

$$e_{ij} = \frac{1}{2}\left(\delta_{kl}\,\frac{\partial q_k}{\partial p_i}\,\frac{\partial q_l}{\partial p_j} - \delta_{ij}\right) \tag{11}$$

For large deformation, e_{ii} and e_{ij} components can all be non-zero. Note that symmetry exists so that $e_{ij} = e_{ji}$ and $\mathring{a}_{ij} = \mathring{a}_{ij}$. Remember ξ, ψ, and ζ (where $\xi = x - x_0$, ...). So, to write ***D*** in terms of these variables we start with:

$$\frac{\partial x_0}{\partial x} = \frac{\partial (x-\xi)}{\partial x} = \frac{\partial x}{\partial x} - \frac{\partial \xi}{\partial x} = 1 - \frac{\partial \xi}{\partial x} = 1 - \frac{\partial u_1}{\partial q_1}$$

$$\frac{\partial x_0}{\partial y} = \frac{\partial (x-\xi)}{\partial y} = \frac{\partial x}{\partial y} - \frac{\partial \xi}{\partial y} = -\frac{\partial \xi}{\partial y} = -\frac{\partial u_1}{\partial q_2} \qquad \textbf{(12)}$$

$$\frac{\partial x_0}{\partial z} = \frac{\partial (x-\xi)}{\partial z} = \frac{\partial x}{\partial z} - \frac{\partial \xi}{\partial z} = -\frac{\partial \xi}{\partial z} = -\frac{\partial u_1}{\partial q_3}$$

and by inspection results in equation 13 where $u_1=\xi$, $u_2=\psi$, and $u_3=\zeta$.

$$\frac{\partial p_j}{\partial q_i} = \delta_{ij} - \frac{\partial u_j}{\partial q_i} \qquad \textbf{(13)}$$

Note that the deformation gradient may be defined in vector notation instead of Einstein notation as done here (Wilhelms-Tricarico, 1994, 1998). Using equation 13, the deformation gradient becomes equation 14.

$$\begin{aligned} \boldsymbol{D}^2 &= ds^2 - {ds_0}^2 = {dq_j}^2 - \left[\left\{\delta_{ij} - \frac{\partial u_j}{\partial q_i}\right\} dq_i\right]^2 \\ &= 2\,\epsilon_{ij}\, dq_i\, dq_j \end{aligned} \qquad \textbf{(14)}$$

Notice that the simplified case is the same but the Almani stress tensor, equation 18, has been defined in terms of other quantities.

$$\boxed{\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial q_j} + \frac{\partial u_j}{\partial q_i} + \frac{\partial u_k}{\partial q_i}\,\frac{\partial u_l}{\partial q_j}\right)}$$

(15)

The deformation can also be written of the resting coordinates. For the *x* coordinate and, by inspection, general case deformation results in equation 16 and 17.

$$\frac{\partial x}{\partial x_0} = \frac{\partial(\xi + x_0)}{\partial x_0} = 1 - \frac{\partial u_1}{\partial p_1} \qquad \textbf{(16)}$$

$$\frac{\partial q_j}{\partial p_i} = \delta_{ij} - \frac{\partial u_j}{\partial p_i} \qquad \textbf{(17)}$$

This results in ***D*** as shown in equation 18.

$$\boldsymbol{D}^2 = 2\, e_{ij}\, dp_i\, dp_j \qquad \textbf{(18)}$$

With the Green-St.Venant strain tensor defined as equation 19.

$$\boxed{e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial p_j} + \frac{\partial u_j}{\partial p_i} - \frac{\partial u_k}{\partial p_i}\,\frac{\partial u_l}{\partial p_j}\right)}$$

(19)

Note that the strains are nonlinear because of the product terms. These product terms are small so they can be neglected if a linear strain tensor is approximated.

Most text denote stress with the Greek letter tau, τ_{ij}. The subscripts define the element

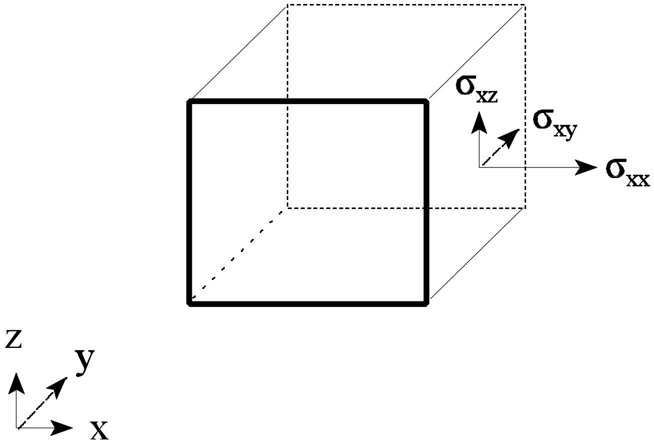


Figure 2. The stresses on the *x* plane of an element of material.

side and direction of the stress. The first subscript denotes the direction (x, y, or z) with the second defining the plane on which it acts. The *x* plane is defined as the plane mapped by the *yz* axis where *x* is normal to the plane. (see figure 2).

Having defined stresses and strains for a material in continuum mechanics, constitutive equations can be defined. A constitutive equation is the relation between stress and strain for a material. In a one-dimensional linear system, a constitutive equation would be $\tau = E\epsilon$.

For the unique stresses and strains above, Hooke's law written in vector notation is (where **S** is the stiffness matrix):

$$\begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{pmatrix} = \begin{pmatrix} S_{11} & \cdot & \cdot & \cdot & \cdot & S_{16} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ S_{61} & \cdot & \cdot & \cdot & \cdot & S_{66} \end{pmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{yz} \\ \epsilon_{zx} \end{pmatrix} \qquad \textbf{(20)}$$

It is often common to label normal stresses, $\hat{o}_{ii}$, as σ_i. Notice that with the use of symmetry, only six terms from equation 20 are unique, which is the same for the strains. This allows the unique stress and stain terms to be written as column vectors for mathematical purposes.