**Sum of a Type of Harmonic Sequence**

**By Erdem Varol**

**Theorem: If n is a positive integer then**

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}$$

**Proof:** The above statement implies

$$1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}$$

Now, let’s investigate the right hand side.

First thing we notice is the pair of $\frac{1}{2n}$. They add up to $\frac{1}{n}$. So, the right hand side turns into:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n-2}+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n-2}+\frac{1}{2n-1}$$

Now, there is a pair of $\frac{1}{2n-2}$. They add up to $\frac{1}{n-1}$.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n-4}+\left.\underline{\frac{1}{n-1}+\frac{1}{n}}\right|+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n-4}+\left|\underline{\frac{1}{2n-3}+\frac{1}{2n-1}}\right.$$

As we continue this process, we will eventually reach the point where the expression turns into:

$$\frac{1}{2}+\frac{1}{4}+\ldots\frac{1}{2m}+\left.\underline{\frac{1}{n-k}+\ldots+\frac{1}{n-1}+\frac{1}{n}}\right|+\left|\underline{\frac{1}{n+1}+\ldots+\frac{1}{2n-3}+\frac{1}{2n-1}}\right.$$

If n is even ( $\frac{1}{n+2}=\frac{1}{2m+2},\frac{1}{n+2}+\frac{1}{2m+2}=\frac{1}{n-k}$ )

$$\frac{1}{2}+\frac{1}{4}+\ldots\frac{1}{2m}+\left.\underline{\frac{1}{n-k}+\ldots+\frac{1}{n-1}+\frac{1}{n}}\right|+\left|\underline{\frac{1}{n+2}+\ldots+\frac{1}{2n-3}+\frac{1}{2n-1}}\right.$$

If n is odd ( $\frac{1}{n+1} = \frac{1}{2m+2}, \frac{1}{n+1} + \frac{1}{2m+2} = \frac{1}{n-k}$ )

We can continue this process with $\frac{1}{2} + \frac{1}{4} + ... \frac{1}{2m} + \frac{1}{n-k} + ... + \frac{1}{n-1} + \frac{1}{n}$ :

(If n is even, $\frac{1}{n} = \frac{1}{2m}$ , $\frac{1}{n-2} = \frac{1}{2m-2} \ldots$

Or if n is odd $\frac{1}{n-1} = \frac{1}{2m}, \frac{1}{n-3} = \frac{1}{2m-2} \ldots$)

by adding 2 of the same fractions in order to only have odd denominators. If there are still even numbered denominators we can keep repeating this process until eventually, they run out. When the entire expression is simplified the residue is:

$$1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} .$$

This means:

$$1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$$

Subtracting $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n}$ from both sides proves the above statement that

$$\sum_{k=1}^{2n} \frac{(-1)^{j+1}}{j} = \sum_{k=n+1}^{2n} \frac{1}{k} . \blacksquare$$