

Fundamental Properties of the Fibonacci Sequence

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In a sequence defined by the recurrence relation $f_0 = 0$ $f_1 = 1$ $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ the following properties hold:

$$1. \sum_{m=1}^n f_m = f_{n+2} - 1$$

$$2. \sum_{m=1}^n f_m^2 = f_{n+1}f_n$$

Proof of 1: If we expand the terms according to the recurrence relation in the following manner, we will find a pattern:

$$\begin{aligned} f_{n+2} &= \underline{f_n} + f_{n+1} \\ f_{n+2} &= \underline{f_n + f_{n-1}} + f_n \\ f_{n+2} &= \underline{f_n + f_{n-1} + f_{n-2}} + f_{n-1} \\ &\vdots \\ f_{n+2} &= \underline{f_n + f_{n-1} + \dots + f_q + f_{q-1}} + f_q \end{aligned}$$

By setting if $q = 2$, we get that $f_{n+2} = \sum_{m=1}^n f_m + 1$ so, $f_{n+2} - 1 = \sum_{m=1}^n f_m$.

Proof of 2: Again, if we expand the terms we find a pattern:

$$\begin{aligned} f_n f_{n+1} &= f_n (f_n + f_{n-1}) = \underline{f_n^2} + f_n f_{n-1} \\ f_n f_{n+1} &= f_n^2 + f_{n-1} (f_{n-1} + f_{n-2}) = \underline{f_n^2 + f_{n-1}^2} + f_{n-1} f_{n-2} \\ &\vdots \\ f_n f_{n+1} &= \underline{f_n^2 + f_{n-1}^2 + \dots + f_q^2} + f_q f_{q-1} \end{aligned}$$

This time, setting $q = 1$ will do the job. Since $f_0 = 0$ the above equation turns into:

$$f_n f_{n+1} = \sum_{m=1}^n f_m^2$$