A Property of the Fibonacci Sequence

Fibonacci sequence is defined with the recurrence relation $F_n = F_{n-1} + F_{n-2}$ where $F_1=F_2=1$. The first few terms are:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 …

By observation, it is noticeable that every 3rd Fibonacci number is divisible by 2, and every 4th Fibonacci number is divisible by 3, etc. It is actually true that nth Fibonacci number divides every nth Fibonacci number.

Theorem: In the Fibonacci sequence where $F_n = F_{n-1}+F_{n-2}$ and $F_1=F_2=1$, F_n divides $F_{m\,n}$.

Proof: By the recurrence relation, $F_{mn} = F_{mn-1} + F_{mn-2} = F_2F_{mn-1} + F_1F_{mn-2}$.

If we only "expand" the first term, we get:

$F_2F_{mn-1} + F_1F_{mn-2} = (F_2+F_1)F_{mn-2} + F_2F_{mn-3} = F_3F_{mn-2} + F_2F_{mn-3}$

If we keep repeating this procedure, we will eventually get:

$F_{mn} = F_nF_{mn-n+1} + F_{n-1}F_{mn-n}$

This means that the divisibility of F_{mn-n} by F_n implies the divisibility of F_{mn} by F_n. This means that, since F_n divides $F_{n(2-1)}$, Fn divides F_{2n}. Since F_n divides $F_{n(3-1)}$, F_n divides F_{3n}…This can be continued infinitely. Therefore, F_n divides F_{mn} for all m $\{m \in N, m \geq 1\}$.