<u>Another Recursive Integer Sequence</u>

The integer sequence with the recurrence relation $A_n = A_{n-1}A_{n-2} + 1$ where $A_1 = A_2 = 1$ has its first few terms as

1,1,2,3,7,22,155,3411,528706,1803416167,953476947989903,
1719515742866809222961802,
1639518622529236077952144318816050685207,
2819178082162327154499022366029959843954512194276761760087463015

Just like in the Fibonacci Sequence, the n^{th} term divides every n^{th} term.

Theorem : In a sequence defined with the recurrence equation, $A_n = A_{n-1}A_{n-2} + 1$ where $A_1 = A_2 = 1$, $A_n \mid A_{mn}$.

Proof: Since $A_n = A_{n-1}A_{n-2} + 1$, If mn = x then $A_x = A_{x-1}A_{x-2} + 1$. Since $A_nA_{n-1} + 1 = A_{n+1}$, $A_{n+1} = 1 \ (\bmod A_n)$. Also, $A_nA_{n+1} + 1 = A_{n+2}$, So $A_{n+2} = 1 \ (\bmod A_n)$. Using this, we get that $A_{x-(n-2)} = A_{x-(n-1)} = 1 \ (\bmod A_{x-n})$. Substituting, we get that $A_{x-(n-2)} = A_2 \ (\bmod A_{x-n})$ and $A_{x-(n-1)} = A_1 \ (\bmod A_{x-n})$.

So $A_{x-(n-2)} A_{x-(n-1)} + 1 = A_{x-(n-3)} = A_2A_1 + 1 = A_3 \ (\bmod A_{x-n})$.

By repeating this process, we will eventually reach:

$A_{x-1}A_{x-2} + 1 = A_x = A_n \ (\bmod A_{x-n})$. By substituting mn with x, we see that if A_n divides A_{mn-n}, then it divides A_{mn}. Again, divisibility of A_n by itself implies the divisibility of A_{mn} by A_n for all positive non-zero m. So A_n divides all A_{mn} $\{m \geq 1, m \in N\}$.

P.S. This sequence is called A007660 by AT&T Integer Sequences Research.