11/09/01 (Tapped Session)

- Summarized Properties of the autocorrelation func. of stationary RP's

  1) $R_X(-\tau) = R_X(\tau) \quad \forall \tau$
  2) $R_X(\tau) \leq R_X(0) \quad \forall \tau$ — Total Avg. Power of $X(t)$
  3) If for some $T \neq 0$ $R_X(T) = R_X(0) \Rightarrow R_X(\tau + T) = R_X(\tau) \; \forall \tau$.
  4) If $R_X(\tau)$ is continuous at $\tau = 0 \Rightarrow R_X(\tau)$ continuous everywhere.

- Summarized Properties of $S_X(f)$ of a stationary RP $X(t)$

  1) $S_X(f)$ is always real.
  2) $S_X(-f) = S_X(f) \quad \forall f$
  3) $S_X(f) \geq 0$

■ Some Extra tools

$$\rightarrow \begin{cases} S_{XX'}(f) = -j2\pi f S_X(f) \longleftrightarrow R_{XX'}(\tau) = -\dfrac{d}{d\tau} R_X(\tau) \\ S_{X'}(f) = \omega^2 S_X(f) \longleftrightarrow R_{X'}(\tau) = -\dfrac{d^2}{d\tau^2} R_X(\tau) \end{cases}$$

- Properties of Cross-correlations of jointly WSS RP's.

  1) $R_{YX}(\tau) = R_{XY}(-\tau)$.
  2) $|R_{XY}(\tau)| \leq \sqrt{R_X(0) R_Y(0)}$

- Extra Properties for Discrete Time WSS processes.

  1) $S_X(f)$ periodic with period 1 (therefore we just need to define it in $(-\frac{1}{2}, \frac{1}{2}]$)
  2) Same thing applies to $S_{XY}(f)$.

- Summary of LTI-related equations

Continuous time RP's:

$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau) \longleftrightarrow S_Y(f) = |H(f)|^2 S_X(f)$

$R_{XY}(\tau) = R_X(\tau) * h(-\tau) \longleftrightarrow S_{XY}(f) = H(f)^* S_X(f)$

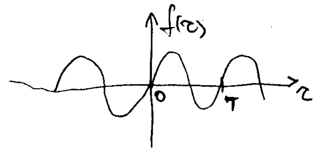
$m_Y = m_X H(0)$

$S_{XY}(f) = S^*_{YX}(f)$

Problem 1

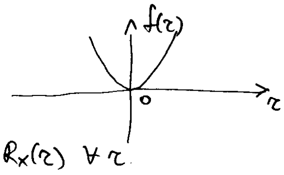
Determine if the following functions qualify as autocorrelations of a WSS RP

a) $f(\tau) = \sin(\omega\tau)$



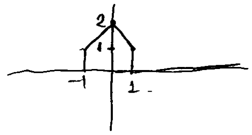
No, because $R_x(-\tau) \neq R_x(\tau)$.

b) $f(\tau) = \tau^2$



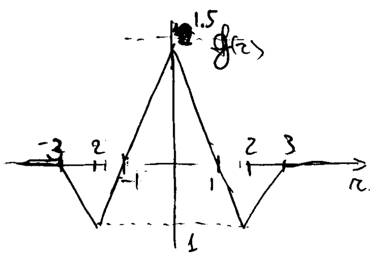
No, because $R_x(0) \leq R_x(\tau) \ \forall \tau$.

c) $f(\tau) = \begin{cases} 2 - |\tau| & |\tau| \leq 1 \\ 0 & |\tau| > 1 \end{cases}$



No, because although it is continuous at $\tau = 0$, it is discontinuous at $\tau = \pm 1$.

d)



$g(\tau)$ looks legitimate, because

i) $g(\tau)$ even

ii) $g(\tau)$ continuous everywhere.

iii) $g(\tau)$ max at $f(0)$

However....

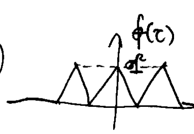
Let us find $\mathcal{F}\{g(\tau)\} = S(f)$

$$g(\tau) = 1.5\,\Lambda(\tau) - \Lambda(\tau-2) - \Lambda(\tau+2) \Rightarrow S(f) = 1.5\,\mathrm{sinc}^2(f) - e^{-4\pi f j}\mathrm{sinc}^2(f) - e^{+4\pi f j}\mathrm{sinc}^2(f)$$

$$S(f) = \left\{1.5 - \left[e^{-4\pi f j} + e^{-4\pi f j}\right]\right\}\mathrm{sinc}^2(f) = \underbrace{\left[1.5 - 2\cos(4\pi f)\right]}_{\text{this can become negative}}\mathrm{sinc}^2(f)$$

therefore $f(\tau)$ is NOT an autocorrelation function

e)



No, it is not because for a real autocorrelation ~~funct~~ if $R_x(\tau_1) = R_x(0)$, then $R_x(\tau)$ would be periodic.

Problem #2

$X(t)$ WSS with $R_X(\tau) = e^{-\alpha|\tau|}$

$X(t) \longrightarrow \boxed{LTI} \longrightarrow Y(t)$ Find $R_Y(\tau)$, $R_{XY}(\tau)$ if.

(α)

$X(t) \longrightarrow \boxed{\text{Delay } \Delta} \longrightarrow Y(t) = X(t-\Delta)$

$$\left.\begin{aligned} h(t) &= \delta(t-\Delta) \Rightarrow H(f) = e^{-j2\pi f\Delta} \Rightarrow |H(f)| = 1 \\ S_Y(f) &= |H(f)|^2 S_X(f) \\ S_{XY}(f) &= H^*(f) S_X(f) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{aligned} S_Y(f) &= S_X(f) \\ S_{XY}(f) &= e^{j2\pi f\Delta} S_X(f) \end{aligned}\right\} \overset{\mathcal{F}^{-1}}{\Longrightarrow} \left\{\begin{aligned} R_Y(\tau) &= R_X(\tau) = e^{-\alpha|\tau|} \\ R_{XY}(\tau) &= R_X(\tau+\Delta) = e^{-\alpha|\tau+\Delta|} \end{aligned}\right.$$

Interpretation: Since $X(t)$ WSS goes through a delay, $Y(t)$ should have the same statistics as $X(t)$

(b) $h(t) = \frac{1}{t} \overset{\mathcal{F}}{\Rightarrow} H(f) = -j\pi \,\mathrm{sgn}(f)$

$$S_{XY}(f) = H^*(f) S_X(f) = j\pi\,\mathrm{sgn}(f) S_X(f) \overset{\mathcal{F}^{-1}}{\Rightarrow} R_{XY}(t) = -\int_{-\infty}^{+\infty} \frac{e^{-\alpha|t|}}{\tau - t}\, dt$$

$$S_Y(f) = |H(f)|^2 S_X(f) = \pi^2 \mathrm{sgn}^2(f) S_X(f) = \pi^2 S_X(f) \overset{\mathcal{F}^{-1}}{\Rightarrow} R_Y(f) = \pi^2 e^{-\alpha|\tau|}$$

(c) LTI described by $\frac{dY(t)}{dt} + Y(t) = \frac{d}{dt}X(t) - X(t)$

$$\left.\begin{aligned} H(f) &= \frac{-1+j2\pi f}{1+j2\pi f} \\ \mathcal{F}\{R_X(\tau)\} &= S_X(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2} \\ S_{XY}(f) &= H^*(f) S_X(f) \end{aligned}\right\} \Rightarrow S_{XY}(f) = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2} \cdot \frac{-1-j2\pi f}{1-j2\pi f} \Rightarrow$$

$$R_{XY}(\tau) = \mathcal{F}^{-1}\{S_{XY}(f)\} = \frac{4\alpha}{1-\alpha^2}\mathcal{F}^{-1}\left\{\frac{1}{1-j2\pi f}\right\} + \frac{\alpha-1}{1+\alpha}\mathcal{F}^{-1}\left\{\frac{1}{\alpha+j2\pi f}\right\} + \frac{1+\alpha}{\alpha-1}\mathcal{F}^{-1}\left\{\frac{1}{\alpha-j2\pi f}\right\} =$$

$$= \frac{4\alpha}{1-\alpha^2} e^{\tau} u(-\tau) + \frac{\alpha-1}{1+\alpha} e^{-\alpha\tau} u(\tau) + \frac{1+\alpha}{\alpha-1} e^{\alpha\tau} u(-\tau)$$

$$S_Y(f) = |H(f)|^2 S_X(f) = S_X(f) \Rightarrow R_Y(\tau) = R_X(\tau) = e^{-\alpha|\tau|}$$

Problem 7.36 p.454

$$Y_n = \alpha_1 Y_{n-1} + \ldots + \alpha_q Y_{n-q} + W_n \qquad \leftarrow ARMA(q,0) = AR(q)$$

$E\{W_n\}=0$ White Noise

this is a hint that W_n is WSS with $R_W(k)=0 \quad k \neq 0$

Y_n will also be WSS.

a) Show that $R_Y(k)$ satisfies

$$R_Y(0) = \sum_{i=1}^{q} \alpha_i R_Y(i) + R_W(0)$$
$$R_Y(k) = \sum_{i=1}^{q} \alpha_i R_Y(k-i)$$

$$R_Y(0) = E\{Y_n^2\} = E\{Y_n[\alpha_1 Y_{n-1} + \ldots + \alpha_q Y_{n-q} + W_n]\} = \sum_{i=1}^{q} \alpha_i \underbrace{E\{Y_n Y_{n-i}\}}_{R_Y(i)} + \underbrace{E\{Y_n W_n\}}_{R_{YW}(0)} \quad ①$$

$$R_{YW}(0) = E\{Y_n W_n\} = E\{W_n[\alpha_1 Y_{n-1} + \ldots + \alpha_q Y_{n-q} + W_n]\} = \sum_{i=1}^{q} \alpha_i \underbrace{E\{W_n Y_{n-i}\}}_{0} + \underbrace{E\{W_n^2\}}_{R_W(0)} \quad ②$$

because Y_{n-i} does not depend on W_n

Therefore, $\overset{①,②}{\Rightarrow} R_Y(0) = \sum_{i=1}^{q} \alpha_i R_Y(i) + R_W(0).$

Next,
$$R_Y(k) = E\{Y_n Y_{n+k}\} \overset{\text{or}}{=} E\{Y_{n-k} Y_n\} = E\{Y_{n-k}[\alpha_1 Y_{n-1} + \ldots + \alpha_q Y_{n-q} + W_n]\} =$$

$$= \sum_{i=1}^{q} \alpha_i \underbrace{E\{Y_{n-k} Y_{n-i}\}}_{R_Y(i-k) = R_Y(k-i)} + \underbrace{E\{Y_{n-k} W_n\}}_{0} \Rightarrow R_Y(k) = \sum_{i=1}^{q} \alpha_i R_Y(k-i).$$

(b) Example 7.20 talks about $Y_n = r Y_{n-1} + W_n$ AR(1)

$a_1 = r$

From the first equation: $R_Y(0) = r R_Y(1) + R_W(0)$ ③

From the second eq. : $R_Y(k) = r R_Y(k-1)$ ④

$\Big\}$ Difference equation with Initial condition

we need to find this

$$\left.\begin{array}{l} k \geq 0 \\ R_Y(k) = r R_Y(k-1) \end{array}\right\} \Rightarrow R_Y(k) = \underbrace{R_Y(0)}_{\sigma_Y^2} r^k$$

$$\left.\begin{array}{l} ④ \Rightarrow R_Y(1) = r R_Y(0) \\ ③ : R_Y(0) = r R_Y(1) + R_W(0) \end{array}\right\} \Rightarrow \underbrace{R_Y(0)}_{\sigma_Y^2} = \frac{\overbrace{R_W(0)}^{\sigma_W^2}}{1-r^2}$$

$\Rightarrow$

$$\Rightarrow R_Y(k) = \left(\frac{R_W(0)}{1-r^2}\right) r^k \quad k \geq 0$$

Since we must have $R_Y(-k) = R_Y(k) \Rightarrow R_Y(k) = \frac{R_W(0)}{1-r^2} r^{|k|}$ ∎