EEL5542 – Fall 2001
HOMEWORK #6

Solution of Problems

.25 a) $\mathcal{E}[M_n] = \frac{1}{n}\mathcal{E}[X_1 + X_2 + ... + X_n] = \frac{n\mathcal{E}[X]}{n} = \mathcal{E}[X]$

$$
\begin{aligned}
C_M(n,k) &= \mathcal{E}[(M_n - \mathcal{E}(X))(M_k - \mathcal{E}(X))] \\
&= \mathcal{E}\left[\frac{1}{n}[S_n - n\mathcal{E}(X)]\frac{1}{k}[S_k - k\mathcal{E}[X]]\right] \\
&= \frac{1}{nk}\mathcal{E}[(S_n - \mathcal{E}[S_n])(S_k - \mathcal{E}[S_k])] \\
&= \frac{1}{nk}C_S(n,k) = \frac{1}{nk}\min(n,k)\sigma_X^2 \\
VAR(M_n) &= C_M(n,n) = \frac{1}{n}\sigma_X^2
\end{aligned}
$$

b) Since $M_{n+1} - M_n = \frac{1}{n+1}X_n - \frac{1}{n+1}M_n$, M_n does not have indep. increments.

6.27 a) $\Phi_{S_n}(\omega) = \Phi_X(\omega)^n = (e^{-\alpha|\omega|})^n = e^{-n\alpha|\omega|}$

$\Rightarrow f_{S_n}(x) = \dfrac{n\alpha/\pi}{x^2 + n^2\alpha^2}$

b) Since S_n has independent and stat. increments

$$
\begin{aligned}
f_{S_n,S_{n+k}}(y_1,y_2) &= f_{S_n}(y_1)f_{S_{n+k-n}}(y_2 - y_1) \\
&= \frac{n\alpha/\pi}{(y_1^2 + n^2\alpha^2)}\frac{k\alpha/\pi}{((y_2 - y_1)^2 + k^2\alpha^2)}
\end{aligned}
$$

6.48 a) From Example 4.32 we know that $Z(t) = X(t) - \alpha X(t-s)$ is a Gaussian RV since $X(t)$ and $X(t-s)$ are jointly Gaussian. Therefore we need only find $m_Z(t)$ and $VAR[Z(t)]$

$$\begin{aligned}
m_Z(t) &= \mathcal{E}[X(t)] - a\mathcal{E}[X(t-s)] = 0 \\
VAR[Z(t)] &= \mathcal{E}[(X(t) - aX(t-s))^2] \\
&= \mathcal{E}[X^2(t)] - 2a\mathcal{E}[X(t)X(t-s)] + a^2\mathcal{E}[X^2(t-s)] \\
VAR[Z(t)] &= \alpha t - 2a(\alpha(t-s)) + a^2\alpha(t-s) \\
&= \alpha t(1 - 2a + a^2) + 2a\alpha s - a^2\alpha s \\
&= \alpha t(a-1)^2 - a\alpha s(a-2) \\
f_{Z(t)}(z) &= \frac{\exp\left\{-\frac{z^2}{2VAR[Z(t)]}\right\}}{\sqrt{2\pi VAR[Z(t)]}}
\end{aligned}$$

b)

$$\begin{aligned}
m_Z(t) &= E[X(t) - aX(t-s)] = 0 \\
C_Z(t_1, t_2) &= E[\{Z(t_1) - m(t_2)\}\{Z(t_2) - m(t_2)\}] \\
&= E[\{X(t_2) - aX(t_1 - s)\}\{X(t_2) - aX(t_2 - s)\}] \\
&= \alpha \min(t_1, t_2) - a\alpha \min(t_1 - s, t_2) \\
&\quad - a\alpha \min(t_1, t_2 - s) + a^2\alpha \min(t_1, t_2)
\end{aligned}$$

6.54 Assume X_n is discrete-valued, for simplicity, so that we can work with pmf's. Consider the third-order joint pmf of Y_n: for $n_1 < n_2 < n_3$ we need to show that for all $\tau > 0$

$$(\star) \quad P[Y_{n_1} = y_1, Y_{n_2} = y_2, Y_{n_3} = y_3] = P[Y_{n_1+\tau} = y_1, Y_{n_2+\tau} = y_2, Y_{n_3+\tau} = y_3]$$

Express the above probabilities in terms of the X_n's:

$$\begin{aligned}
&P[Y_{n_1} = y_1, Y_{n_2} = y_2, Y_{n_3} = y_3] \\
&= P\left[\frac{1}{2}(X_{n_1} + X_{n_1-1}) = y_1, \frac{1}{2}(X_{n_2} + X_{n_2-1}) = y_2, \frac{1}{2}(X_{n_3} + X_{n_3-1}) = y_3\right] \\
&= P\left[\frac{1}{2}(X_2 + X_1) = y_1, \frac{1}{2}(X_{n_2-n_1+2} + X_{n_2-n_1+1}) = y_2,\right. \\
&\left.\frac{1}{2}(X_{n_3-n_1+2} + X_{n_3-n_1+1}) = y_3\right]
\end{aligned}$$

Since the joint pdf of $(X_{n_1-1}, X_{n_1}, X_{n_2-1}, X_{n_2}, X_{n_3-1}, X_{n_3})$ is identical to that of $(X_1, X_2, X_{n_2-n_1+1}, X_{n_2-n_1+2}, \ldots, X_{n_3-n_1+2})$ if X_n is a stationary process.

Similarly we have that

$$\begin{aligned}
&P[Y_{n_1+\tau} = y_1, Y_{n_2+\tau} = y_2, Y_{n_3+\tau} = y_3] \\
&= P\left[\frac{1}{2}(X_{n_1+\tau} + X_{n_1+\tau-1}) = y_1, \ldots \frac{1}{2}(X_{n_3+\tau} + X_{n_3+\tau-1}) = y_3\right] \\
&= P\left[\frac{1}{2}(X_2 + X_1) = y_1, \frac{1}{2}(X_{n_2-n_1+2} + X_{n_2-n_1+1}) = y_2, \frac{1}{2}(X_{n_3-n_1+2} + X_{n_3-n_1+1}) = y_3\right]
\end{aligned}$$

$\therefore$ $(\star)$ holds if X_n is a stationary random process and in particular if X_n is an iid process.

6.56 a)

$$\begin{aligned}
\mathcal{E}[Y(t)] &= \mathcal{E}[X(t) - aX(t+s)] = \mathcal{E}[X(t)] - a\mathcal{E}[X(t+s)] \\
&= m_X(1-a) \qquad \text{since } X(t) \text{ is WSS} \\
C_Y(t_1, t_2) &= \mathcal{E}[Y(t_1)Y(t_2)] - m_X^2(1-a)^2 \\
&= \mathcal{E}[X(t_1)X(t_2)] - a\mathcal{E}[X(t_1+s)X(t_2)] \\
&\quad -a\mathcal{E}[X(t_2+s)X(t_1)] + a^2\mathcal{E}[X(t_1+s)X(t_2+s)] - m_X^2(1-a)^2 \\
&= C_X(t_1, t_2) + a^2 C_X(t_1+s, t_2+s) - aC_X(t_1+s, t_2) \\
&\quad -aC_X(t_2+s, t_1) \\
C_Y(t_1, t_2) &= (1+a^2)C_X(t_2-t_1) - aC_X(t_2-t_1-s) - aC_X(t_2-t_1+s) \\
&= C_Y(t_2-t_1) \Rightarrow Y(t) \text{ is WSS}
\end{aligned}$$

b)

$$\begin{aligned}
VAR[Y(t)] &= C_Y(t,t) = (1+a^2)C_X(0) - aC_X(-s) - aC_X(s) \\
&= (1+a^2)C_X(0) - 2aC_X(s)
\end{aligned}$$

$\Rightarrow$ $Y(t)$ is a Gaussian random process with mean $(1-a)m_X$ and variance $(1+a^2)C_X(0) - 2aC_X(s)$.

6.71 $R_X(\tau) = \sigma^2 e^{-\alpha\tau^2}, \quad R_X(t_1, t_2) = \sigma^2 e^{-\alpha(t_1-t_2)^2}$

a) Yes since $R_X(\tau)$ is continuous at τ.

b) Yes since $R_X(\tau)$ has derivatives of all orders at $\tau = 0$.

$$
\begin{aligned}
E\left[\frac{d}{dt}X(t)\right] &= \frac{d}{dt}[E[X(t)]] = 0, \qquad \text{because } E[X(t)] = \text{constant} \\
R_{X'}(\tau) &= -\frac{d}{d\tau^2}R_X(\tau) \\
&= 2\alpha\sigma^2 e^{-\alpha\tau^2}(1 - 2\alpha\tau^2)
\end{aligned}
$$

c) Yes since $R_X(\tau)$ is M.S. continuous.
Consider $Y(t) = \int_0^t X(t)dt$.

$$
\begin{aligned}
E[Y(t)] &= \int_0^t E[X(t)]dt = m_X t \\
R_Y(t_1, t_2) &= \int_0^{t_1}\int_0^{t_2} R_X(u, v)dudv
\end{aligned}
$$

If $t_2 \leq t_1$

$$
\begin{aligned}
R_Y(t_1, t_2) &= \int_0^{t_1}\int_0^{t_2} R_X(u - v)dudv \\
&= \int_{t_2}^{0}(t_2 + \tau)\sigma^2 e^{-\alpha\tau^2}d\tau + \int_0^{t_1-t_2} t_2\sigma^2 e^{-\alpha\tau^2}d\tau \\
&\quad + \int_{t_1-t_2}^{t_1}(t_1 - \tau)\sigma^2 e^{-v\tau^2}d\tau
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma^2 t_2 \int_{-t_2}^{0} e^{-\alpha\tau^2} d\tau + \frac{\sigma^2}{2\alpha} \int_{-t_2}^{0} e^{-\alpha\tau^2} d\alpha\tau^2 + \sigma^2 t_2 \int_{0}^{t_1-t_2} e^{-\alpha\tau^2} d\tau \\
&\quad + \sigma^2 t_1 \int_{t_1-t_2}^{t_1} e^{-\alpha\tau^2} d\tau - \frac{\sigma^2}{2\alpha} \int_{t_1-t_2}^{t_1} e^{-\alpha\tau^2} d\sigma\tau^2 \\
&= \sigma^2 t_2 \int_{-t_2}^{t_1-t_2} e^{-\alpha\tau^2} d\tau - \frac{\sigma^2}{2\alpha}(t - e^{-\alpha t_2^2}) \\
&\quad + \sigma^2 + 1 \int_{t_1-t_2}^{t_1} e^{-\alpha\tau^2} d\tau + \frac{\sigma^2}{2\alpha}(e^{-\alpha t_1^2} - e^{-\alpha(t_1-t_2)^2})
\end{aligned}
$$

If $t_2 > t_1$

$$
\begin{aligned}
R_Y(t_1, t_2) &= \sigma^2 t_1 \int_{-t_1}^{t_2-t_1} e^{-\alpha\tau^2} d\tau + \sigma^2 t_2 \int_{t_2-t_1}^{t_2} e^{-\alpha\tau^2} d\tau \\
&\quad + \frac{\sigma^2}{2\alpha}(-1 + e^{-\alpha t_1^2} + e^{-\alpha t_2^2} - e^{-\alpha(t_1-t_2)^2})
\end{aligned}
$$

d) The fact that the autocorrelation has the shape of a Gaussian pdf does not imply that the process is Gaussian.

6.76

$$
\begin{aligned}
R_{XX'}(t_1, t_2) &= \frac{\partial}{\partial t_2} R_X(t_1, t_2) \\
&= \frac{\partial}{\partial t_2} R_X(t_1 - t_2) \\
&= -\frac{dR_X(\tau)}{d\tau}, \qquad \tau = t_1 - t_2
\end{aligned}
$$

$$R_X(\tau) \le R_X(0), \qquad R_X(0) \text{ is the max of } R_X(\tau)$$

$$\therefore \left. \frac{dR_X(\tau)}{d\tau} \right|_{\tau=0} = 0$$

$$\text{i.e.,} \qquad R_{XX'}(t, t) = 0$$