EEL5542 – Fall 2001
HOMEWORK #5

Solution of Problems

6.3 **a)**

		...	$n=0$	$n=1$	$n=2$	...	
If $\xi =$ Heads	X_n	...	1	-1	1	-1	...
If $\xi =$ Tails	X_m	...	-1	1	-1	1	...

b) n even $\quad P[X_n = 1] = P[\text{Heads}] = \frac{1}{2}$

n odd $\quad P[X_m = 1] = P[\text{Tails}] = \frac{1}{2}$

c) k even

$$\begin{aligned} P[X_n = 1, X_{n+k} = 1] &= P[\text{Heads}] = \frac{1}{2} \\ P[X_n = -1, X_{n+k} = -1] &= P[\text{Tails}] = \frac{1}{2} \\ P[X_n = \pm 1, X_{n+k} = \mp 1] &= 0 \end{aligned}$$

k odd

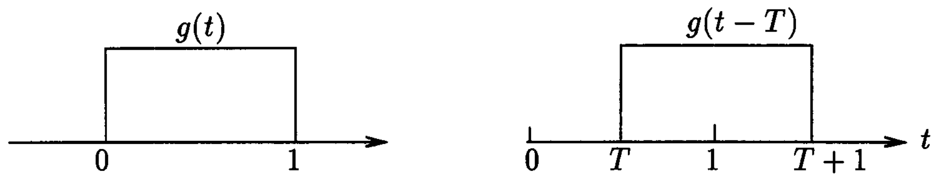
$$\begin{aligned} P[X_n = 1, X_{n+k} = -1] &= P[\text{Heads}] = \frac{1}{2} \\ P[X_n = -1, X_{n+k} = 1] &= P[\text{Tails}] = \frac{1}{2} \\ P[X_n = \pm 1, X_{n+k} = \pm 1] &= 0 \end{aligned}$$

d) $\mathcal{E}[X_n] = 1\left(\frac{1}{2}\right) + (-10)\frac{1}{2} = 0$

k even $\quad \mathcal{E}[X_n X_{n+k}] = (1)^2\frac{1}{2} + (-1)^2\frac{1}{2} = 1$

k odd $\quad \mathcal{E}[X_n X_{n+k}] = (1)(-1)\frac{1}{2} + (-1)(1)\frac{1}{2} = -1$

6.6



a)

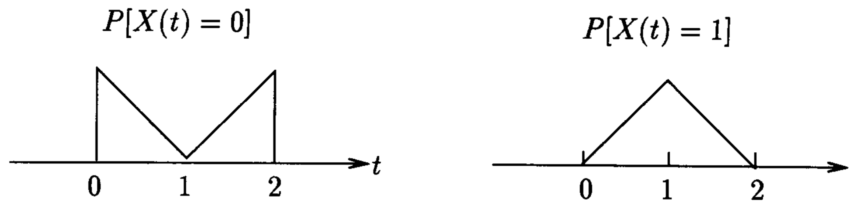
$$P[Y(t)=1] = P[g(t-T)=1]$$

For $0 < t < 1$

$$P[Y(t)=0] = P[t < T] = 1 - t = 1 - P[Y(t)=1]$$

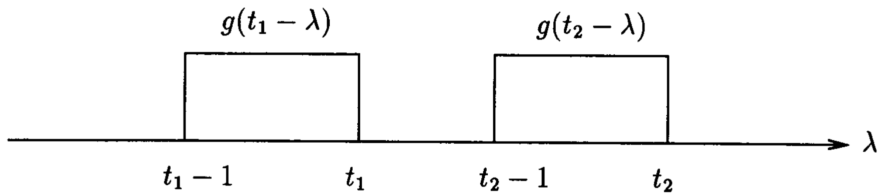
For $1 < t < 2$

$$\begin{aligned} P[Y(t)=1] &= P[t < T+1] = P[T > t-1] = 1-(t-1) \\ &= 2-t \\ P[Y(t)=0] &= 1 - P[Y(t)=1] = t-1 \end{aligned}$$



b) $\mathcal{E}[Y(t)] = 1 \cdot P[Y(t) = 1] = P[Y(t) = 1]$

$$\begin{aligned} E[Y(t_1)Y(t_2)] &= \int_0^1 E[g(t_1 - T)g(t_2 - T)|T = \lambda] f_T(\lambda) d\lambda \\ &= \int_0^1 g(t_1 - \lambda) g(t_2 - \lambda) d\lambda \end{aligned}$$



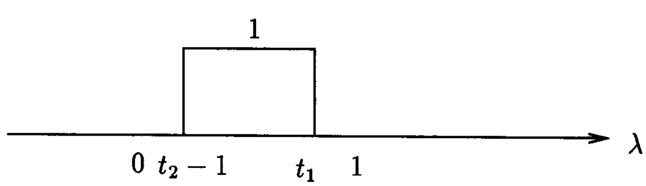
$$g(t_1 - \lambda)g(t_2 - \lambda) = 0 \quad \text{for} \quad t_1 < t_2 - 1$$

$$\Rightarrow \ R_Y(t_1, t_2) = 0 \quad \text{for} \quad t_2 - t_1 > 1$$

If $t_2 - 1 < t$, then

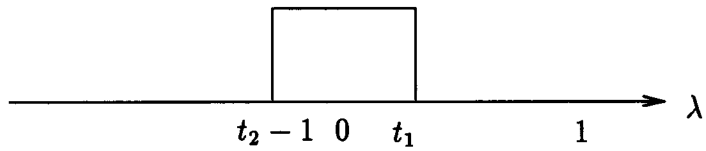
$$g(t_1 - \lambda)g(t_2 - \lambda) = \begin{cases} 1 & t_2 - 1 < \lambda < t_1 \\ 0 & \text{elsewhere} \end{cases}$$

Case 1



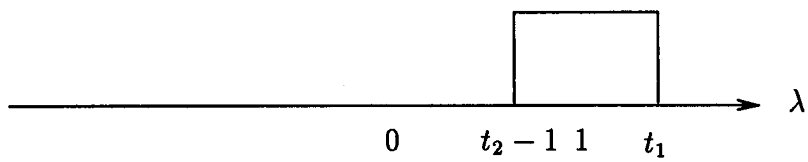
$$R_Y(t_1, t_2) = t_1 - (t_2 - 1) = 1 - (t_2 - t_1) \qquad t_1 < 1, \ \ 0 < t_2 - 1, \ \ t_2 - t_1 < 1$$

Case 2



$$R_Y(t_1, t_2) = t_1 \qquad t_1 < 1, \quad t_2 < 1, \quad t_2 - t_1 < 1$$

Case 3



$$R_Y(t_1, t_2) = 2 - t_2 \qquad t_1 > 1, \quad t_2 < 2$$

6.10 a) $P[H(t)=1] = P[X(t)\geq 0] = P[\xi\cos 2\pi t \geq 0] = \frac{1}{2} = P[H(t)=-1]$

$$\begin{aligned}
\mathcal{E}[H(t)] &= 1\cdot P[H(t)=1] + (-1)P[H(t)=-1] = 0 \\
C_H(t,t+\tau) &= \mathcal{E}[H(t)H(t+\tau)] \\
&= 1\cdot P[\underbrace{H(t)H(t+\tau)=1}_{\substack{H(t)\ \&\ H(t+\tau)\\ \text{same sign}}}] + (-1)P[\underbrace{H(t)H(t+\tau)=-1}_{\substack{H(t)\ \&\ H(t+\tau)\\ \text{opposite sign}}}]
\end{aligned}$$

$$\begin{aligned}
H(t)H(t+\tau) &= 1 \Leftrightarrow \cos 2\pi t \text{ and } \cos 2\pi(t+\tau) \text{ have same sign} \\
H(t)H(t+\tau) &= -1 \Leftrightarrow \cos 2\pi t \text{ and } \cos 2\pi(t+\tau) \text{ have different sign}
\end{aligned}$$

$$\therefore C_H(t,t+\tau) = \begin{cases} 1 & \text{for } t,\tau \text{ such that } \cos 2\pi t\cos 2\pi(t+\tau) = 1 \\ -1 & \text{for } t,\tau \text{ such that } \cos 2\pi t \cos 2\pi(t+\tau) = -1 \end{cases}$$

b) $P[H(t)=1] = P[X(t)\geq 0] = P[\cos(\omega t+\Theta)\geq 0] = \frac{1}{2} = P[H(t)=-1]$

$$\begin{aligned}
\mathcal{E}[H(t)] &= 1\left(\frac{1}{2}\right) + (-1)\frac{1}{2} = 0 \\
\mathcal{E}[H(t)H(t+\tau)] &= 1\cdot \underbrace{P[X(t)X(t+\tau)>0]}_{1-P[X(t)X(t+\tau)<0]} + (-1)P[X(t)X(t+\tau)<0] \\
&= 1 - 2P[X(t)X(t+\tau)<0]
\end{aligned}$$

$$\begin{aligned}
P[X(t)X(t+\tau)<0] &= P[\cos(\omega t+\Theta)\cos(\omega(t+\tau)+\Theta)<0] \\
&= \left[\frac{1}{2}\cos\omega\tau + \frac{1}{2}\cos(2\omega t+\omega\tau+2\Theta) < 0\right] \\
&= P[\cos(2\omega t+\omega\tau+2\Theta) < \cos\omega\tau] \\
&= 1 - \frac{\text{shaded region in figure}}{2\pi}
\end{aligned}$$

c) $P[H(t)=1] = P[X(t)\geq 0] = 1 - F_{X(t)}(0^-) = 1 - P[H(t)=-1]$

$$\begin{aligned}\mathcal{E}[H(t)] &= 1 \cdot P[H(t)=1] + (-1)P[H(t)=-1] \\ &= 1 - F_{X(t)}(0^-) - F_{X(t)}(0^-) \\ &= 1 - 2F_{X(t)}(0^-)\end{aligned}$$

d) $\mathcal{E}[H(t)X(t)] = \mathcal{E}[|X(t)|]$

since

$$H(t)X(t) = \begin{cases} +X(t) & X(t) \geq 0 \\ -X(t) & X(t) < 0. \end{cases}$$

6.17 a) $\mathcal{E}[Y(t)] = \mathcal{E}[X(t) - aX(t+d)] = m_X(t) - am_X(t+d)$

$$\begin{aligned}\mathcal{E}[Y(t_1)Y(t_2)] &= \mathcal{E}[(X(t_1) - aX(t_1+d))(X(t_2) - aX(t_2+d))] \\ &= \mathcal{E}[X(t_1)X(t_2)] - a\mathcal{E}[X(t_1+d)X(t_2)] \\ &\quad -a\mathcal{E}[X(t_1)X(t_2+d)] + a^2\mathcal{E}[X(t_1+d)X(t_2+d)] \\ C_Y(t_1,t_2) &= \mathcal{E}[Y(t_1)Y(t_2)] - m_Y(t_1)m_Y(t_2) \\ &= C_X(t_1,t_2) - aC_X(t_1,t_2+d) - aC_X(t_1+d,t_2) + a^2C_X(t_1+d,t_2+d)\end{aligned}$$

b) Since $X(t)$ and $X(t+d)$ are jointly Gaussian RV's, $Y(t)$ is also Gaussian with mean $m_Y(t)$ and variance:

$$\sigma^2_{Y(t)} = C_Y(t,t) = \sigma^2_X(t) - 2aC_X(t,t+d) + a^2\sigma^2_X(t+d)$$

$$f_{Y(t)}(y) = \frac{\exp\left\{-\frac{(y - m_X(t) + am_X(t+d))^2}{2\sigma^2_Y(t)}\right\}}{\sqrt{2\pi}\sigma_Y(t)}$$

c) $Y(t)$ and $Y(t+s)$ are jointly Gaussian RV's defined by the linear transformation

$$\begin{aligned}Y(t) &= X(t) - aX(t+d) \\ Y(t+s) &= X(t+s) - aX(t+s+d)\end{aligned}$$

$$f_{Y(t)Y(t+s)}(y_1,y_2) = \frac{\exp\left\{-\frac{\left(\frac{y_1 - m_Y(t)}{\sigma_Y(t)}\right)^2 - \frac{2C_Y(t,t+s)}{\sigma^2_Y(t)\sigma^2_Y(t+s)}(y_1 - m_Y(t))(y_2 - m_Y(t+s)) + \left(\frac{y_2 - m_Y(t+s)}{\sigma_Y(t+s)}\right)^2}{2(1 - C^2_Y(t,t+s)/\sigma^2_Y(t)\sigma^2_Y(t+s))}\right\}}{2\pi\sigma_Y(t)\sigma_Y(t+s)\sqrt{1 - C^2_Y(t,t+s)/\sigma^2_Y(t)\sigma^2_Y(t+s)}}$$

where $X_Y(t,t+s)$ is given in part a) and $\sigma^2_Y(t)$ is given in part b).

d) Any n time samples of $Y(t)$ are defined by a linear transformation of time samples of $X(t)$.

6.19 $\mathcal{E}[Y(t)] = \mathcal{E}[X^2(t)] = C_X(t,t)$
$\mathcal{E}[Y(t_1)Y(t_2)] = \mathcal{E}[X^2(t_1)X^2(t_2)]$
To proceed further we need the result from Problem 4.69

$$\begin{aligned} \mathcal{E}[X^2(t_1)X^2(t_2)] &= \mathcal{E}[X^2(t_1)]\mathcal{E}[X^2(t_2)] + 2\mathcal{E}[X(t_1)X(t_2)]^2 \\ \Rightarrow \mathcal{E}[Y(t_1)Y(t_2)] &= C_X(t_1,t_1)C_X(t_2,t_2) + 2C_X^2(t_1,t_2) \end{aligned}$$

$$\begin{aligned} \therefore C_Y(t_1,t_2) &= \mathcal{E}[Y(t_1)Y(t_2)] - m_Y(t_1)m_Y(t_2) \\ &= 2C_X^2(t_1,t_2) \end{aligned}$$