EEL5542 – Fall 2001
HOMEWORK #4

Solution of Problems

5.3 Proceeding as in previous problem:

$$
\begin{aligned}
\mathcal{E}[S_n] &= n\mu \\
K &= \begin{bmatrix} \sigma^2 & \rho\sigma^2 & \rho^2\sigma^2 & \dots & \rho^{n-1}\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \rho\sigma^2 & \dots & \rho^{n-2}\sigma^2 \\ \vdots & & & & \\ \rho^{n-1}\sigma^2 & & \dots & & \sigma^2 \end{bmatrix} \\
VAR(S_n) &= n\sigma^2 + 2\rho\sigma^2 \sum_{j=1}^{n-1}\sum_{k=0}^{j-1} \rho^k \\
&= n\sigma^2 + 2\rho\sigma^2 \sum_{j=1}^{n-1} \frac{1-\rho^j}{1-\rho} \\
&= n\sigma^2 + 2\rho\sigma^2 \left[\frac{n-1}{1-\rho} - \left(\frac{\rho}{1-\rho}\right)\frac{1-\rho^{n-1}}{1-\rho}\right]
\end{aligned}
$$

5.6 **a)** From Ex. 3.26 X_i^2 is chi-square with one degree of freedom. From Prob. 5.5, S_n is then chi-square with n degrees of freedom

b)

$$
\begin{aligned}
T_n &= \sqrt{S_n} \\
\Rightarrow f_{T_n}(x) &= \frac{f_{S_n}(x^2)}{\frac{1}{2}|(x^2)^{-\frac{1}{2}}|} = 2x f_{X_n}(x^2)
\end{aligned}
$$

Now use fact that S_n is chi-square:

$$
= \frac{2x(x^2)^{\frac{n-2}{2}} e^{-x^2/2}}{2^{n/2}\Gamma(n/2)} = \frac{x^{n-1} e^{-x^2/2}}{2^{n/2-1}\Gamma\left(\frac{n}{2}\right)} \qquad x > 0
$$

c) $f_{T_2}(x) = xe^{-x^2/2} \quad x > 0$

d) $f_{T_3}(x) = \dfrac{x^2 e^{-x^2/2}}{2^{1/2}\Gamma\left(\frac{3}{2}\right)} = \dfrac{x^2 e^{-x^2/2}}{\sqrt{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \sqrt{\dfrac{2}{\pi}}x^2 e^{-x^2/2} \quad x > 0$

5.10

$$\begin{aligned} G_{S_k}(z) &= \mathcal{E}[z^{X_1+\ldots+X_k}] = \mathcal{E}[z^{X_1}]\ldots\mathcal{E}[z^{X_k}] = G_{X_1}(z)\ldots G_{X_k}(z) \\ &= [pz+q]^{n_1}[pz+q]^{n_2}\ldots[pz+q]^{n_k} \\ &= [pz+q]^{n_1+\ldots+n_k} \end{aligned}$$

where the second equality follows from the independence of the X_i's. The result states that S_k is Binomial with parameters $n_1 + \ldots + n_k$ and p. This is obvious since S_k is the number of heads in $n_1 + \ldots + n_k$ tosses.

5.21 a)

$$\begin{aligned} LHS &= \sum_{j=1}^{n}(X_j^2 - 2\mu X_j + \mu^2) = \sum_{j=1}^{n} X_j^2 - 2\mu(nM_n) + n\mu^2 \\ RHS &= \sum_{j=1}^{n}(X_j^2 - 2M_nX_j + M_n^2) + n(M_n-\mu)^2 \\ &= \sum_{j=1}^{n} X_j^2 - 2M_n(nM_n) + nM_n^2 + nM_n^2 - 2n\mu M_n + n\mu^2 \\ &= \sum_{j=1}^{n} X_j^2 - 2n\mu M_n + n\mu^2 = LHS \quad \surd \end{aligned}$$

b)

$$\begin{aligned} \mathcal{E}\left[k\sum_{j=1}^{n}(X_j - M_n)^2\right] &= k\mathcal{E}\left[\underbrace{\sum_{j=1}^{n}(X_j-\mu)^2 - n(M_n-\mu)^2}_{\text{from part a}}\right] \\ &= k\sum_{j=1}^{n}\mathcal{E}[(X_j-\mu)^2] - kn\mathcal{E}[(M_n-\mu)^2] \\ &= kn\sigma^2 - kn\frac{\sigma^2}{n} \\ &= k(n-1)\sigma^2 \quad \text{since } VAR[M_n] = \tfrac{\sigma^2}{n}. \end{aligned}$$

c) If $k = \frac{1}{n-1}$ then $\mathcal{E}[V_n^2] = \sigma^2$

d) if $k = \frac{1}{n}$ then

$$\mathcal{E}\left[\frac{1}{n}\sum_{j=1}^{n}(X_j - M_n)^2\right] = \left(1 - \frac{1}{n}\right)\sigma^2 = \sigma^2 - \underbrace{\frac{1}{n}\sigma^2}_{\text{bias}}$$

5.26

$$\begin{aligned} \mathcal{E}[S_n] &= n\mathcal{E}[X_i] = n \cdot 1 = n \\ VAR[S_n] &= n\sigma_{x_i}^2 = n \cdot 1^2 = n \end{aligned}$$

Assuming S_n approximately Gaussian:

$$P[S_n > 15] = P\left[\frac{S_n - n}{\sqrt{n}} > \frac{15-n}{\sqrt{n}}\right] \approx Q\left(\frac{15-n}{\sqrt{n}}\right) = 0.99$$

From Table 3.4

$$\frac{15-n}{\sqrt{n}} = -2.3263$$

$\Rightarrow n - 2.3263\sqrt{n} - 15 = 0 \Rightarrow n = 27.04$

$\Rightarrow$ by 28 pens