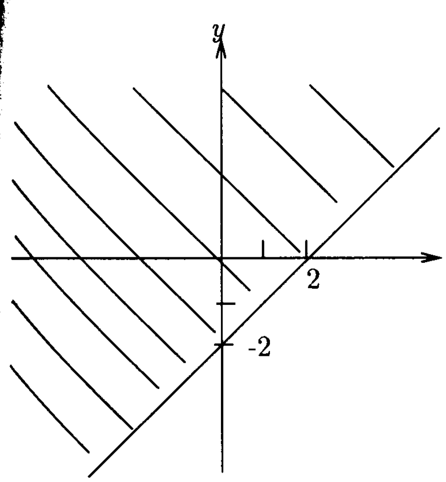
EEL5542 – Fall 2001
HOMEWORK #3

Solution of Problems

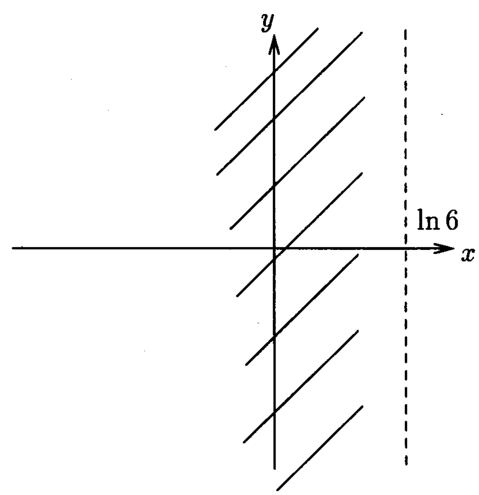
1.1

a) $\{Y \geq X - 2\}$



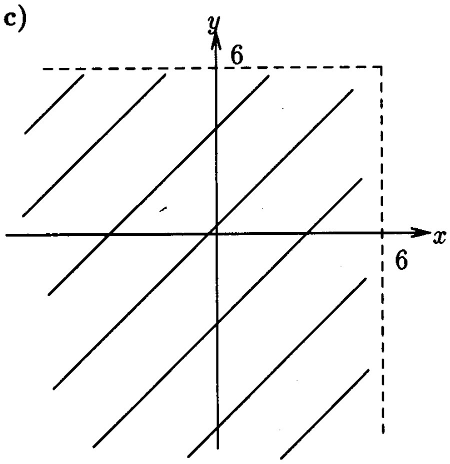
Not product-form

b) $\{X < \ln b\}$



product-form

c)



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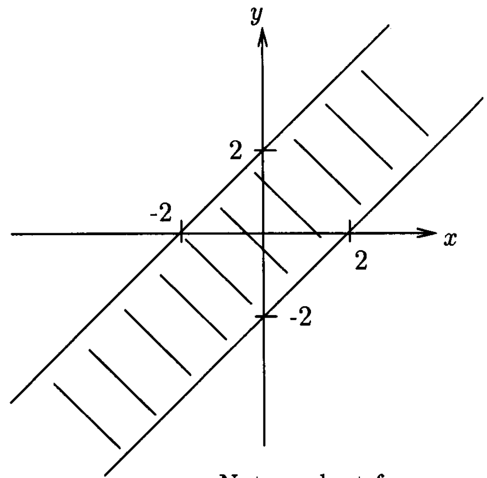
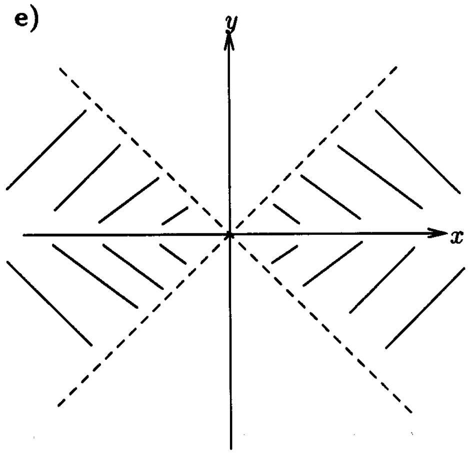
d)



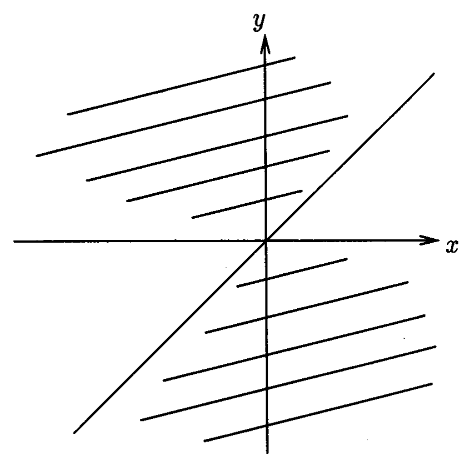
Not product-form

e)



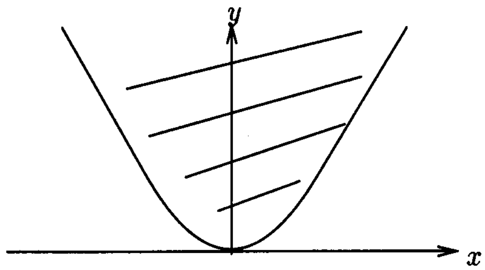
Not product-form

f)



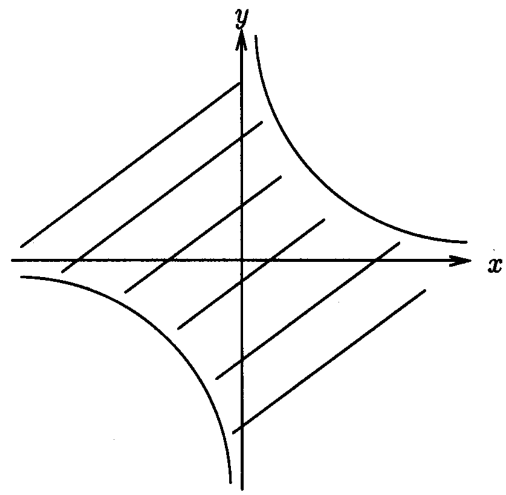
Not product-form

g)

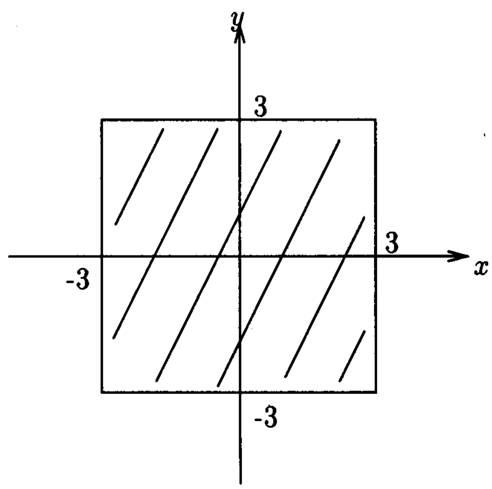


Not product-form

h)



Not product-form



product-form

4.17 a) $P[X=i, Y \leq y] = P[Y \leq y | X=i] P[X=i]$

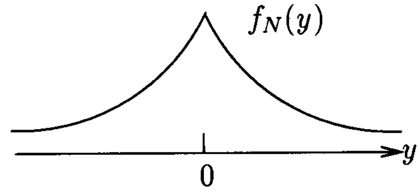
If $X = 1$:

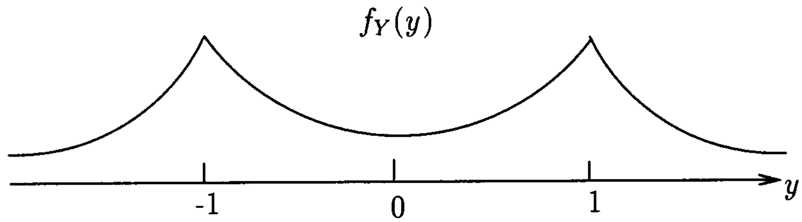
$$\begin{aligned} P[Y \leq y|X=1]P[X=1] &= P[N+1 \leq y]\frac{1}{2} = \frac{1}{2}\int_{-\infty}^{y-1} \frac{\alpha}{2} e^{-\alpha|z|} dz \\ &= \begin{cases} \frac{1}{4}e^{\alpha(y-1)} & y < 1 \\ \frac{1}{4}(2 - e^{-\alpha(y-1)}) & y > 1 \end{cases} \end{aligned}$$

$X = -1$ is obtained in similar fashion:

$$P[Y \leq y|X=1]P[X=-1] = \begin{cases} \frac{1}{4}e^{\alpha(y+1)} & y < -1 \\ \frac{1}{4}(2 - e^{-\alpha(y+1)}) & y \geq -1 \end{cases}$$

b)
$$\begin{aligned} F_Y(y) &= P[Y \leq y|X=]P[X=1] + P[Y \leq y|X=-1]P[X=-1] \\ &= P[N+1 \leq y]\frac{1}{2} + P[N-1 \leq y]\frac{1}{2} \\ f_Y(y) &= \frac{d}{dy}F_Y(y) = \frac{1}{2}\frac{d}{dy}F_N(y-1) + \frac{1}{2}\frac{d}{dy}F_N(y+1) \\ &= \frac{1}{2}f_N(y-1) + \frac{1}{2}f_N(y+1) \end{aligned}$$





c) $P[X=1|Y>0] = \dfrac{P[Y>0|X=1]P[X=1]}{P[Y>0]} = \dfrac{1-\frac{1}{4}e^{-\alpha}}{2P[Y>0]}$

$P[X=-1|Y>0] = \dfrac{P[Y>0|X=-1]P[X=-1]}{P[Y>0]} = \dfrac{\frac{1}{2}(1+\frac{1}{2}e^{-\alpha})}{2P[Y>0]}$

$P[X=1|Y>0] - P[X=-1|Y>0] = \dfrac{\frac{1}{2}(1-e^{-\alpha})}{2P[Y>0]} > 0$

$\Rightarrow X = 1$ is more likely

4.29 a)

X \ Y		-1	0	1
		$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
-1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$
0	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{6}$	$\frac{1}{16}$

X^2 \ Y^2		-1	0
		$\frac{1}{2}$	$\frac{1}{2}$
0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

b)

X \ Y		-1	0	1
		$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
-1	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
0	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
1	$\frac{1}{2}$	0	$\frac{1}{8}$	0

X^2 \ Y^2		0	1
		$\frac{1}{2}$	$\frac{1}{2}$
0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

4.46 For $k_j \geq 0$ such that $k_1 + k_2 + k_3 \leq$n

$$p(k_1, k_2, k_3) = \frac{1}{\binom{n+3}{3}}$$

Note: $\binom{n+3}{3}$ is the number of ways of distributing n identical balls in 4 cells: See Sampling with Replacement and Without Ordering in Section 2.3.

a) $p(k_1, k_2) = \sum_{k_3=0}^{n-k_1-k_2} p(k_1, k_2, k_3) = \dfrac{n - k_1 - k_2 + 1}{\binom{n+3}{3}}$

b) $p(k_1) = \sum_{k_2=0}^{n-k_1} \dfrac{n - k_1 - k_2 + 1}{\binom{n+3}{3}} \qquad j = n - k_1 - k_2 + 1$

$$= \sum_{j=1}^{n-k_1+1} \frac{j}{\binom{n+3}{3}} = \frac{(n-k_1+2)(n-k_1+1)}{2\binom{n+3}{3}}$$

Check

$$\sum_{k_1=0}^{n} p(k_1) = \frac{1}{2\binom{n+3}{3}} \sum_{k_1=0}^{n} (n-k_1+2)(n-k_1+1) \qquad j = n-k_1+1$$

$$= \frac{1}{2\binom{n+3}{3}} \sum_{j=1}^{n+1} j(j+1) \qquad \sum_{j=1}^{n+1} j = \frac{(n+2)(n+1)}{2} \qquad \sum_{j=1}^{n+1} j^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{1}{2\binom{n+3}{3}} \underbrace{\left[\frac{(n+2)(n+1)}{2} + \frac{(n+1)(n+2)(2n+3)}{6}\right]}$$

$$= \frac{1}{2\binom{n+3}{3}} \frac{(n+1)(n+2)(n+3)}{3} = 1 \quad \surd$$

c) $$p(k_2, k_3 | k_1) = \frac{p(k_1, k_2, k_3)}{p(k_1)} = \frac{\left[\frac{(n-k_1+2)(n-k_1+1)}{2\binom{n+3}{3}}\right]^{-1}}{\binom{n+3}{3}}$$

$$= \left[\frac{(n-K_1 \text{[redacted]} + 2)(n-k_1+1)}{2}\right]^{-1}$$

.58 Use spherical coordinates:

$$X = R\cos\Theta\sin\phi \qquad Y = R\sin\Theta\sin\phi \qquad Z = R\cos\phi$$

$$J(r,\theta,\phi) = \begin{vmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -r\sin\theta\sin\phi & r\cos\theta\sin\phi & 0 \\ r\cos\theta\cos\phi & r\sin\theta\cos\phi & -r\sin\phi \end{vmatrix} = |-r^2\sin\phi|$$

$$\begin{aligned} f_{R,\Theta,\phi}(r,\theta,\phi) &= f_{\underline{X}}(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)r^2\sin\phi \\ &= \frac{e^{-r^2/2}}{\sqrt{2\pi}^3} r^2 \sin\phi \end{aligned}$$

$$
\begin{aligned}
f_R(r) &= \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \frac{e^{-r^2/2}}{\sqrt{2\pi}^3} r^2 \sin\phi \\
&= \int_0^{\pi} \frac{r^2 e^{-r^2/2}}{\sqrt{2\pi}} \sin\phi d\phi \\
&= \sqrt{\frac{2}{\pi}} r^2 e^{-r^2/2} \qquad r > 0
\end{aligned}
$$