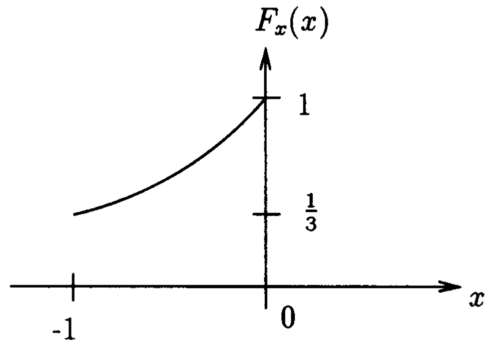
EEL5542 – Fall 2001
HOMEWORK #2

Solution of Problems

3.13



$$P[\{X > \frac{1}{3}\}] = 0,\ \ P[\{|X| \geq 1\}] = P[X = -1] = \frac{1}{3}$$

$$|X - \frac{1}{3}| < 1 \Leftrightarrow -\frac{2}{3} < X < \frac{4}{3}$$

$$P[\{|X - \frac{1}{3}| < 1\}] = F_X\left(\frac{4}{3}\right) - F_X\left(\frac{-2}{3}\right) = 1 - \frac{2}{3} \cdot \left(\frac{1}{3}\right)^2 = \frac{16}{27}$$

$$P[\{X < 0\}] = F_X(0) = 1$$

3.14 a) X is a random variable of mixed type since it is continuous except for discontinuity at 0 and at 1.

b)

$$
\begin{aligned}
P\left[X < -\frac{1}{2}\right] &= F_X\left(-\frac{1^-}{2}\right) = 0 \\
P[X < 0] &= F_X(0^-) = 0 \quad \text{The point } x = 0 \text{ is excluded from the problem} \\
P[X \le 0] &= F_X(0) = \frac{1}{4} \quad \text{The point } x = 0 \text{ is included in the problem} \\
P\left[\frac{1}{4} \le X < 1\right] &= F_X(1^-) - F_X\left(\frac{1^-}{4}\right) \quad \text{Since } x = 1 \text{ is excluded} \\
&= \frac{1}{2} - \left(\frac{1}{4} + \frac{1}{4}\left(\frac{1}{4}\right)\right) = \frac{3}{16} \quad x = \tfrac{1}{4} \text{ is included} \\
P\left[\frac{1}{4} \le X \le 1\right] &= F_X(1) - F_X\left(\frac{1^-}{4}\right) = 1 - \frac{5}{16} = \frac{11}{16} \\
P\left[X > \frac{1}{2}\right] &= 1 - P\left[X \le \frac{1}{2}\right] = 1 - F_X\left(\frac{1}{2}\right) = 1 - \left(\frac{1}{4} + \frac{1}{4}\frac{1}{2}\right) = \frac{5}{8} \\
P[X \ge 5] &= 1 - P[X < 5] = 1 - F_X(5^-) = 0 \\
P[X < 5] &= F_X(5^-) = 1
\end{aligned}
$$

3.21 a) $f_X(x) = \begin{cases} 0 & |x| > a \\ c\left(1 - \frac{|x|}{a}\right) & |x| \le a \end{cases}$

$$\begin{aligned} 1 &= \int_{-a}^{a} f_X(x)dx = \text{ Area of Triangle } = \frac{c(2a)}{2} = ac \\ &\Rightarrow \ c = \frac{1}{a} \end{aligned}$$

b) $F_X(x) = 0$ for $x < -a$; $F_X(x) = 1$ for $x > a$
For $-a \le x \le 0$

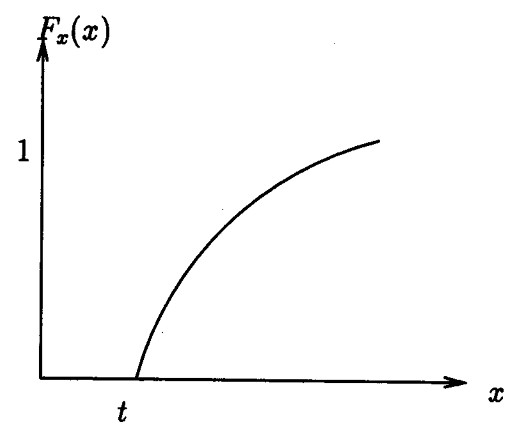
$$F_X(x) = \frac{1}{a}\int_{-a}^{x}\left(1 + \frac{x'}{a}\right)dx' = \frac{1}{2} + \frac{1}{a}\left(x + \frac{x^2}{2a}\right)$$

For $0 \le x \le a$

$$\begin{aligned} F_X(x) &= \int_{-a}^{0} f_X(x')dx' + \int_{0}^{x} f_X(x')dx' = \frac{1}{2} + \int_{0}^{x}\frac{1}{a}\left(1 - \frac{x'}{a}\right)dx' \\ &= \frac{1}{2} + \frac{1}{a}\left(x - \frac{x^2}{2a}\right) \end{aligned}$$

c) $P[|X| < b] = \frac{1}{2} = F_X(b) - F_X(-b) = \frac{2}{a}(b - \frac{b^2}{2a}) \Rightarrow b = a(1 - \frac{1}{\sqrt{2}})$.

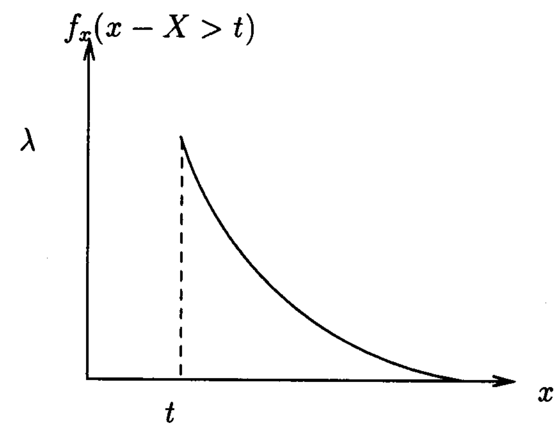
3.27 a)



$$
\begin{aligned}
F_X(x|X > t) &= \frac{P[\{X \le x\} \cap \{X > t\}]}{P[X > t]} \\
&= \begin{cases} 0 & x < t \\ \frac{F_X(x) - F_X(t)}{1 - F_X(t)} & x \ge t \end{cases} \\
&= \begin{cases} 0 & x < t \\ \frac{(1 - e^{-\lambda x}) - (1 - e^{-\lambda t})}{1 - (1 - e^{-\lambda t})} & x \ge t \end{cases} \\
&= \begin{cases} 0 & x < t \\ \frac{e^{-\lambda t} - e^{-\lambda x}}{e^{-\lambda t}} & x \ge t \end{cases}
\end{aligned}
$$

$F_x(x|x > t)$ is delayed version of $F_x(x)$

b) $f_x(x|x > t) = \dfrac{f_x(x)}{1 - F_x(t)} = \dfrac{\lambda e^{-\lambda x}}{e^{-\lambda t}} = \lambda e^{-\lambda(x-t)}, \quad x \ge t$



c)

$$
\begin{aligned}
P &= [X > t + x|X > t] \qquad x \ge 0 \\
&= \frac{P[\{X > t + x\} \cap \{X > t\}]}{P[X > t]} \\
&= \frac{1 - F_X(t + x)}{1 - F_X(t)} \\
&= \frac{1 - (1 - e^{-\lambda(t+x)})}{1 - (1 - e^{-\lambda t})} \\
&= e^{-\lambda x} \\
&= P[X > x]
\end{aligned}
$$

The probability of waiting additional x seconds doesn't depend on the previous waiting time t . It is the same as when one begins to wait.

3.36 The memoryless property states that for $j, k \geq 1$.

$$\begin{aligned} P[M \geq k] &= P[M \geq k+j | M > j] \\ &= \frac{P[M \geq k+j]}{P[M > j]} = \frac{P[M \geq k+j]}{P[M \geq j+1]} \end{aligned}$$

$\Longrightarrow$

$$P[M \geq k+j] = P[M \geq k]P[M \geq j+1]$$

Let

$$a_k = P[M \geq k],$$

then we have

$$(*) \qquad a_{k+j} = a_k a_{j+1} \qquad j \geq 1, k \geq 1$$

where $a_1 = 1$ and $a_2 = 1 - P[M = 1] = 1 - p$.
Equation (*) with $j = 1$ becomes

$$a_{k+1} = a_2 a_k \qquad k \geq 1$$

$$\Rightarrow a_k = a_2^{k-1} \qquad k \geq 1$$

$$\Rightarrow P[M \geq k] = (1-p)^{k-1} \qquad k \geq 1$$

$$\begin{aligned} P[M = k] &= P[M \geq k] - P[M \geq k+1] \\ &= (1-p)^{k-1} - (1-p)^k \\ &= (1-p)^{k-1}(1-(1-p)) \\ &= (1-p)^{k-1} p \end{aligned}$$

3.45
$$\begin{aligned} P[\text{error}|v = -1] &= P[Y \geq 0 | v = -1] \\ &= P[-1 + N \geq 0] = P[N \geq 1] = Q(1) = 0.159 \end{aligned}$$

$$\begin{aligned} P[\text{error}|v = -1] &= P[Y < 0 | v = 1] = P[1 + N < 0] = P[N < -1] \\ &= 1 - Q(-1) = Q(1) = 0.159 \end{aligned}$$

3.51

$$P[Y=3.5d] = P[Y=-3.5d] = \int_{-\infty}^{-3d} \frac{ae^{ax}}{2}dx = \frac{1}{2}e^{-3ad}$$

$$P[Y=2.5d] = P[Y=-2.5d] = \int_{-3d}^{-2d} \frac{ae^{ax}}{2}dx = \frac{1}{2}\{e^{-2ad} - e^{-3ad}\}$$

$$P[Y=1.5d] = P[Y=-1.5d] = \int_{-2d}^{-d} \frac{ae^{ax}}{2}dx = \frac{1}{2}\{e^{-ad} - e^{-2ad}\}$$

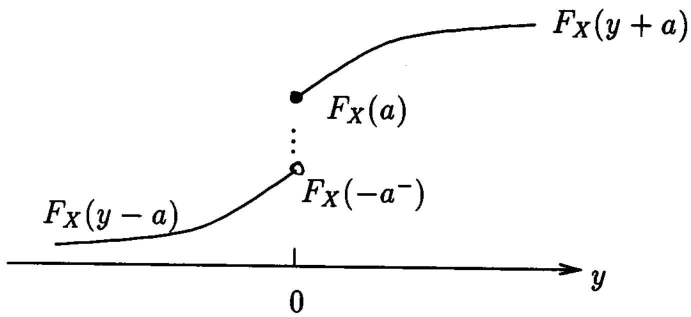
$$P[Y=.5d] = P[Y=-.5d] = \int_{-d}^{0} \frac{ae^{ax}}{2}dx = \frac{1}{2}\{1 - e^{-ad}\}$$

$$P[|Y|>4d] = 2\int_{4d}^{\infty} \frac{ae^{-ax}}{2}dx = e^{-4ad}$$

3.58 a) For $y<0$ $\quad P[Y \le y] = P[X + a \le y] = F_X(y-a)$

For $y=0$ $\quad P[Y \le 0] = P[X \le a] = F_X(a)$

For $y>0$ $\quad P[Y \le y] = P[X - a \le y] = F_X(y+a)$



$$f_Y(y) = \begin{cases} f_X(y-a) & y<0 \\ (F_X(a) - F_X(-a^-))\delta(y) & y=0 \\ f_X(y+a) & y>0 \end{cases}$$

b) If $f_X(x) = \frac{\beta}{2}e^{-\beta|x|}$ as in 3.46b, then

$$F_Y(y) = \begin{cases} \frac{1}{2}e^{\beta(y-a)} & y<0 \\ 1 - \frac{1}{2}e^{-\beta(y+a)} & x \ge 0 \end{cases} \qquad f_Y(y) = \begin{cases} \frac{\beta}{2}e^{-\beta(a-y)} & y<0 \\ (1-e^{-\beta a})\delta(y) & y=0 \\ \frac{\beta}{2}e^{-\beta(y+a)} & y>0 \end{cases}$$

3.59 a) For $y \leq 0$ $\quad P[Y \leq y] = 0$

For $y > 0$ $\quad P[Y \leq y] = P[e^X \leq y] = P[X \leq \ln y] = F_X(\ln y)$

$$\therefore \quad F_Y(y) = \begin{cases} 0 & y \leq 0 \\ F_X(\ln y) & y > 0 \end{cases}$$

For $y > 0$

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) = F'_X(\ln y) \frac{d}{dy} \ln y \\ &= \frac{1}{y} f_X(\ln y) \end{aligned}$$

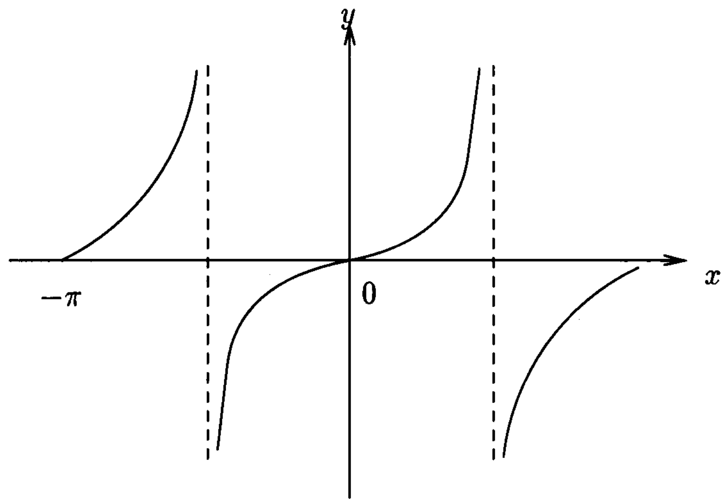
b) If X is a Gaussian random variable, then

$$f_Y(y) = \begin{cases} 0 & y \leq 0 \\ \dfrac{e^{-(\ln y - m)^2/2\sigma^2}}{y\sqrt{2\pi}\sigma} & y > 0 \end{cases}$$

3.62 $Y = a \tan X$.

$$x = \tan^{-1}(y/a), \; -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\frac{dx}{dy} = \frac{1}{1 + (y/a)^2}\frac{1}{a} = \frac{a}{y^2 + a^2}$$



$$\begin{aligned} fx(y) &= \sum_k fx(x)|\frac{dx}{dy}|\,|_{x=x_k} \\ &= 2 \cdot \frac{1}{2\pi}\frac{a}{y^2 + a^2} \\ &= \frac{a/\pi}{y^2 + a^2} \end{aligned}$$

Y is a Cauchy RV.