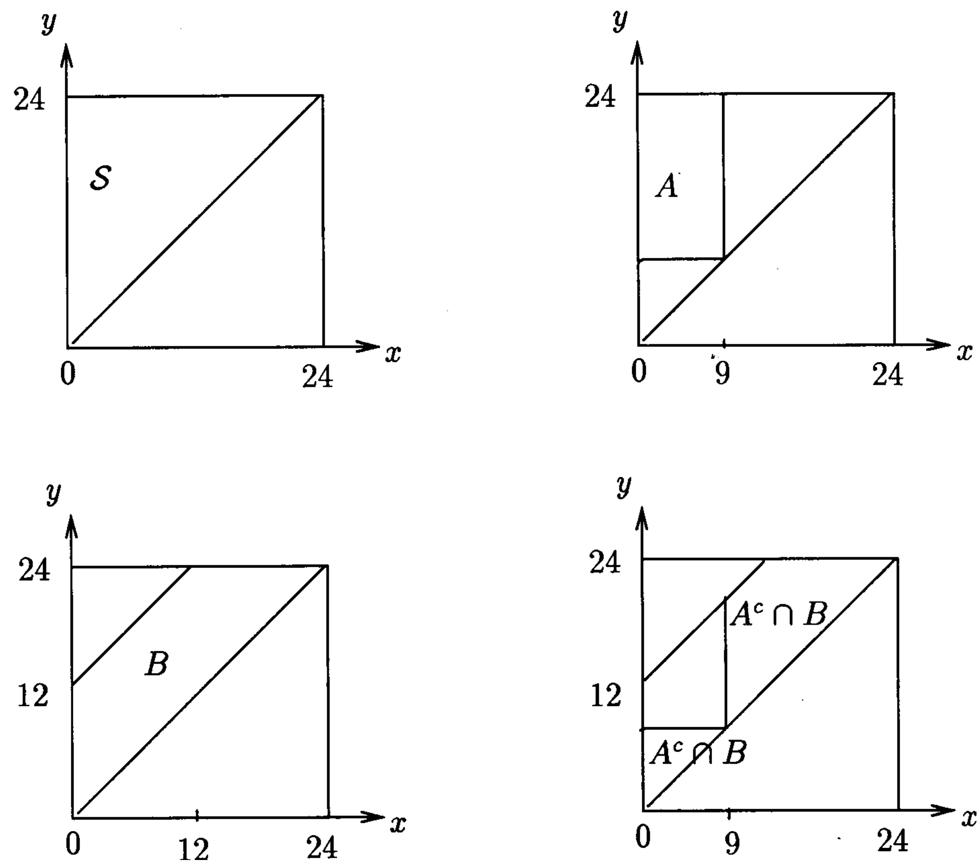
EEL5542 – Fall 2001
HOMEWORK #1

Solution of Problems

2.14 a) In Figure P2.1, there is a connection across the network if and only one switch on each stage is closed. If we let each stage represent a subsystem condition "system up" corresponds to "a connection exists across the network"

b)



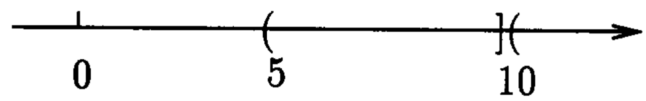
2.26 Let $A_i = \{i\text{th character is in error}\}$

$$P[\text{any error in document}] = P\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} P[A_i] = np$$

2.30 a) Since $(-\infty, r] \subset (-\infty, s]$ when $r < s$

$$P[(-\infty, r]] \leq P[(-\infty, s]] \text{ by Corollary 7.}$$

b)



$$\begin{aligned} P[(-\infty, s]] &= P[(-\infty, r] \cup (r, s]] \\ &= P[(-\infty, r]] + P[(r, s]] \\ \Rightarrow P[(r, s]] &= P[(-\infty, s]] - P[(-\infty, r]] \end{aligned}$$

2.52 The events B and C are shown below.

It then follows from the definition of conditional probability that:

$$\begin{aligned} P[B|C] &= \frac{P[B \cap C]}{P[C]} = \frac{P[C]}{P[C]} = 1 \\ P[C|B] &= \frac{P[B \cap C]}{P[B]} = \frac{P[C]}{P[B]} = \frac{\frac{1}{4}}{\frac{3}{2}} = \frac{1}{6} \end{aligned}$$

2.60 Let X denote the input and Y the output.

a)
$$\begin{aligned} P[Y=0] &= P[Y=0|X=0]P[X=0] + P[Y=0|X=1]P[X=0] \\ &\quad +P[Y=0|X=2]P[X=2] \\ &= (1-\varepsilon)\tfrac{1}{2}+)\cdot\tfrac{1}{4}+\varepsilon\cdot\tfrac{1}{4} \\ &= \tfrac{1}{2}-\tfrac{\varepsilon}{4} \end{aligned}$$

Similarly

$$\begin{aligned} P[Y=1] &= \varepsilon\cdot\frac{1}{2}+(1-\varepsilon)\frac{1}{4}+0\varepsilon\frac{1}{4} = \frac{1}{4}+\frac{\varepsilon}{4} \\ P[Y=2] &= 0\cdot\frac{1}{2}+\varepsilon\cdot\frac{1}{4}+(1-\varepsilon)\frac{1}{4} = \frac{1}{4} \end{aligned}$$

b) Using Bayes' Rule

$$\begin{aligned} P[X=0|Y=1] &= \frac{P[Y=1|X=0]P[X=0]}{P[Y=1]} = \frac{\frac{1}{2}\varepsilon}{\frac{1}{4}+\frac{\varepsilon}{4}} = \frac{2\varepsilon}{1+\varepsilon} \\ P[X=1|Y=1] &= \frac{P[Y=1|X=1]P[X=1]}{P[Y=1]} = \frac{(1-\varepsilon)\frac{1}{4}}{\frac{1}{4}+\frac{\varepsilon}{4}} = \frac{1-\varepsilon}{1+\varepsilon} \\ P[X=2|Y=1] &= 0 \end{aligned}$$

2.68 Events A and B are independent iff

$$P[A\cap B] = P[A]P[B]$$

In terms of relative frequencies we expect

$$\underbrace{f_{A\cap B}(n)} = f_A(n)f_B(n)$$

rel. freq. of joint occurrence of A and B — rel. freq.'s of A and B in isolation

2.70 From Example 2.26 $P[A_0|B_1] = P[A_1|B_1] = \frac{1}{2}$ when $\varepsilon = \frac{1}{2}$. Thus either input is equally likely after observing the output. Thus the channel conveys no useful information about the input to the channel.

2.71

$$\begin{aligned} P[\text{3 or more errors}] &= 1 - P[\text{2 or fewer errors}] \\ &= 1 - \sum_{k=0}^{2} \binom{100}{k} p^k (1-p)^{100-k} \qquad p = 10^{-3} \\ &= 1 - \left\{ (1-p)^{100} + 100(1-p)^{99} p + \frac{100.99}{2} (1-p)^{98} p^2 \right\} \\ &= 1 - 0.99985 = 1.5 \times 10^{-4} \end{aligned}$$

2.78 a)

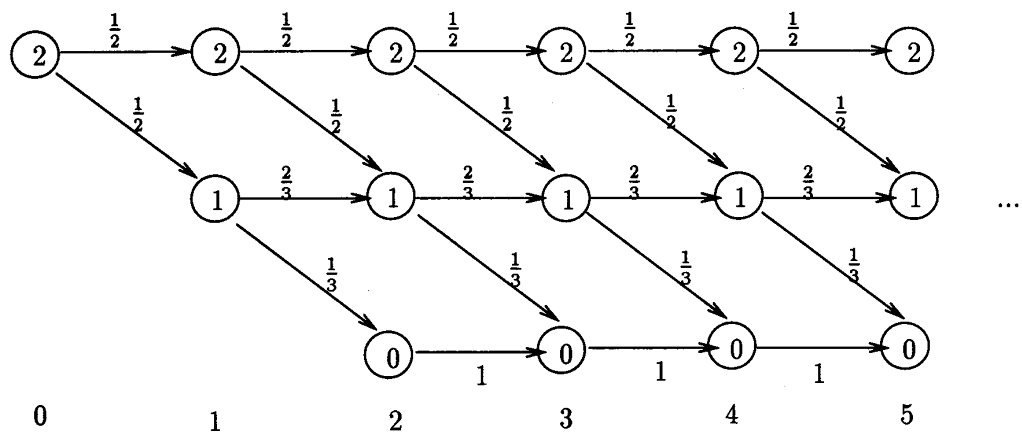
$$\begin{aligned} P[k=0] &= p \\ P[k=1] &= (1-p)p \\ P[k=2] &= (1-p)^2 p \\ P[k=3] &= 1 - P[k=0] - P[k=1] - P[k=2] = (1-p)^3 \end{aligned}$$

b)

$$\begin{aligned} P[k] &= (1-p)^k p \qquad 0 \le k < m \\ P[m] &= 1 - \sum_{k=0}^{m-1} P[k] \\ &= 1 - \sum_{k=0}^{m-1} (1-p)^k p \\ &= 1 - p \frac{1-(1-p)^m}{1-(1-p)} = (1-p)^m \end{aligned}$$

2.81 a) The number of black balls in the urn completely determines the probability of the outcomes of the next experiment;

Let s_n denote the number of black balls in the urn after n repetitions, the trellis diagram for s_n is then:



downward transitions occur when a black ball is selected
horizontal transitions occur when a white ball is selected

b)

$$
\begin{aligned}
P[\omega\omega\omega] &= \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8} \\
P[b\omega\omega] &= \frac{1}{2}\frac{2}{3}\frac{2}{3} = \frac{2}{9} \\
P[bb\omega] &= \frac{1}{2}\frac{1}{3}1 = \frac{1}{6} \\
P[bb\omega\omega\omega] &= \frac{1}{2}\frac{1}{3}1\ 1\ 1 = \frac{1}{6}
\end{aligned}
$$

c) There are 3 paths that lead to the zero states after 3 draws; The probability that there are not blacks balls in the urn after 3 draws is the sum of the three path probabilities:

$$P[bb\omega] + P[\omega bb] + P[b\omega b] = \frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{1}{2}\frac{1}{3} + \frac{1}{2}\frac{2}{3}\frac{1}{3} = \frac{13}{36}$$