Midterm Exam II

EEL 5542 – Fall 2002

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Name:________________________________

(please print your name)

Instructions

For each problem show your work in detail and circle your final answer. Also start each problem on a new sheet. This exam consists of 5 problems; each is worth 20 points. When you turn in your exam, please staple the 3 pages of this document to your work. Midterm Exam I is due by Wednesday, November 21st 2001 at 12:00pm. For students that take this course from remote sites: please fax or send your work via mail by the aforementioned deadline. For all other students: please turn in your work to the department office (Room 407, CEBA-I building) by the aforementioned deadline and have one of the administrative assistants date your work and place it into my mailbox. If you have any questions concerning this exam, don't hesitate to get in touch with me. Please refer to my contact information at

http://www.geocities.com/eel5542/contact_info.htm

Good Luck!

Please, read carefully, then date and sign the following statement:

Pertaining to this take-home exam I have received no help from any other than the instructor of my class.

Signature:____________________________ Date:______________

Problem 1

A WSS random process $X(t)$ features a mean m_x and an autocorrelation of $R_x(\tau)$.

a) Find an upper bound for the probability $\Pr\{|X(t)-m_x|\geq \varepsilon\}$, where $\varepsilon>0$.

b) Find an upper bound for the probability $\Pr\{|X(t+T)-X(t)|\geq \varepsilon\}$, where for $\varepsilon>0$.
In both cases the upper bound is a function of autocorrelation function values and ε.

Problem 2

Assume the $X(t)$ is a WSS, zero mean, differentiable in the mean square sense random process.

a) Show that $X(t)$ and $X'(t)=\frac{dX(t)}{dt}$ are uncorrelated.

b) Let the output of a system be

$$Y(t) = X(t) + \frac{dX(t)}{dt}$$

Find the power spectrum of $Y(t)$ and show that it is the sum of the power spectra of X(t) and $X'(t)$.

Problem 3

A WSS, zero mean, Gaussian random process $X(t)$ acts as an input to a system giving an output of $Y(t)=h(X(t))$, where

$$h(x) = \begin{cases} +1 & \text{when } x \geq 0 \\ -1 & \text{when } x < 0 \end{cases}$$

a) Show that $E\{Y(t)\}=0$.

b) Next, show that the autocorrelation function of $Y(t)$ is given as

$$R_Y(\tau) = \Pr\{X(t+\tau)X(t) > 0\} - \Pr\{X(t+\tau)X(t) < 0\}$$

Problem 4

Let X_1(t) and X_2(t) be two jointly WSS and jointly Gaussian random processes. Each process acts as an input to a different linear time invariant system. X_1(t) enters a system with impulse response function h_1(t) and transfer function $H_1(f)=F\{h_1(t)\}$ giving as an output a random process Y_1(t). Similarly, X_2 acts as an input for a system with impulse response function h_2(t) and transfer function $H_2(f)=F\{h_2(t)\}$ giving as an output a random process Y_2(t). Here, F{ } denotes the Fourier Transform operator.

a) Find the cross-correlation and the cross-power spectral density of Y_1(t) and Y_2(t).

b) Show that Y_1(t) and Y_2(t) are jointly WSS and jointly Gaussian random processes.

c) Assume that the transfer functions of the two systems have no overlapping spectra, that is, $|H_1(f)||H_2(f)|=0$ for every f. Show that Y_1(t) and Y_2(t) are independent random processes.

d) If X_1(t) and X_2(t) were non-stationary jointly Gaussian random processes, would the above results (b) and (c) still hold?

Problem 5

Consider a discrete time system described by the following difference equation

$$X_n = aX_{n-1} + W_n \qquad n \geq 0$$

where $|a|<1$ and W_n is white Gaussian noise with zero mean and variance of σ_w^2. Furthermore, the initial condition X_{-1} is normally distributed with mean m and variance σ_x^2 and is independent of W_n for all values of n.

a) Show that X_n is not stationary in any sense.

Hint: First solve the difference equation, which will give you X_n as a function of X_{-1} and lagged versions of W_n (that is, W_n, W_{n-1}, etc.).

b) Show that X_n becomes WSS as $n \rightarrow \infty$.

Hint: One of the things you have to show is that the autocorrelation $R_x(n,m)$ of X_n depends only on the difference $k=m-n$ as $n \rightarrow \infty$. Or you can show the same thing for the autocovariance $C_x(n,m)$ of X_n, which is slightly easier (it involves less calculations). In any case though, be aware of the fact that by definition $R_x(n,m)= R_x(m,n)$ as well as $C_x(n,m)= C_x(m,n)$ for all n and m.

c) If $a \geq 1$, is there any possibility that X_n might be stationary, when $n \rightarrow \infty$? Why or why not?

d) Show that X_n becomes an AR(1) (first-order autoregressive) random process for $n \to \infty$.

Hint: If Y_n is an AR(1) process, then it is described by $Y_n = \boldsymbol{b} Y_{n-1} + V_n$, where $|\boldsymbol{b}|<1$ and V_n is a white Gaussian process with zero mean and variance σ^2. Most importantly, it satisfies E{Y_n}=0 and

$$R_Y(k) = \boldsymbol{s}^2 \frac{\boldsymbol{b}^{|k|}}{1-\boldsymbol{b}^2}$$

e) Find the pdf of X_n as $n \to \infty$ (determine type of pdf as well as what its parameters' values are).

Note: Baffled by the fact that a linear time-invariant (LTI) system with WSS input might give you a non-stationary output? The above equation describes a first-order *Gauss-Markov*, which resembles a first-order autoregressive (AR(1)) model except that the output X_n starts at $n=0$ with some initial condition X_{-1} and therefore X_n may not be WSS.

Problem 1

Tchebycheff Inequality: $Pr\{|X - E\{X\}| \geq \varepsilon\} \leq \frac{Var(X)}{\varepsilon^2} \quad \forall \varepsilon > 0$ (TI)

(a) We are given that $E\{X(t)\} = \mu_x$ and $Var(X(t)) = R_x(0) - \mu_x^2$

By applying T.I. we readily get

$$Pr\{|X(t) - \mu_x| \geq \varepsilon\} \leq \frac{R_x(0) - \mu_x^2}{\varepsilon^2} \quad \forall \varepsilon > 0$$

■

(b) Let us define the R.P. $Y(t) \triangleq X(t+T) - X(t)$ ①

$$\mu_Y = E\{Y(t)\} = E\{X(t+T)\} - E\{X(t)\} = \mu_x - \mu_x = 0 \quad ②$$

$$Var(Y(t)) = E\{Y(t)^2\} \overset{①}{=} E\{[X(t+T) - X(t)]^2\} =$$

$$= E\{X^2(t+T)\} - 2E\{X(t)X(t+T)\} + E\{X^2(t)\} \quad ③$$

Since $X(t)$ is WSS ④ $\Rightarrow Var(Y(t)) = R_x(0) - 2R_x(T) + R_x(0) \Rightarrow$

$$\Rightarrow Var(Y(t)) = 2[R_x(0) - R_x(T)] \quad ⑤$$

Applying the T.I. for $Y(t)$ we get

$$Pr\{|Y(t) - E\{Y(t)\}| \geq \varepsilon\} \leq \frac{Var(Y(t))}{\varepsilon^2} \underset{② ⑤}{\overset{①}{\Longrightarrow}}$$

$$\Rightarrow Pr\{|X(t+T) - X(t)| \geq \varepsilon\} \leq \frac{2[R_x(0) - R_x(T)]}{\varepsilon^2} \quad \forall \varepsilon > 0$$

■

Problem 2

(a) We need to show $E\{X(t)X'(t)\} = E\{X(t)\}E\{X'(t)\}$

But $E\{X'(t)\} = \mu_{x'} = \frac{d}{dt}E\{X(t)\}$

$X(t)$ is WSS $\Rightarrow E\{X(t)\} = \mu_x = \text{constant}$

$\Rightarrow E\{X'(t)\} = 0 \Rightarrow$

$\Rightarrow$ Therefore it suffices to show that $E\{X(t)X'(t)\} = 0$ (E)

By definition $R_{xx'}(0) = E\{X(t)X'(t)\}$ (1)

We also know that $R_{xx'}(\tau) = -\frac{dR_x(\tau)}{d\tau}$ (2)

One of the properties of $R_x(\tau)$ is that $R_x(0) \geq R_x(\tau)\ \forall \tau$, which means that $R_x(\tau)$ has a maximum at $\tau = 0$.

Since $R_x(\tau)$ is a continuous function of τ, a necessary condition for $R_x(\tau)$ to have a maximum at $\tau = 0$ is

$$\left.\frac{dR_x(\tau)}{d\tau}\right|_{\tau=0} = 0 \quad (3)$$

(1), (2), (3) $\Rightarrow E\{X(t)X'(t)\} = 0 \overset{(E)}{\Rightarrow} X(t)$ and $X'(t)$ are uncorrelated.

(b) $Y(t)$ is the output of a Linear Time Invariant (LTI) system in steady state (the effect of initial conditions, if any, has already diminished) due to a WSS input $X(t)$. This implies that $Y(t)$ is WSS. Following a similar rational, $X'(t)$ is also WSS and $X(t)$, $X'(t)$ are jointly WSS.

Therefore,

$$E\{Y(t)Y(t+\tau)\} = R_Y(\tau) \quad ④$$
$$E\{X(t)X'(t+\tau)\} = R_{XX'}(\tau) \quad ⑤$$
$$E\{X'(t)X'(t+\tau)\} = R_{X'}(\tau) \quad ⑥$$

← depend only on τ

We first start calculating

$$\left.\begin{array}{l} R_Y(\tau) = E\{Y(t)Y(t+\tau)\} \\ Y(t) = X(t) + X'(t) \end{array}\right\} \Rightarrow R_Y(\tau) = E\{X(t)X(t+\tau)\} + E\{X(t)X'(t+\tau)\} +$$

$$+ E\{X'(t)X(t+\tau)\} + E\{X'(t)X'(t+\tau)\} \Rightarrow$$

$$\overset{④⑤⑥}{\Longrightarrow} R_Y(\tau) = R_X(\tau) + R_{XX'}(\tau) + R_{X'X}(\tau) + R_{X'}(\tau) \quad ⑦$$

However, we can easily show that

$$\left.\begin{array}{l} R_{YX}(\tau) = R_{XY}(-\tau) \\ Y = X' \end{array}\right\} \Rightarrow R_{X'X}(\tau) = R_{XX'}(-\tau) \quad ⑧$$

Also, we know that $R_{XX'}(\tau) = -\dfrac{dR_X(\tau)}{d\tau}$

But $R_X(\tau) = R_X(-\tau)$ (even function), therefore its derivative $\dfrac{dR_X(\tau)}{d\tau}$ must be an odd function. Thus, $R_{XX'}(-\tau) = -R_{XX'}(\tau)$ ⑨

$$⑦ \overset{⑧}{\underset{⑨}{\Longrightarrow}} R_Y(\tau) = R_X(\tau) + R_{X'}(\tau) \quad \Rightarrow$$

$$\Rightarrow \quad S_Y(f) = \mathcal{F}\{R_Y(\tau)\} = S_X(f) + S_{X'}(f) \quad \blacksquare$$

Problem 3

(a) Due to $h(x)$, $Y(t)$ can take only two values: ± 1

Therefore
$$E\{Y(t)\} = 1 \cdot Pr\{Y(t)=1\} + (-1)\, Pr\{Y(t)=-1\} \quad ①$$

However
$$Pr\{Y(t)=1\} = Pr\{X(t) \geq 0\} = 1 - F_{X(t)}(0) \quad ②$$
$$Pr\{Y(t)=-1\} = Pr\{X(t) < 0\} = F_{X(t)}(0^-) \quad ③$$

where $F_{X(t)}(x)$ is the CDF of $X(t)$

$$①,②,③ \Rightarrow E\{Y(t)\} = 1 - F_{X(t)}(0) - F_{X(t)}(0^-)$$

But $X(t)$ is a Gaussian RP $\Rightarrow F_{X(t)}(0) = F_{X(t)}(0^-) = \frac{1}{2}$ $\Bigg\} \Rightarrow$

$$\Rightarrow E\{Y\} = 0 \quad \blacksquare$$

(b) By definition $R_Y(\tau) = E\{Y(t)\,Y(t+\tau)\}$ ④

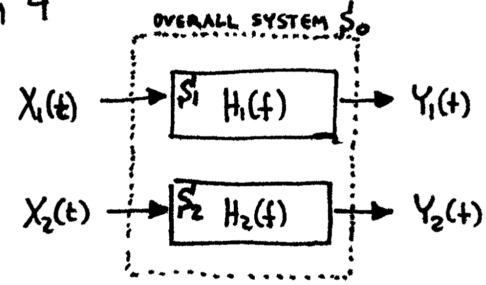
Again, due to $h(\cdot)$, $Y(t)Y(t+\tau)$ can only take values ± 1

Therefore
$$④ \Rightarrow E\{Y(t)\,Y(t+\tau)\} = 1 \cdot Pr\{Y(t)Y(t+\tau) > 0\} + (-1) \cdot$$
$$\cdot Pr\{Y(t)Y(t+\tau) < 0\} \quad ⑤$$

However
$$Pr\{Y(t)\,Y(t+\tau) < 0\} = Pr\{X(t)X(t+\tau) < 0\} \quad ⑥$$
$$Pr\{Y(t)\,Y(t+\tau) > 0\} = Pr\{X(t)X(t+\tau) > 0\} \quad ⑦$$

$$④,⑤,⑥,⑦ \Rightarrow R_Y(\tau) = Pr\{X(t)X(t+\tau) > 0\} - Pr\{X(t)X(t+\tau) < 0\} \quad \blacksquare$$

Problem 4



(α) $Y_1(t)$ and $Y_2(t)$ can be thought to be the output of an LTI system S_o in steady state (there's no effect due to initial conditions, if any) due to the inputs $X_1(t)$ and $X_2(t)$.

Therefore, since $X_1(t), X_2(t)$ jointly WSS, $Y_1(t), Y_2(t)$ are also jointly WSS. Thus,

$$R_{Y_1Y_2}(t, t+\tau) = R_{Y_1Y_2}(\tau) \quad \leftarrow \text{depends only on } \tau, \text{ not on } t.$$

$$\left.\begin{array}{l} R_{Y_1Y_2}(\tau) = E\{Y_1(t)Y_2(t+\tau)\} \\ Y_1(t) = h_1(t) * X_1(t) = \int_{-\infty}^{+\infty} h_1(u) X_1(t-u)\,du \\ Y_2(t) = h_2(t) * X_2(t) = \int_{-\infty}^{+\infty} h_2(v) X_2(t+\tau-v)\,dv \end{array}\right\} \Rightarrow$$

$$\Rightarrow R_{Y_1Y_2}(\tau) = \iint_{-\infty}^{+\infty} h_1(u) h_2(v)\, E\{X_1(t-u) X_2(t+\tau-v)\}\,du\,dv =$$

$$= \iint_{-\infty}^{+\infty} h_1(u) h_2(v)\, R_{X_1X_2}(\tau-v+u)\,du\,dv \Rightarrow$$

$$\Rightarrow R_{Y_1Y_2}(\tau) = h_1(\tau) * h_2(-\tau) * R_{X_1X_2}(\tau-v+u) \quad (1)$$

■

$$S_{Y_1Y_2}(f) = \mathcal{F}\{R_{Y_1Y_2}(\tau)\} \overset{①}{\Longrightarrow} S_{Y_1Y_2}(f) = H_1(f)H_2^*(f)\, S_{X_1X_2}(f) \quad ②$$

■

(b) $X_1(t)$ and $X_2(t)$ are jointly Gaussian R.P.'s linearly transformed by system S_0 in steady state, therefore $Y_1(t)$, $Y_2(t)$ are also jointly Gaussian R.P.'s. $Y_1(t)$ and $Y_2(t)$ are also jointly WSS (see part a)

(c) From ② we get $|S_{Y_1Y_2}(f)| = |H_1(f)|\,|H_2^*(f)|\,|S_{X_1X_2}(f)|$

$|H_2^*(f)| = |H_2(f)|$

By the hypothesis $|H_1(f)|\,|H_2(f)| = 0 \ \forall f$ $\Rightarrow$

$\Rightarrow |S_{Y_1Y_2}(f)| = 0 \ \forall f \Rightarrow R_{Y_1Y_2}(\tau) = 0 \ \forall \tau$

$C_{Y_1Y_2}(\tau) = R_{Y_1Y_2}(\tau) - E\{Y_1(t)\}E\{Y_2(t)\}$ $\Rightarrow$

$\Rightarrow C_{Y_1Y_2}(\tau) = -E\{Y_1(t)\}\,E\{Y_2(t)\}$

But we know that $E\{Y_1(t)\} = H_1(0)\,E\{X_1(t)\}$

$E\{Y_2(t)\} = H_2(0)\,E\{X_2(t)\}$

$\Rightarrow C_{Y_1Y_2}(\tau) = -H_1(0)H_2(0)\,E\{X_1(t)\}E\{X_2(t)\}$ ③

By the hypothesis $|H_1(f)|\,|H_2(f)| = 0 \ \forall f$, therefore $C_{Y_1Y_2}(\tau) = 0 \Rightarrow$

$\Rightarrow Y_1(t), Y_2(t)$ uncorrelated

Part (b) $\Rightarrow Y_1(t), Y_2(t)$ jointly Gaussian

$\Rightarrow Y_1(t), Y_2(t)$ Independent ■

(d) The only result that will hold is that $Y_1(t), Y_2(t)$ are still jointly Gaussian R.P.'s. ■

Problem 5

(a) We will solve the difference equation:

$$\left.\begin{array}{ll} n=0 & X_0 = \alpha X_{-1} + W_0 \\ n=1 & X_1 = \alpha X_0 + W_1 \\ n=2 & X_2 = \alpha X_1 + W_2 \\ & \vdots \\ & \text{etc.} \end{array}\right\} \Rightarrow X_n = \alpha^{n+1} X_{-1} + \sum_{i=0}^{n} \alpha^i W_{n-i} \quad (1)$$

$$(1) \Rightarrow E\{X_n\} = \alpha^{n+1} E\{X_{-1}\} + \sum_{i=0}^{n} \alpha^i E\{W_{n-i}\}$$

W_n is white Gaussian noise $\Rightarrow$ W_n is WSS

$E\{W_n\} = 0$

From the hypothesis $E\{X_{-1}\} = \mu$ $\Rightarrow$

$$\Rightarrow E\{X_n\} = \alpha^{n+1} \mu \quad (2)$$

We observe that $E\{X_n\}$ depends on n (we assume $\mu \neq 0$), therefore X_n is neither WSS or SSS R.P. ∎

(b) From ② we have:

$$② \Rightarrow \left.\begin{array}{l} \lim_{n\to+\infty} E\{X_n\} = \mu \lim_{n\to+\infty} \alpha^{n+1} \\ |\alpha|<1 \Rightarrow \lim_{n\to+\infty} \alpha^n = 0 \end{array}\right\} \Rightarrow \lim_{n\to+\infty} E\{X_n\} = 0 \quad ③$$

Therefore, X_n is mean-stationary in the limit $n \to +\infty$.

In order to show that X_n becomes WSS when $n \to +\infty$, we also need to show that $R_X(n,m) = R_X(m-n)$ or alternatively (due to ③) that $C_X(n,m) = C_X(m-n)$ in the limit $n \to +\infty$.

$$C_X(n,m) = E\{[X_n - E\{X_n\}][X_m - E\{X_m\}]\} =$$

$$= \alpha^{m+n+2}\sigma_X^2 + \sum_{i=0}^{n}\sum_{j=0}^{m} \alpha^{i+j} E\{W_{n-i}W_{m-j}\}$$

Since W_n is white noise $E\{W_{n-i}W_{m-j}\} = \begin{cases} \sigma_W^2 & \text{when } i=j+n-m \\ 0 & \text{otherwise} \end{cases}$ $\Rightarrow$

$$\Rightarrow C_X(n,m) = \alpha^{n+m+2}\sigma_X^2 + \sigma_W^2 \alpha^{m-n} \sum_{i=0}^{n} \alpha^{2i} \quad ④$$

where we have assumed that $m \geq n$.

Since $m \geq n \Rightarrow$ we can write $m = n+k$ and ④ becomes

$$④ \Rightarrow C_X(n,n+k) = \alpha^{2(n+1)+k}\sigma_X^2 + \sigma_W^2 \alpha^k \sum_{i=0}^{n} \alpha^{2i} \quad ⑤$$

Taking the limit of (5) for $n \to +\infty$ we get

$$\lim_{n\to+\infty} C_x(n,n+k) = \alpha^k \sigma_x^2 \lim_{n\to+\infty} \alpha^{2(n+1)} + \sigma_w^2 \alpha^k \lim_{n\to+\infty} \sum_{i=0}^{n} \alpha^{2i}$$

$$|\alpha|<1 \Rightarrow \lim_{n\to+\infty} \alpha^n = 0$$

$$|\alpha|<1 \Rightarrow \sum_{i=0}^{+\infty} [\alpha^2]^i = \frac{1}{1-\alpha^2}$$

$\Rightarrow$

$$\Rightarrow \lim_{n\to+\infty} C_x(n,n+k) = \frac{\sigma_w^2 \alpha^k}{1-\alpha^2} = C_x(k) \quad \text{for } k \geq 0$$

$$\text{For } E\{X_n\}=0 \Rightarrow C_x(k) = C_x(-k)$$

$\Rightarrow$

$$\Rightarrow \lim_{n\to+\infty} C_x(n,n+k) = \frac{\sigma_w^2 \alpha^{|k|}}{1-\alpha^2} = C_x(k) \quad \forall k \quad (6)$$

From (3) and (6) we conclude that X_n becomes WSS as $n \to +\infty$. ∎

(c) If $\alpha \geq 1$ $\quad \lim_{n\to+\infty} \alpha^n = +\infty$, (2) $\Rightarrow \lim_{n\to+\infty} E[X_n] = +\infty$

therefore X_n cannot be stationary in any sense. ∎

(d) Due to ③ and ⑥ X_n is AR(1) R.P. with $\sigma^2 = \sigma_w^2$ and parameter $\beta = \alpha$. ■

(e) Due to ① X_n is a Gaussian R.P.

When $n \to +\infty$ $\begin{cases} ③ \Rightarrow E\{X_n\} = 0 \\ ⑥ \Rightarrow Var\{X_n\} = C_X(0) = \dfrac{\sigma_w^2}{1-\alpha^2} \end{cases}$

Hence, X_n is Gaussian with $X_n \sim \mathcal{N}(0, \frac{\sigma_w^2}{1-\alpha^2})$. ■