Midterm Exam I
EEL 5542 – Fall 2002

Instructor: Georgios C. Anagnostopoulos, Ph.D.

Name:__________________________________
(please print your name)

Instructions

For each problem show your work in detail and circle your final answer. Also start each problem on a new sheet. This exam consists of 5 problems; each is worth 20 points. When you turn in your exam, please staple the 3 pages of this document to your work. Midterm Exam I is <u>due by Monday, October 15th 2001 at 16:00</u>. For students that take this course from remote sites: please fax or send your work via mail by the aforementioned deadline. For all other students: please turn in your work to the department office (Room 407, CEBA-I building) by the aforementioned deadline and have one of the administrative assistants date your work and place it into my mailbox. If you have any questions concerning this exam, don't hesitate to get in touch with me. Please refer to my contact information at

http://www.geocities.com/eel5542/contact_info.htm

Good Luck!

Please, read carefully, then date and sign the following statement:

Pertaining to this take-home exam I have received <u>no help</u> from any other than the instructor of my class.

Signature:______________________________ Date:________________

Problem 1

The actual time to failure of a communications satellite is distributed exponentially with parameter λ=1/3. What is the conditional mean lifetime of the satellite given that it has survived 2 years?

Problem 2

The random variables X and Y follow a distribution with pdf[1] given as

$$f_{XY}(x,y) = Cx^2 e^{-x-xy} u(x)u(y)$$

where C is a constant and u(·) is the step function.

a) Find the value of C. Are X and Y mutually independent? Explain your answer.
b) Find the marginal pdf of Y.
c) Find the conditional pdf of X given Y

Problem 3

Let X and Y be two independent random variables following a normal distribution with mean μ=0 and variance σ^2. A system transforms these two random variables into two new random variables U and V as shown below

$$\mathbf{U} = \mathbf{X}^2 - \mathbf{Y}^2$$
$$\mathbf{V} = 2\mathbf{XY}$$

Find the joint pdf of U and V.

[1] Note that *pdf* refers to *probability density function*.

Problem 4

Let the random variables X and Y follow a joint pdf of

$$f_{XY}(x, y) = \begin{cases} \frac{1}{p} & 0 \le x^2 + y^2 \le 1 \\ 0 & 1 < x^2 + y^2 \end{cases}$$

and Z, W is another pair of random variables expressed in terms of X and Y as

$$\mathbf{Z} = \mathbf{X} + \mathbf{Y}$$
$$\mathbf{W} = \mathbf{X} - \mathbf{Y}$$

a) Are X and Y mutually independent? Explain your answer.
b) Find the joint pdf of Z and W.
c) Find the pdf of Z.
d) Calculate the variance of Z.

Problem 5

If X and Y are random variables with joint pdf

$$f_{XY}(x, y) = \frac{2}{p} e^{-\frac{x^2+y^2}{2}} u(x)u(y)$$

where u(·) is the step function and Z=Y/X, find the pdf of Z.

Problem 1

Let T be the R.V. for the failure time with pdf

$$f_T(t) = \frac{1}{3} e^{-t/3} u(t) \quad ①$$

We are asked to calculate $E\{T \mid T>2\} = \int_{t=-\infty}^{+\infty} t\, f_T(t \mid T>2)\, dt$ ②

Before we can do that, we need to find $f_T(t \mid T>2)$

From Example 3.10 on page 99 of the Textbook, we have

$$f_T(t \mid T>2) = \frac{f_T(t)}{1 - F_T(t)}\, u(t-2) \quad ③$$

$$②,③ \Rightarrow E\{T \mid T>2\} = \frac{1}{1-F_T(2)} \int_{t=2}^{+\infty} t\, f_T(t)\, dt \overset{①}{=} \frac{1}{1-F_T(2)} \int_{t=2}^{+\infty} \frac{t}{3} e^{-t/3}\, dt =$$

$$= \frac{5e^{-2/3}}{1-F_T(2)} \quad ④$$

It remains to compute $F_T(2)$:

$$F_T(t) = \int_{-\infty}^{t} f_T(t')\,dt' \overset{①}{=} \frac{1}{3} \int_{-\infty}^{t} e^{-t'/3} u(t')\,dt' = \frac{1}{3} \int_{0}^{t} e^{-t'/3}\,dt' = (1-e^{-t/3})\,u(t) \Rightarrow$$

$$\Rightarrow F_T(2) = 1 - e^{-2/3} \quad ⑤$$

$$④,⑤ \Rightarrow E\{T \mid T>2\} = 5 \quad ■$$

Problem 2

(a) It should hold:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy = 1 \Rightarrow G\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 e^{-x} e^{-xy} u(x)u(y)\,dx\,dy = 1 \Leftrightarrow$$

$$\Leftrightarrow G\int_{x=0}^{+\infty} x^2 e^{-x}\int_{y=0}^{+\infty} e^{-xy}\,dy\,dx = 1 \Rightarrow G = \frac{1}{\int_{x=0}^{+\infty} x e^{-x}\,dx} = 1 \Rightarrow G = 1 \quad ■$$

X and Y are not mutually independent, since their joint pdf cannot be written as a product of factors each one depending only on one variable. That is, $f_{XY}(x,y)$ cannot be written as a product of the form $\varphi_1(x)\varphi_2(y)$. ■

(b) $$f_Y(y) = \int_{x=-\infty}^{+\infty} f_{XY}(x,y)\,dx = u(y)\int_{x=0}^{+\infty} x^2 e^{-(1+y)x}\,dx = \frac{2}{(1+y)^3}u(y) \quad ■$$

(c) $$f_X(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \overset{\text{Part (b)}}{\Longrightarrow} f_X(x|y) = \frac{x^2 e^{-x} e^{-xy}}{\frac{2}{(1+y)^3}} u(x)u(y) \Rightarrow$$

$$\Rightarrow f_X(x|y) = \frac{x^2}{2}(1+y)^3 e^{-x} e^{-xy} u(x)u(y) \quad ■$$

Problem 3

$$\left.\begin{aligned} U &= X^2 - Y^2 \\ V &= 2XY \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} X^2 &= \frac{U + \sqrt{U^2+V^2}}{2} \\ Y^2 &= \frac{\sqrt{U^2+V^2} - U}{2} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} X_1 &= \sqrt{\frac{U+\sqrt{U^2+V^2}}{2}}, \quad Y_1 = \sqrt{\frac{\sqrt{U^2+V^2}-U}{2}} \\ X_2 &= \sqrt{\frac{U+\sqrt{U^2+V^2}}{2}}, \quad Y_2 = -\sqrt{\frac{\sqrt{U^2+V^2}-U}{2}} \\ X_3 &= -\sqrt{\frac{U+\sqrt{U^2+V^2}}{2}}, \quad Y_3 = \sqrt{\frac{\sqrt{U^2+V^2}-U}{2}} \\ X_4 &= -\sqrt{\frac{U+\sqrt{U^2+V^2}}{2}}, \quad Y_4 = -\sqrt{\frac{\sqrt{U^2+V^2}-U}{2}} \end{aligned}\right\} \quad (2)$$

System (1)

In other words, we came up with 4 solutions $(X_i, Y_i)_{i=1..4}$ as a function of U and V. Therefore, using the Theorem for RV transforms, we get

$$\left.\begin{aligned} f_{UV}(u,v) &= \sum_{i=1}^{4} \frac{f_{XY}(x,y)}{|J(x,y)|}\Bigg|_{(x,y)=(x_i,y_i)} \\ |J(x,y)| &= \left|\det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}\right| \overset{(1)}{=} 4|x^2+y^2| = 4(x^2+y^2) \end{aligned}\right\} \overset{(2)}{\Longrightarrow}$$

$$\Rightarrow f_{UV}(u,v) = \frac{e^{-\frac{\sqrt{U^2+V^2}}{2\sigma^2}}}{2\pi\sigma^2\sqrt{U^2+V^2}}$$

■

Problem 4

(a) We can write the pdf as $f_{XY}(x,y) = \frac{1}{\pi} u(1-x^2-y^2)$

Since we cannot factor $f_{XY}(x,y)$ as $\varphi_1(x)\varphi_2(y)$, X and Y are dependent RV's ■

(b) We are going to use the Theorem for R.V. transforms

$$\left.\begin{aligned} z &= x+y \\ w &= x-y \end{aligned}\right\} \text{ (E)} \Rightarrow \left\{\begin{aligned} x &= \frac{z+w}{2} \\ y &= \frac{z-w}{2} \end{aligned}\right. \quad (1) \quad \text{(we got a unique solution)}$$

$$f_{ZW}(z,w) = \left.\frac{f_{XY}(x,y)}{|J(x,y)|}\right|_{x=\frac{z+w}{2},\ y=\frac{z-w}{2}}$$

$$|J(x,y)| \overset{(E)}{=} \left|\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\right| = 2$$

$$\Rightarrow f_{ZW}(z,w) = \frac{1}{2} f_{XY}\left(\frac{z+w}{2}, \frac{z-w}{2}\right) \Rightarrow$$

$$\Rightarrow f_{ZW}(z,w) = \frac{1}{2\pi} u\left(1 - \frac{z^2+w^2}{2}\right) \Rightarrow f_{ZW}(z,w) = \frac{1}{2\pi} u(2-z^2-w^2) \quad (2)$$ ■

(c) $$f_Z(z) = \int_{w=-\infty}^{+\infty} f_{ZW}(z,w)\,dw \overset{(2)}{=} \frac{1}{2\pi}\int_{w=-\infty}^{+\infty} u(2-z^2-w^2)\,dw$$

$u(2-z^2-w^2) \neq 0$ for $-\sqrt{2-z^2} < w < \sqrt{2-z^2}$ with $|z| \leq 2$ (3)

$$\Rightarrow f_Z(z) = \frac{1}{2\pi}\int_{-\sqrt{2-z^2}}^{\sqrt{2-z^2}} dw = \frac{1}{\pi}\sqrt{2-z^2} \overset{(3)}{\Longrightarrow} f_Z(z) = \frac{1}{\pi}\sqrt{2-z^2}\, u(2-z^2) \quad (4)$$ ■

(d)

$$Var(Z) = E\{Z^2\} - E^2\{Z\} \quad (5)$$

$$E\{Z\} = \int_{-\infty}^{+\infty} z f_Z(z)\, dz \overset{(4)}{=} \frac{1}{\pi} \int_{-\sqrt{2}}^{\sqrt{2}} z\sqrt{2-z^2}\, dz = 0 \quad (6)$$

since $z\sqrt{2-z^2}$ is an odd function integrated over symmetric limits

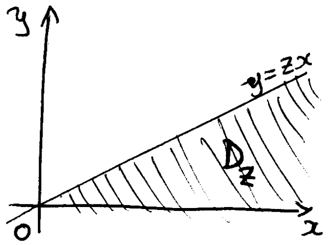
$$E\{Z^2\} = \int_{-\infty}^{+\infty} z^2 f_Z(z)\, dz \overset{(4)}{=} \frac{1}{\pi} \int_{-\sqrt{2}}^{\sqrt{2}} z^2\sqrt{2-z^2}\, dz = \frac{2}{\pi} \int_{0}^{\sqrt{2}} z^2\sqrt{2-z^2}\, dz = \frac{1}{2} \quad (7)$$

$$(5) \underset{(7)}{\overset{(6)}{\Longrightarrow}} Var(Z) = \frac{1}{2}$$

Problem 5

$$F_Z(z) = \Pr\left\{ \frac{Y}{X} \leq z \right\} = \iint_{D_z} f_{XY}(x,y)\,dx\,dy \quad (1)$$

The region D_z is depicted below in the figure.



$$(1) \Rightarrow F_Z(z) = \frac{2}{\pi} \int_{x=0}^{+\infty} \int_{y=0}^{zx} e^{-\frac{x^2+y^2}{2}}\,dx\,dy = \frac{2}{\pi} \int_{x=0}^{+\infty} e^{-\frac{x^2}{2}} \int_{y=0}^{zx} e^{-\frac{y^2}{2}}\,dy\,dx \Rightarrow$$

$$\Rightarrow f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{2}{\pi} \int_{x=0}^{+\infty} e^{-\frac{x^2}{2}} \left[\frac{\partial}{\partial z} \int_{y=0}^{zx} e^{-y^2/2}\,dy \right] dx$$

By Leibniz differentiation rule.

$$\frac{\partial}{\partial z} \int_{y=0}^{zx} e^{-y^2/2}\,dy = x\,e^{-\frac{z^2x^2}{2}}$$

$\Rightarrow$

$$\Rightarrow f_Z(z) = \frac{2}{\pi} \int_{x=0}^{+\infty} x\,e^{-\frac{(1+z^2)x^2}{2}}\,dx = \frac{2}{\pi}\,\frac{1}{1+z^2}$$

Since $X \geq 0$, $Y \geq 0$ and $Z = \frac{X}{Y} \Rightarrow Z \geq 0$

$$\Rightarrow f_Z(z) = \frac{2}{\pi}\,\frac{1}{(1+z^2)}\,u(z) \quad \blacksquare$$