Final Exam
EEL 5542 – Fall 2002

Instructor: Georgios C. Anagnostopoulos, Ph.D.

Please PRINT the following entries:

TODAY'S DATE:______________
(please write today's date)

FIRST NAME: ____
(please PRINT your first name)

LAST NAME: ____
(please PRINT your last name)

YOUR SSN: ____
(please write your SSN)

YOUR EMAIL: ____
(please PRINT your email address)

MAJOR: ____
(please PRINT your major – please NO ACRONYMS)

If you are a graduate Electrical Engineering student,
are you specializing in the area of Communications?:__________________________
(please PRINT YES or NO)

Instructions

First of all, take a deep breath and relax. It won't be that hard!

This Final Exam consists of two parts. The first part titled CONCEPTS contains some true/false questions and some other ones that are simple questions and which require a short answer. The purpose of this part is, first, to warm you up and, secondly, for me to evaluate how well you have understood some important concepts related to probability, random variables and random processes. The points you will gain for each question is clearly indicated. Notice, that not all questions carry the same credit.

The second part of the Exam, titled PROBLEMS, contains 5 problems, so I can assess your course-related, analytical skills that you have developed during this last semester. The problems are manageable and are much easier than the ones that were part of the Midterm Exams. Each problem is worth 20 points. Notice, that parts of the same problem may not be equally weighted. At the end of this handout you will find some Fourier Transform related tables that might turn out useful to you.

In each part of the Final Exam you will find specific instruction. Please, follow them!

Moreover, this Final Exam is closed book and its duration is 2 hours and 50 minutes. You may use any notes you have taken during the class and the cheat-sheet that you have (hopefully) prepared. Make sure you don't loose this cheat-sheet, since it may become useful to you in case you have to take the Electrical Engineering Qualifying Exams in the future.

And now the good news…
Don't waste your time counting the total points of this exam. They are actually more than 100!!!
To be more specific, if you answer everything correctly, you can get up to 130 points. The extra 30 points will be counted as extra credit with weight 35% of your total grade!

Finally, let me know if you have any questions.

GOOD LUCK!

CONCEPTS

Instructions
The CONCEPTS part consists of two segments. The first one has true/false questions. For each question circle either TRUE or FALSE but not both. In the second segment for each question you need to provide a short but sufficient answer. Write your answer below each question. Total points from Concepts: 30

I. True/False Questions

1) Two mutually exclusive events are <u>always</u> independent. *(1 point)*

TRUE FALSE

2) Two independent events are <u>always</u> mutually exclusive. *(1 point)*

TRUE FALSE

3) For events A, B and C we have that Pr{A∩B∩C}=Pr{A} Pr{B} Pr{C}. Are these events mutually independent? *(1 point)*

TRUE FALSE

4) Let $F_X(x)$ be the CDF of X. If $F_X(x_1) = 0.25$ and $F_X(x_2) = 0.5$ for some x_1 and x_2, then it <u>may</u> be the case that $x_2 < x_1$. *(1 point)*

TRUE FALSE

5) Let $f_X(x)$ be the PDF of X. If $f_X(x_1) = 0.25$ and $f_X(x_2) = 0.5$ for some x_1 and x_2, then it <u>may</u> be the case that $x_1 < x_2$. *(1 point)*

TRUE FALSE

6) The conditional joint PDF $f_{XY}(x, y \mid Y > 1)$ is always equal to zero for $y \leq 1$. *(1 point)*

TRUE FALSE

7) Two dependent random variables X and Y have marginal PDF's that are Gaussian. Are X and Y also jointly Gaussian? *(1 point)*

TRUE FALSE

8) The joint PDF of X and Y is given as $f_{XY}(x, y) = Ce^{-x-2y}u(x-y)$. Therefore, X and Y are mutually dependent. *(1 point)*

TRUE FALSE

9) A random process with independent increments is not also a process with stationary increments. *(1 point)*

TRUE FALSE

10) A random process with stationary increments is not also a process with independent increments. *(1 point)*

TRUE FALSE

11) Two jointly Gaussian random processes X(t) and Y(t) are uncorrelated. Then, are they also independent? *(1 point)*

TRUE FALSE

12) Two jointly Gaussian random processes X(t) and Y(t) are independent. Then, they are also uncorrelated? *(1 point)*

TRUE FALSE

13) The autocorrelation function $R_X(t)$ of a WSS random process X(t) is always an even function of $\boldsymbol{t}$. *(1 point)*

TRUE FALSE

14) A random process X(t) may be WSS, even if $E\{X^3(t_1)X^2(t_2)\}$ is not only a function of t_2-t_1. *(1 point)*

TRUE FALSE

15) A random process X(t) may be WSS, even if $E\{X^3(t_1)X^2(t_2)\}$ is a function only of t_2-t_1. *(1 point)*

TRUE FALSE

16) The power spectral density $S_X(f)$ of a WSS random process X(t) is always an odd function of f. *(1 point)*

TRUE FALSE

<END OF TRUE/FALSE QUESTIONS>

II. Questions with short answers

1) Let X be a random variable distributed uniformly in [0,1]. In this case we know that Pr{X=2}=0 and Pr{X=0.5}=0, although X can take the value 0.5. How would you explain to somebody that has not taken a course in Probability/Random Variables what the difference is between the events {X=0.5} and {X=2}? *(2 points)*

2) If a random variable X takes <u>only</u> the value X=-1, then $f_X(x)=$? *(2 points)*

3) Under which strict condition does it hold that $\Pr\{a<X<b\}=\Pr\{a\le X\le b\}$ for a random variable X that takes values in $(-\infty,+\infty)$?*(2 points)*

4) Let X(t) be a WSS random process with $\lim_{t\to\infty} R_X(t)=3$. Then, E{X(t)}=? *(2 points)*

5) What are the sufficient conditions for a system to generate a WSS random process Y(t), if it responds to an input that is a WSS random process X(t)? *(2 points)*

6) Two random processes X(t) and Y(t) are orthogonal. Under which sufficient conditions X(t) and Y(t) are mutually independent? *(2 points)*

7) If you randomly select the answers from the true/false question segment above, what is the probability that you will score more than 10 points? *(2 points)*

<END OF SHORT ANSWER QUESTIONS>

PROBLEMS

Instructions
Start each problem on a new page, where you mark clearly to which problem you are referring.
Also, mark clearly the part of the problem you answer.
Show your work in detail and explain all your answers.
Circle your final result in each part/problem.
Total points from Problems: 100

Problem 1 *(20 points total)*

Box 1 contains 3 black balls and 2 white ones, while box 2 contains 3 white and 2 black balls. Consider the following experiment: A fair coin is flipped. If the outcome is *heads* we randomly remove one ball from box 1, otherwise we randomly remove one from box 2. In either case, the ball is removed permanently from the box. Next, we toss the coin for a second time. If the outcome is *heads* we pick at random a ball from box 1, otherwise one from box 2.

Now, assume that we actually perform the above experiment. We flip the coin the first time, remove a ball from the appropriate box and then we toss the coin a second time and we pick again a ball from the appropriate box. Assume, that the ball we picked after the second coin toss is black. What is the probability that on the second coin toss we had an outcome of *heads*?

Problem 2 *(20 points total)*

Let X be a continuous random variable with a CDF of $F_X(x)$ that is continuous everywhere. Let Y=g(X), where

$$g(x) = \begin{cases} 0 & x \in [0,1] \\ x & \text{otherwise} \end{cases}$$

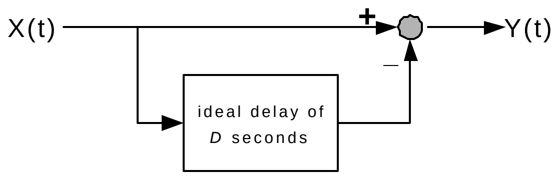
a) Find the CDF of Y, $F_Y(y)$, as a function of values of $F_X(x)$. *(7 points)*
b) Plot $F_Y(y)$, if X is uniformly distributed in [-2,2]. *(2 points)*
c) Find the PDF of Y, $f_Y(y)$, as a function of values of $f_X(x)$ and $F_X(x)$. *(7 points)*
d) Plot the $f_Y(y)$, if X is uniformly distributed in [-2,2]. *(2 points)*
e) Express Pr{Y=1} as a function of values of $F_X(x)$. *(1 point)*
f) Express Pr{Y=0} as a function of values of $F_X(x)$. *(1 point)*

Problem 3 *(20 points total)*

Let X be a random variable uniformly distributed in [0,1] and Y another random variable independent of X with PDF $f_Y(y) = e^{-y}u(y)$. Calculate the probability $\Pr\{0 \le X + Y < 1\}$.

Problem 4 *(20 points total)*

The system shown below accepts as an input a zero mean, Gaussian random process X(t), which generates an output Y(t). The autocorrelation function of X(t) is given as $R_X(t_1,t_2) = ae^{-b|t_1-t_2|}$, where a, b>0.



a) Calculate the cross-correlation function $R_{X,Y}(t_1,t_2)$. *(4 points)*

b) Find the power cross-spectral density $S_{X,Y}(f)$, if it can be defined. *(4 points)*

c) Calculate the autocorrelation function $R_Y(t_1,t_2)$ of Y(t). *(4 points)*

d) Find the power spectral density $S_Y(f)$ of Y(t), if it can be defined. *(4 points)*

e) Find the PDF $f_{Y(t)}(y)$ of Y(t). *(4 points)*

Problem 5 *(20 points)*

A random process X_n is of the form $X_n = n + W_n$, where W_n is zero mean, Gaussian white noise with variance σ^2. Furthermore, X_n acts as an input to a system with output Y_n that is described by the difference equation $Y_n = X_n - aX_{n-1}$, where a is some constant.

a) Is X_n a WSS process? Why or why not? *(3 points)*

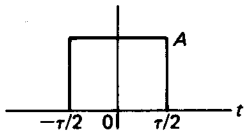
b) For what value of a is Y_n a WSS process? *(3 points)*

c) For the value of a you found in part (a) find the PDF $f_{Y_n}(y)$ of Y_n. *(4 points)*

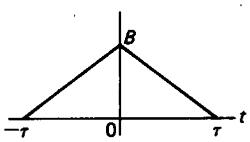
d) Up to this point you have seen how to "stationarize" a process of the form $X_n = n + W_n$ using a simple linear system. Now, come up with a difference equation of a linear system with output Z_n that "stationarizes" $V_n = n^2 + W_n$. Hint: The difference equation will involve V_n, V_{n-1} and V_{n-2}. *(10 points)*

<END OF PROBLEMS>

C.1 Fourier Transform Pairs

1. 

$\triangleq A\Pi(t/\tau)$ — $A\tau \dfrac{\sin \pi f\tau}{\pi f\tau} \triangleq A\tau \operatorname{sinc} f\tau$

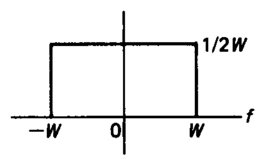
2. 

$\triangleq B\Lambda(t/\tau)$ — $B\tau \dfrac{\sin^2 \pi f\tau}{(\pi f\tau)^2} \triangleq B\tau \operatorname{sinc}^2 f\tau$

3. $e^{-\alpha t}u(t)$ — $\dfrac{1}{\alpha + j2\pi f}$

4. $\exp(-|t|/\tau)$ — $\dfrac{2\tau}{1 + (2\pi f\tau)^2}$

5. $\exp[-\pi(t/\tau)^2]$ — $\tau \exp[-\pi(f\tau)^2]$

6. $\dfrac{\sin 2\pi Wt}{2\pi Wt} \triangleq \operatorname{sinc} 2Wt$ — 

$\triangleq \dfrac{\Pi(f/2W)}{2W}$

7. $\exp[j(\omega_c t + \phi)]$ — $\exp(j\phi)\delta(f - f_c),\ \omega_c = 2\pi f_c$

8. $\cos(\omega_c t + \phi)$ — $\frac{1}{2}\delta(f - f_c)\exp(j\phi) + \frac{1}{2}\delta(f + f_c)\exp(-j\phi)$

9. $\delta(t - t_0)$ — $\exp(-j2\pi f t_0)$

10. $\displaystyle\sum_{m=-\infty}^{\infty} \delta(t - mT_s)$ — $\displaystyle\frac{1}{T_s}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_s}\right)$

11. $\operatorname{sgn} t = \begin{cases} +1, & t > 0 \\ -1, & t < 0 \end{cases}$ — $-\dfrac{j}{\pi f}$

12. $u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$ — $\frac{1}{2}\delta(f) + \dfrac{1}{j2\pi f}$

13. $\hat{x}(t)$ — $-j\operatorname{sgn}(f)X(f)$

C.2 Fourier Transform Theorems

Name of Theorem	Signal*	Fourier Transform
1. Superposition	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(f) + a_2X_2(f)$
2. Time delay	$x(t - t_0)$	$X(f)e^{-j\omega t_0}$
3a. Scale change	$x(at)$	$\|a\|^{-1}X(f/a)$
b. Time reversal	$x(-t)$	$X(-f) = X^*(f)$
4. Duality	$X(t)$	$x(-f)$
5a. Frequency translation	$x(t)e^{j\omega_0 t}$	$X(f - f_0),\ \omega_0 = 2\pi f_0$
b. Modulation	$x(t)\cos\omega_0 t$	$\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$
6. Differentiation	$\frac{d^n x(t)}{dt^n}$	$(j2\pi f)^n X(f)$
7. Integration	$\int_{-\infty}^{t} x(t')dt'$	$(j2\pi f)^{-1}X(f) + \frac{1}{2}X(0)\delta(f)$
8. Convolution	$\int_{-\infty}^{\infty} x_1(t - t')x_2(t')dt'$ $= \int_{-\infty}^{\infty} x_1(t')x_2(t - t')dt'$	$X_1(f)X_2(f)$
9. Multiplication	$x_1(t)x_2(t)$	$\int_{-\infty}^{\infty} X_1(f - f')X_2(f')\,df'$ $= \int_{-\infty}^{\infty} X_1(f')X_2(f - f')\,df'$

*All signals are assumed to be real.

CONCEPTS

I. True/False Questions

1) FALSE It is only true if at least one event is impossible

$$\left.\begin{array}{l} P(A\cap B)=0 \\ P(A\cap B)=P(A)P(B) \end{array}\right\} \Rightarrow P(A)=0 \text{ or } P(B)=0 \text{ or both}$$

2) FALSE It is only true if at least one event is impossible (see above)

3) FALSE because A, B, C need also to be pair-wise mutually independent.

4) FALSE because $F_X(x)$ must be a non-decreasing function of x

5) TRUE because there are no requirements for $f_X(x)$ to be monotonic

6) TRUE since we have conditioned on the event $\{Y>1\}$

7) FALSE since, in general, this is not the case

8) TRUE because it seems that we cannot write $f_{XY}(x,y)$ as a product of the form $f_X(x)f_Y(y)$

9) FALSE Independent increments imply stationary increments

10) TRUE In general, the converse of the statement in Question 9 is not true.

11) TRUE (see book for more details)

12) TRUE (see book for more details)

13) TRUE since it is one of the properties of $R_X(\tau)$

14) TRUE For a WSS R.P. $X(t)$ it must hold

(i) $E\{X(t)\} = \text{const}$

(ii) $R_X(t_1,t_2) = R_X(t_2-t_1) = E\{X(t_1)X(t_2)\}$

What happens with $E\{X^3(t_1)\,X^2(t_2)\}$ is completely irrelevant.

15) TRUE (see above)

16) FALSE $S_X(f)$ cannot be an odd function of f, otherwise we would have $S_X(-f) = -S_X(f) < 0$ and power cannot be negative.

CONCEPTS

II. Questions with short answers

1) $\{X=2\}$ is an impossible event while $\{X=0.5\}$ is a rare event, whose relative frequency of occurrance goes to zero when the number of experiments goes to infinity.

2) $f_X(x) = \delta(x+1)$

3) $$\left.\begin{array}{l} Pr\{a < X < b\} = F_X(b^-) - F_X(a) \\ Pr\{a \leq X \leq b\} = F_X(b) - F_X(a^-) \end{array}\right\} \Rightarrow$$ $F_X(x)$ must be continuous at a and b.

4) $E\{X(t)\} = \pm\sqrt{\lim_{\tau \to \infty} R_X(\tau)} = \pm\sqrt{3}$

5) The system must be Linear Time Invariant and in steady-state (initial conditions, if any, have no influence any more)

6) $X(t)$ and $Y(t)$ must be jointly Gaussian R.P.'s and at least one of them must have zero mean.

7) $$Pr\{S > 10\} = Pr\{F \leq 5\} = \sum_{k=0}^{5} \binom{16}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{16-k} =$$

$$= \frac{1}{2^{16}} \sum_{k=0}^{5} \binom{16}{k} = \frac{3609}{2^{16}} \simeq 0.0551$$

Problem 1

After flipping the first coin and removing one ball from the corresponding box we can have the following configurations:

Heads & 1 B removed from "1" — C_1
Heads & 1 W removed from "1" — C_2
Tails & 1 B removed from "2" — C_3
Tails & 1 W removed from "2" — C_4

	"1"	"2"	$P(C_i)$
C_1	2B 2W	3W 2B	$P(H)\cdot P(B_1)=\frac{1}{2}\frac{3}{5}=\frac{3}{10}$
C_2	3B 1W	3W 2B	$P(H)\cdot P(W_1)=\frac{1}{2}\frac{2}{5}=\frac{2}{10}$
C_3	3B 2W	3W 1B	$P(T)\cdot P(B_2)=\frac{1}{2}\frac{2}{5}=\frac{2}{10}$
C_4	3B 2W	2W 2B	$P(T)\cdot P(W_2)=\frac{1}{2}\frac{3}{5}=\frac{3}{10}$

($P(B_1)$: $P(B|H)$; $P(W_2)$: $P(W|T)$)

$$P(H|B)=\frac{P(H\cap B)}{P(B)}, \quad P(B)=P(B\cap H)+P(B\cap T) \Rightarrow P(H|B)=\frac{P(B\cap H)}{P(B\cap H)+P(B\cap T)} \quad ①$$

$$P(B\cap H)=\sum_{i=1}^{4}P(B\cap H\cap C_i)=\sum_{i=1}^{4}P(B|HC_i)P(H|C_i)P(C_i)$$

$$H, C_i \text{ indep} \Rightarrow P(H|C_i)=P(H)$$

$$\Rightarrow P(B\cap H)=P(H)\sum_{i=1}^{4}P(B|HC_i)P(C_i) \quad ②$$

Similarly $P(B\cap T)=P(T)\sum_{i=1}^{4}P(B|TC_i)P(C_i)$ ③

fair coin $\Rightarrow P(H)=P(T)$ ④

$$① \xRightarrow[④]{②③} P(H|B)=\frac{\sum_{i=1}^{4}P(B|HC_i)P(C_i)}{\sum_{i=1}^{4}P(B|HC_i)P(C_i)+\sum_{i=1}^{4}P(B|TC_i)P(C_i)} \quad ⑤$$

①

Probability of B if we start from config. G_1 and then flip a Head, therefore we draw one from "1"

$$P(B|HG_1) = \frac{2}{4} = \frac{10}{20} \qquad P(B|TG_1) = \frac{2}{5} = \frac{8}{20}$$

$$P(B|HG_2) = \frac{3}{4} = \frac{15}{20} \qquad P(B|TG_2) = \frac{2}{5} = \frac{8}{20}$$

$$P(B|HG_3) = \frac{3}{5} = \frac{12}{20} \qquad P(B|TG_3) = \frac{1}{4} = \frac{5}{20}$$

$$P(B|HG_4) = \frac{3}{5} = \frac{12}{20} \qquad P(B|TG_4) = \frac{2}{4} = \frac{10}{20}$$

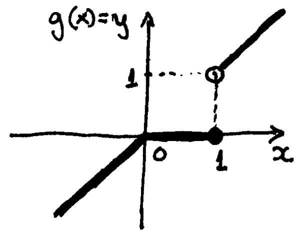
$$\sum_{i=1}^{4} P(B|HG_i)P(G_i) = \frac{10}{20}\cdot\frac{3}{10} + \frac{15}{20}\frac{2}{10} + \frac{12}{20}\frac{2}{10} + \frac{12}{20}\frac{3}{10} = \frac{120}{200} = \frac{6}{10} = 0.6 = \frac{3}{5} \quad (6)$$

$$\sum_{i=1}^{4} P(B|TG_i)P(G_i) = \frac{8}{20}\frac{3}{10} + \frac{8}{20}\frac{2}{10} + \frac{5}{20}\frac{2}{10} + \frac{10}{20}\frac{3}{10} = \frac{80}{200} = \frac{4}{10} = 0.4 = \frac{2}{5} \quad (7)$$

$$(5) \xRightarrow[(7)]{(6)} \boxed{P(H|B) = 0.6}$$

(2)

Problem 2



(a)[7] $F_Y(y) \triangleq Pr\{Y \le y\}$

Because of g(x)'s form we need to consider three cases

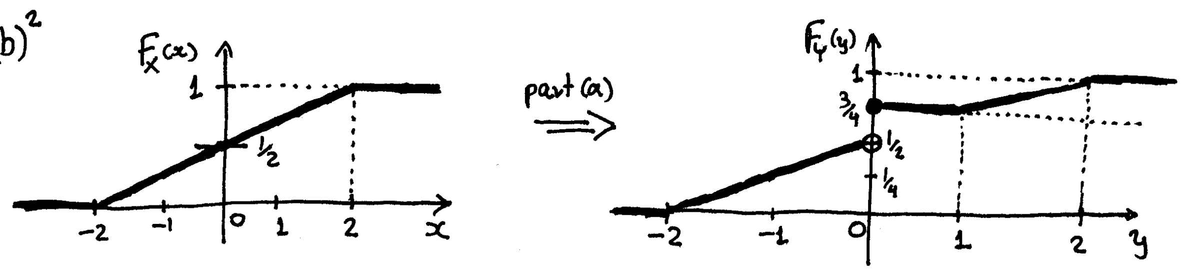
(i) $y<0$ $\quad Pr\{Y \le y\} = Pr\{X \le y\} = F_X(y)$

(ii) $0 \le y < 1$ $\quad Pr\{Y \le y\} = Pr\{X \le 1\} = F_X(1)$

(iii) $1 \le y$ $\quad Pr\{Y \le y\} = Pr\{X \le y\} = F_X(y)$

Therefore $\quad F_Y(y) = \begin{cases} F_X(1) & 0 \le y < 1 \\ F_X(y) & \text{otherwise} \end{cases}$

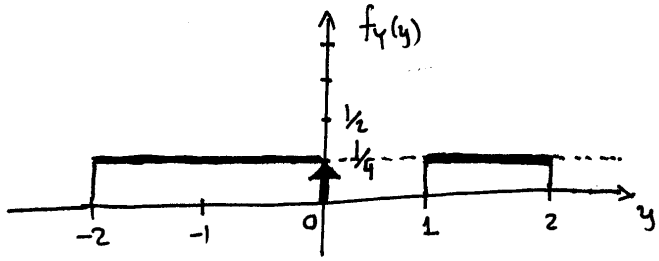
(b)[2]



(c)[7] From part (a) we notice that $F_Y(y)$ is discontinuous at $y=0$.
More precisely, $F_Y(0^-) = F_X(0)$ and $F_Y(0) = F_X(1)$
Therefore: .

$$f_Y(y) = \frac{d F_Y(y)}{dy} \xrightarrow{\text{part}(a)} f_Y(y) = \begin{cases} 0 & 0<y<1 \\ [F_X(1) - F_X(0)]\,\delta(y) & y=0 \\ f_X(y) & \text{otherwise} \end{cases}$$

(d)[2] From part (b) we have



(e)[1] From part (a) $F_Y(y)$ is continuous at $y=1$, therefore $Pr\{Y=1\}=0$

(f)[1] From part (a) $F_Y(y)$ is discontinuous at $y=0$, therefore

$Pr\{Y=0\} = [F_X(1) - F_X(0)]$

Problem 3

Let us define $Z \triangleq X+Y$

Being the sum of two independant RV's, it holds

$$f_Z(z) = f_X(x) * f_Y(y) = \int_{x=-\infty}^{+\infty} f_X(x) f_Y(z-x)\,dx$$

$$X \text{ uniform in } [0,1] \Rightarrow f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \end{cases}$$

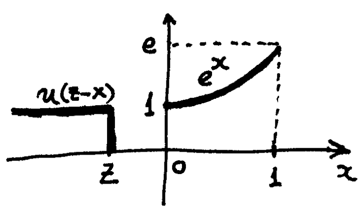
$$\Rightarrow f_Z(z) = \int_{x=0}^{1} f_Y(z-x)\,dx$$

$$f_Y(y) = e^{-y}u(y)$$

$$\Rightarrow f_Z(z) = e^{-z}\int_{x=0}^{1} e^{x} u(z-x)\,dx \quad (1)$$

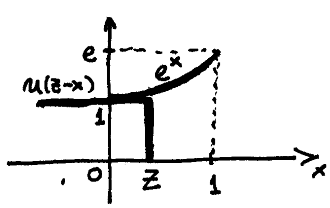
Let us define $I(z) \triangleq \int_{x=0}^{1} e^{x} u(z-x)\,dx$. In order to evaluate this integral we need to consider the following three cases:

(i) $z<0$



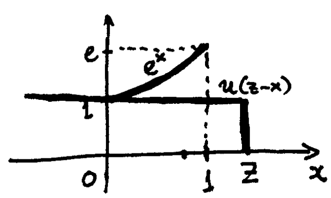
$I(z) = 0$

(ii) $0 \le z \le 1$



$$I(z) = \int_{x=0}^{z} e^{x}dx = e^{z}-1$$

(iii) $1<z$



$$I(z) = \int_{x=0}^{1} e^{x}dx = e-1$$

In summary
$$I(z) = \begin{cases} 0 & z<0 \\ e^{z}-1 & 0\leq z\leq 1 \\ e-1 & 1<z \end{cases} \overset{①}{\Longrightarrow}$$

$$\Rightarrow \quad f_Z(z) = \begin{cases} 0 & z<0 \\ 1-e^{-z} & 0\leq z\leq 1 \\ (e-1)e^{-z} & 1<z \end{cases} \qquad ②$$

Since $f_Z(z)$ has no delta function and, therefore, $F_Z(z)$ will be everywhere continuous

$$Pr\{0\leq X+Y<1\} = Pr\{0<X+Y\leq 1\} = Pr\{0<Z\leq 1\} =$$
$$= \int_0^1 f_Z(z)\,dz \overset{②}{=} \int_0^1 (1-e^{-z})\,dz = e^{-1} \Rightarrow$$

$$\Rightarrow Pr\{0\leq X+Y<1\} = e^{-1}$$

Problem 4

$$E\{X(t)\} = 0 = \text{const}$$
$$R_X(t_1,t_2) = \alpha e^{-b|t_1-t_2|} = \alpha e^{-b|\tau|} = R_X(\tau) \quad \text{where } \tau \triangleq t_2 - t_1 \quad \Big\} \Rightarrow X(t) \text{ WSS}$$

(0) marks the equation $\alpha e^{-b|\tau|}$.

From the diagram $Y(t) = X(t) - X(t-D)$ (1)

which describes a Linear Time Invariant system, $X(t)$ WSS $\Rightarrow$ $Y(t)$ WSS; $X(t), Y(t)$ jointly WSS

Therefore, $R_{X,Y}(t_1,t_2) = R_{X,Y}(\tau)$ and $S_{X,Y}(f)$ is defined

$R_Y(t_1,t_2) = R_Y(\tau)$ and $S_Y(f)$ is defined

(a)[4] $$R_{XY}(t_1,t_2) = R_{XY}(\tau) \triangleq E\{X(t)Y(t+\tau)\} \overset{(1)}{=} E\{X(t)X(t+\tau)\} - E\{X(t)X(t+\tau-D)\} =$$

$$= R_X(\tau) - R_X(\tau-D) \overset{(0)}{=} \alpha\left(e^{-b|\tau|} - e^{-b|\tau-D|}\right) \quad (2)$$

(b)[4] $$S_{XY}(f) = \mathcal{F}\{R_{XY}(\tau)\} \quad (3)$$

From Table C.1 $\mathcal{F}\{e^{-b|\tau|}\} = \dfrac{2}{b + \frac{4\pi^2}{b}f^2}$

From Table C.2 $\mathcal{F}\{f(\tau-D)\} = \mathcal{F}\{f(\tau)\}e^{-j2\pi fD}$ $\Bigg\} \Rightarrow$

$$\Rightarrow \begin{cases} \mathcal{F}\{e^{-b|\tau|}\} = \dfrac{2}{b + \frac{4\pi^2}{b}f^2} & (4) \\[2ex] \mathcal{F}\{e^{-b|\tau-D|}\} = \dfrac{2}{b + \frac{4\pi^2}{b}f^2}e^{-j2\pi fD} & (5) \end{cases}$$

$$\left.\begin{matrix} ③ \\ ② \end{matrix}\right\} \overset{④}{\underset{⑤}{\Longrightarrow}} S_{XY}(f) = \frac{2\alpha\left(1-e^{-j2\pi fD}\right)}{b+\frac{4\pi^2}{b}f^2}$$

(c)[4] $R_Y(t_1,t_2) = R_Y(\tau) \triangleq E\{Y(t)Y(t+\tau)\} \overset{①}{=} E\{X(t)X(t+\tau)\} - E\{X(t)X(t+\tau-D)\} -$

$$- E\{X(t-D)X(t+\tau)\} + E\{X(t-D)X(t+\tau-D)\} =$$

$$= R_X(\tau) - R_X(\tau-D) - R_X(\tau+D) + R_X(\tau) \overset{③}{=}$$

$$= \alpha\left(2e^{-b|\tau|} - e^{-b|\tau-D|} - e^{-b|\tau+D|}\right) \quad ⑥$$

(d)[4] $S_Y(f) = \mathcal{F}\{R_Y(\tau)\} \overset{⑥}{=} \frac{2\alpha}{b+\frac{4\pi^2}{b}f^2}\left(2 - e^{-j2\pi fD} - e^{j2\pi fD}\right) =$

$$= \frac{4\alpha\left[1-\cos(2\pi fD)\right]}{b+\frac{4\pi^2}{b}f^2} \quad ⑦$$

(e)[4] $X(t)$ is a Gaussian R.P. $\Rightarrow$ $X(t), X(t-D)$ jointly Gaussian $\forall t, D$
$① \Rightarrow Y(t) = X(t) - X(t-D)$ $\Big\} \Rightarrow$

$\Rightarrow Y(t)$ Gaussian R.P.

$$\left.\begin{matrix} E\{Y(t)\} \overset{①}{=} E\{X(t)\} - E\{X(t-D)\} \\ E\{X(t)\} = 0 \ \forall t \end{matrix}\right\} \Rightarrow E\{Y(t)\} = 0 \quad ⑧$$

$$Var(Y(t)) = C_Y(0) \overset{⑧}{=} R_Y(0) \overset{⑥}{=} 2\alpha\left(1-e^{-b|D|}\right) \quad ⑨$$

$$⑧, ⑨ \Rightarrow Y(t) \sim \mathcal{N}\left(0, \sqrt{2\alpha\left(1-e^{-b|D|}\right)}\right)$$

Problem 5

$$X_n = n + W_n \quad ①$$

(a)[3]

$$\left.\begin{array}{l} E\{X_n\} \overset{①}{=} n + E\{W_n\} \\ E\{W_n\} = 0 \ \forall n \end{array}\right\} \Rightarrow E\{X_n\} = n \ \text{(not constant)} \Rightarrow X_n \text{ not WSS.}$$

(b)[3]

$$\left.\begin{array}{l} Y_n = X_n - \alpha X_{n-1} \\ X_n = n + W_n \end{array}\right\} \Rightarrow Y_n = (1-\alpha)n + \alpha + W_n - \alpha W_{n-1} \quad ②$$

$$\left.\begin{array}{l} E\{Y_n\} \overset{②}{=} (1-\alpha)n + \alpha + E\{W_n\} - \alpha E\{W_{n-1}\} \\ E\{W_n\} = 0 \ \forall n \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow E\{Y_n\} = (1-\alpha)n + \alpha \\ \text{For } Y_n \text{ to be WSS } E\{Y_n\} = \text{const} \end{array}\right\} \Rightarrow \alpha = 1$$

With $\alpha = 1$ from ② we get $Y_n = 1 + W_n - W_{n-1}$ ③

We still need to show that for $\alpha = 1$ $R_Y(n,m) = R_Y(m-n) = R_Y(k)$

$$\left.\begin{array}{l} R_Y(n,m) \overset{③}{=} E\{Y_n Y_m\} = 1 + E\{W_m\} - E\{W_{m-1}\} + E\{W_n\} + E\{W_n W_m\} \\ \qquad - E\{W_n W_{m-1}\} - E\{W_{n-1}\} - E\{W_{n-1} W_m\} + E\{W_{n-1} W_{m-1}\} \\ E\{W_n\} = 0 \ \forall n \\ W_n \text{ white noise} \Rightarrow E\{W_n W_m\} = \begin{cases} E\{W_n^2\} & n = m \\ 0 & n \neq m \end{cases} \end{array}\right\} \Rightarrow$$

$$\Rightarrow R_Y(n,m) = 1 + R_W(m-n) - R_W(m-1-n) - R_W(m-n+1) + R_W(m-n) \quad ④$$

Since W_n is white noise $R_W(n,m) = R_W(m-n) = R_W(k) = E\{W_n W_{n+k}\}$

Therefore, $R_Y(n,m)$ is a function only of $k = m-n$

Thus, for $\alpha = 1$ Y_n becomes WSS.

(c)[4] X_n is a Gaussian R.P. $\Rightarrow$ X_n, X_{n-1} jointly Gaussian, $Y_n = X_n - \alpha X_{n-1}$ $\Rightarrow$ Y_n Gaussian R.P.

$\Uparrow$

W_n Gaussian R.P.

From part (b) if $\alpha = 1$ $E\{Y_n\} = 1$

$Var(Y_n) = C_Y(0) = R_Y(0) - [E\{Y_n\}]^2$ (4)

$\Rightarrow Var(Y_n) = 2R_W(0) - R_W(-1) - R_W(1)$

W_n white noise $\Rightarrow R_W(k) = \begin{cases} E\{W_n^2\} & k=0 \\ 0 & k \neq 0 \end{cases}$ $\Rightarrow$

$$\Rightarrow Var(Y_n) = 2E\{W_n^2\}$$

Thus, $Y_n \sim \mathcal{N}\left(1, \sqrt{2E\{W_n^2\}}\right)$ if $\alpha = 1$.

(d)[10] $Z_n = \alpha_1 V_n + \alpha_2 V_{n-1} + \alpha_3 V_{n-2}$ (5)

We need to identify $\alpha_1, \alpha_2, \alpha_3$ so that Z_n is WSS if $V_n = n^2 + W_n$

The first condition that must be satisfied is

$E\{Z_n\} = \text{const } \forall n \overset{(5)}{\Rightarrow} E\{Z_n\} = \alpha_1 E\{V_n\} + \alpha_2 E\{V_{n-1}\} + \alpha_3 E\{V_{n-2}\}$ $\Rightarrow$

$V_n = n^2 + W_n$, $E\{W_n\} = 0 \ \forall n$ $\Rightarrow$ $E\{V_n\} = n^2$

$$\Rightarrow \quad E\{Z_n\} = \alpha_1 n^2 + \alpha_2 (n-1)^2 + \alpha_3 (n-2)^2 = \text{const } \forall n \quad \Rightarrow$$

$$\Rightarrow \quad (\alpha_1+\alpha_2+\alpha_3)n^2 - 2(\alpha_2+2\alpha_3)n + \alpha_2+4\alpha_3 = \text{const } \forall n \quad \Rightarrow$$

$$\Rightarrow \begin{cases} \alpha_1+\alpha_2+\alpha_3 = 0 \\ \alpha_2+2\alpha_3 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_2 = -2\alpha_1 \\ \alpha_3 = \alpha_1 \end{cases} \quad (6)$$

For the values found in (6) Z_n becomes

$$\left.\begin{array}{l} Z_n = \alpha_1 (V_n - 2V_{n-1} + V_n) \quad (8) \\ V_n = n^2 + W_n \end{array}\right\} \Rightarrow Z_n = \alpha_1 \left[2 + W_n - 2W_{n-1} + W_{n-2} \right] \quad (7)$$

The second condition that must be met for Z_n to be WSS is $R_Z(n,m) = R_Z(k) \quad k \triangleq m-n$

As we did in part (b), because of (7), $R_Z(n,m)$ is going to be a function of $R_W(n,m) = R_W(k)$; therefore the second condition is satisfied for any $\alpha_1 \neq 0$.

The final answer is

$$(8) \Rightarrow Z_n = \alpha_1 (V_n - 2V_{n-1} + V_n) \quad \text{where } \alpha_1 \neq 0 \text{ arbitrary.}$$