Problem 5

For the 1s state of hydrogen, show that $<1/r> = 1/(n^2 a_0)$. You may use

$$\int_0^\infty dx x^n e^{-ax} = \frac{n!}{a^{n+1}}$$

$$\left\langle \frac{1}{r} \right\rangle = \int_0^\infty \frac{1}{r} P(r) = \int_0^\infty r \left| R(r) \right|^2$$

Since $R(r) = \frac{2}{a_0^{3/2}} e^{-r/a_0}$

$$\left\langle \frac{1}{r} \right\rangle = \frac{4}{a_0^3} \int_0^\infty dr\, r e^{-2r/a_0}$$

Now let $a \equiv \frac{2}{a_0}$, and $r = x$

then

$$\left\langle \frac{1}{r} \right\rangle = \frac{4}{a_0^3} \int_0^\infty dx\, x e^{-ax}$$

which, by the integral given ($n=1$)

is

$$\left\langle \frac{1}{r} \right\rangle = \frac{4}{a_0^3} \frac{1!}{a^2} = \frac{4}{a_0^3} \left(\frac{a_0}{2} \right)^2 = \frac{1}{a_0}$$