# Wave envelope examples for progressive waves

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SUMMARY
The wave envelope method, due to Astley (**1**), is applied to some simple propagating wave problems governed by the Helmholtz equation and similar equations. It is shown that the method yields solutions which for some of the problems are exact and for the others are approximate and complex. In these cases however, when the number of elements is increased the trial function becomes a better approximation.

## 1 Introduction

Astley (**1**) (**2**) and co-workers have used wave envelope elements to solve wave problems of large extent. The method was extended by Bettess (**3**) to unbounded wave type problems. The goal now is to use wave elements, which include information about the wavelength, to solve short wave problems. By short waves we mean the wave length is much smaller than the dimensions of the problem. We envisage using elements which contain many wavelengths of the field variable, rather than several elements per wavelength. In this note, results using the Astley methodology are obtained for a number of one dimensional problems as a first step. Our attention is directed to problems governed by the Helmholtz equation and similar equations. These will be described in detail later.

The Astley approach for the acoustic problem involves first removing the harmonic time dependence and then modifying the shape and weighting functions (which then become complex conjugates of each other) within a standard Galerkin finite element scheme. This is done in order to accommodate the fine harmonic detail of the solution. This has the advantage that the progressive wave can be modelled using finite elements of length much greater than the ten nodes per wavelength necessary in the usual finite element scheme. Hence the method is very economical for short wave problems.

In this paper we also consider an equivalent approach to Astley's in which both the harmonic time dependence and fine harmonic detail are first removed from the problem before discretization. Thus the wave envelope amplitude is modelled. We shall then find that there is then no need to alter the shape and weighting functions: the standard shape and weighting functions within the Galerkin finite element scheme (**4**, p215) are used in the equivalent approach.

The above conclusion raises an important point with regard to mapped elements: it shall be seen that the derivation (in Section 2) of the equivalent approach does not necessarily depend upon the use of the standard Galerkin shape and weighting functions. In particular, shape and weighting functions for infinite elements (**5**, Ch4 p 53) (**4**, Ch 8.13 p 183) could be considered. (These functions may be used in modelling the wave envelope amplitude because it is not subject to the oscillatory behaviour of the potential.) It would therefore appear from the above conclusion that the resulting matrix equation set should be equivalent to that obtained using mapped wave envelope elements (**3**) (**6**). However, this is not obviously the case and further investigation of this point is necessary, but shall not be done in the present paper.

The Astley method is tested for some simple propagating wave examples. We shall find that for these examples the stiffness matrix is complex and non-symmetric and for some examples the trial solution is also complex.

The Astley approach is considered next.

## 2 The Astley Approach

In this section we shall consider the Astley approach for progressive waves: this gives an approximation $\tilde{\Phi}$ to the wave potential $\Phi$. We shall also show that the Astley approach is equivalent to using the standard Galerkin finite element scheme in order to find an approximation $\tilde{\phi}$ to the wave envelope amplitude $\phi$.

The progressive wave examples are dependent upon one variable in 1-D, 2-D and 3-D space; letting the cartesian co-ordinate system be given by $(x, y, z)$, then in 1-D space we consider the variable $x$, in 2-D space we consider the variable $\rho = [x^2 + y^2]^{1/2}$ and in 3-D space the variable $r = [x^2 + y^2 + z^2]^{1/2}$.

The wave equation governing an acoustic wave disturbance is given by

$$\frac{\partial^2 \Psi}{\partial t^2} = c^2 \nabla^2 \Psi \tag{1}$$

where $\Psi$ is the velocity potential, $t$ is time and $\nabla^2$ is the Laplacian operator. We consider time periodic wave disturbances of frequency $2\pi\omega$, and so $c$ is the wave propagation velocity and the wave potential $\Phi^*$ satisfies the Helmholtz equation

$$(\nabla^2 + k^2)\Phi^* = 0 \tag{2}$$

where $\Psi = \Phi^* e^{-i\omega t}$ and $k = \frac{\omega}{c}$. However, there may be forcing, damping and other effects acting upon the system. We shall therefore consider wave type solutions from the second order differential equation

$$\left(\frac{d^2}{dx^2} + \alpha^*(x)\frac{d}{dx} + \beta^*(x)\right)\Phi = \gamma^*(x) \tag{3}$$

where $\alpha^*(x)$ is related to damping and $\gamma^*(x)$ is related to forcing.

The progressive waves in 1-D space are considered first.

**1-D space**

The Astley method gives the approximation $\tilde{\Phi}$ in the form

$$\tilde{\Phi} = \sum_i \Phi_i N_i e^{ikx} \tag{4}$$

where $N_i$ denotes the conventional shape function associated with node $i$ and $\Phi_i$ are the unknowns. Astley also considers a weighting function, different from the conventional weighting function, to be the complex conjugate of his shape function. Hence the Astley weighting function $w_i^*$ is given as

$$w_i^* = N_i e^{-ikx} \tag{5}$$

This choice of weighting function eliminates the exponential term with its dependence upon the wave number $k$ and makes possible the use of Gauss-Legendre integration.

The Astley approximation $\tilde{\Phi}$ for $\Phi$ using the Astley method is now given.

Suppose the function $\Phi(x)$ satisfies the differential equation

$$D^*[\Phi(x)] - \gamma(x)e^{ikx} = 0 \tag{6}$$

for some function $\gamma(x)$ and second order differential operator

$$D^* = \frac{d^2}{dx^2} + \alpha^*(x)\frac{d}{dx} + \beta^*(x) \tag{7}$$

for two functions $\alpha^*(x)$ and $\beta^*(x)$ dependent upon the particular wave example considered. (Thus the 1-D Helmholtz equation is such that $D^* = \frac{d^2}{dx^2} + k^2$ and $\gamma(x) = 0$.)

The finite element method (**4**, p214-215) is now followed; in the 1-D domain $x_0 \leq x \leq x_{n+1}$,

$$\int_{x_0}^{x_{n+1}} w_j^*(D^*[\tilde{\Phi}(x)] - \gamma(x)e^{ikx})dx = 0 \tag{8}$$

$1 \leq j \leq n$ for a domain divided into $n+1$ finite elements. Integrating by parts gives

$$\int_{x_0}^{x_{n+1}} \{-\frac{dw_j^*}{dx}\frac{d\tilde{\Phi}}{dx} + w_j^*(\alpha^*\frac{d\tilde{\Phi}}{dx} + \beta^*\tilde{\Phi} - \gamma e^{ikx})\}dx + [w_j^*\frac{d\tilde{\Phi}}{dx}]_{x_0}^{x_{n+1}} = 0 \tag{9}$$

(8) and also (9) give an approximation $\tilde{\Phi}$ for $\Phi$ using the Astley finite element method: given $\Phi(x_0)$ and $\Phi(x_{n+1})$ then (8) and also (9) represent a system of $n$ equations in $n$ unknowns $\Phi_i$ ; solving the system for $\Phi_i$ gives the approximation for $\Phi$ such that $\tilde{\Phi} = \sum_i \Phi_i N_i e^{ikx}$.

Substituting for $\tilde{\Phi}$ and $w_j^*$ with the shape and weighting functions given by (4) and (5) respectively gives (9) explicitly as:

$$\begin{aligned}\{\int_{x_0}^{x_{n+1}}( &- [k^2 N_j N_i - ikN_j\frac{dN_i}{dx} + ikN_i\frac{dN_j}{dx} + \frac{dN_j}{dx}\frac{dN_i}{dx}] \\ &+ \alpha^*(ikN_jN_i + N_j\frac{dN_i}{dx}) + \beta^* N_j N_i)dx \\ &+ [ikN_jN_i + N_j\frac{dN_i}{dx}]_{x_0}^{x_{n+1}}\}\Phi_i = \int_{x_0}^{x_{n+1}} \gamma N_j dx \qquad \text{(9a)}\end{aligned}$$

We now show that the Astley method, used to determine an approximation for $\Phi$ given by (8) and in weak form by (9), is equivalent to the standard Galerkin method used to determine an approximation for the wave envelope amplitude $\phi$.

Let us consider the differential operator $D$ such that

$$D[\phi(x)] = e^{-ikx}D^*[\Phi(x)] \qquad (10)$$

where $\Phi = \phi e^{ikx}$ and $\phi$ is the wave envelope amplitude. Substituting (10) into the expression for the differential operator $D^*$ given by (7) gives an expression for the differential operator $D$ such that

$$D[\phi(x)] = \frac{d^2}{dx^2} + \alpha(x)\frac{d}{dx} + \beta(x) \qquad (11)$$

where $\alpha(x) = 2ik + \alpha^*(x)$ and $\beta(x) = -k^2 + ik\alpha^*(x) + \beta^*(x)$.

Similarly, substituting (10) into the differential equation (3) for $\Phi$ gives the differential equation for $\phi$

$$D[\phi(x)] - \gamma(x) = 0 \qquad (12)$$

The approximation $\tilde{\phi}$ for $\phi$ used in the standard Galerkin scheme is

$$\tilde{\phi}(x) = \sum_i \phi_i N_i \qquad (13)$$

where $N_i$ are the conventional shape functions and $\phi_i$ are the unknown constants. The conventional weighting functions for $\phi$ are given by

$$w_i(x) = N_i \qquad (14)$$

We rewrite the integral expression for $\tilde{\Phi}$ given by (8). Substituting for $D^*[\Phi(x)] = e^{ikx}D[\phi(x)]$, $\tilde{\Phi}(x) = \tilde{\phi}(x)e^{ikx}$, $w_j^* = w_j e^{-ikx}$ into the integral expression for $\tilde{\Phi}$ given by (8), gives

$$\int_{x_0}^{x_{n+1}} w_j(D[\tilde{\phi}(x)] - \gamma(x))dx = 0. \tag{15}$$

Integrating (15) by parts gives

$$\int_{x_0}^{x_{n+1}} \{-\frac{dw_j}{dx}\frac{d\tilde{\phi}}{dx} + w_j(\alpha\frac{d\tilde{\phi}}{dx} + \beta\tilde{\phi} - \gamma)\}dx + [w_j\frac{d\tilde{\phi}}{dx}]_{x_0}^{x_{n+1}} = 0 \tag{16}$$

(15) and (16) are also the integrals used to determine an approximation $\tilde{\phi}$ for $\phi$ using the standard Galerkin finite element scheme such that (16) is the weak form of (15). However, we consider the piecewise linear Galerkin shape and weighting functions (**4**, p219) such that

$$\begin{aligned} N_i &= \frac{x - [x_0 + L(i-1)]}{L} \quad x_0 + L(i-1) \leq x \leq x_0 + iL \\ &= -\frac{x - [x_0 + L(i+1)]}{L} \quad x_0 + iL \leq x \leq x_0 + L(i+1) \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{17}$$

where $L$ is the finite element length such that $x_{n+1} - x_0 = (n+1)L$ and $x_0$, $x_{n+1}$ are the domain boundaries $x_0 \leq x \leq x_{n+1}$. Thus, letting $x_i = x_0 + iL$, then

$$\tilde{\phi}(x_i) = \phi_i \tag{18}$$

For the problems in 2-D and 3-D space, the variable $x$ is replaced by $\rho$ and $r$ respectively in (17) and (18).

For this choice of triangle shape function, the integrals (8) and (15) are indeterminate. However, we may consider the shape function $N_i(x, \delta)$ where the discontinuity (in the first derivative ) is replaced by a variable continuous (in the first derivative ) in a small distance $\delta$. (Hence the 'corner' of $N_i$ is rounded off with radius of curvature order $\delta$.) Thus choosing the shape function $N_i(x, \delta)$, the integrals (8) and (15) are now determinate and give (9) and (16) respectively. In the limit as $\delta \to 0$, $N_i(x, \delta) \to N_i(x)$ and (8) and (15) tend to (9) and (16) respectively. Hence (9) and (16) are equal and determinate for $N_i$ defined by (17).

Thus we have shown that the approximation $\tilde{\Phi}$ for $\Phi$ determined from the Astley method is $\tilde{\phi}e^{ikx}$ such that the approximation $\tilde{\phi}$ for $\phi$ is determined from the standard Galerkin finite element scheme. (Hence $\Phi_i = \phi_i$, where $\tilde{\Phi} = \Phi_i N_i e^{ikx}$ and $\tilde{\phi} = \phi_i N_i$.)

This result is checked and seen more clearly by substituting for $w_j$ and $\tilde{\phi}$ with shape and weighting functions given by (14) and (18) respectively. This gives

$$\begin{aligned}\{\int_{x_0}^{x_{n+1}}( &- [k^2 N_j N_i + 2ikN_j\frac{dN_i}{dx} + \frac{dN_j}{dx}\frac{dN_i}{dx}] \\ &+ \alpha^*(ikN_jN_i + N_j\frac{dN_i}{dx}) + \beta^* N_j N_i)dx \\ &+ [N_j\frac{dN_i}{dx}]_{x_0}^{x_{n+1}}\}\phi_i = \int_{x_0}^{x_{n+1}} \gamma N_j dx \end{aligned} \qquad (16a)$$

which is equation (16) given explicitly. By comparison we see that (9a) and (16a) are equivalent, as expected, since integration by parts gives

$$\int_{x_0}^{x_{n+1}} N_j\frac{dN_i}{dx}dx = \int_{x_0}^{x_{n+1}} -N_i\frac{dN_j}{dx}dx + [N_jN_i]_{x_0}^{x_{n+1}}$$

However, we shall also consider particular examples where $\Phi$ is dependent upon one variable in 2-D and 3-D space. Such examples are considered next.

**2-D space**

In cylindrical polar co-ordinates $(\rho, \theta)$ we shall consider a solution for $\Phi$ independent of $\theta$ such that

$$\Phi(\rho) = \phi(\rho)e^{ik\rho} \qquad (19)$$

in the 2-D domain $\rho_0 \leq \rho \leq \rho_{n+1}$. the Astley approximation for $\Phi$ is therefore

$$\int_{\rho_0}^{\rho_{n+1}} \rho w_j^*(\rho)(D^*[\tilde{\Phi}(\rho)] - \gamma(\rho)e^{ik\rho})d\rho = 0 \qquad (20)$$

where $\tilde{\Phi}$, $w_j^*$ and $D^*$ are defined by (4), (5) and (6) respectively. The integrand (20) differs from that for the Astley approximation for $\Phi$ in 1-D space by the multiplication of the factor $\rho$. Integrating by parts gives

$$\int_{\rho_0}^{\rho_{n+1}}\{-\frac{d(\rho w_j^*)}{d\rho}\frac{d\tilde{\Phi}}{d\rho} + \rho w_j^*(\alpha^*\frac{d\tilde{\Phi}}{d\rho} + \beta^*\tilde{\Phi} - \gamma e^{ik\rho})\}d\rho + [\rho w_j^*\frac{d\tilde{\Phi}}{d\rho}]_{\rho_0}^{\rho_{n+1}} = 0 \quad (21)$$

Substituting $w_j^*$, $D^*$ and $\tilde{\Phi}$ for $w_j$, $D$ and $\tilde{\phi}$ respectively, then (20) becomes

$$\int_{\rho_0}^{\rho_{n+1}} \rho w_j(D[\tilde{\phi}(\rho)] - \gamma(\rho))d\rho = 0. \tag{22}$$

Integrating (22) by parts gives

$$\int_{\rho_0}^{\rho_{n+1}} \{-\frac{d(\rho w_j)}{d\rho}\frac{d\tilde{\phi}}{d\rho} + \rho w_j(\alpha\frac{d\tilde{\phi}}{d\rho} + \beta\tilde{\phi} - \gamma)\}d\rho + [\rho w_j \frac{d\tilde{\phi}}{d\rho}]_{\rho_0}^{\rho_{n+1}} = 0 \tag{23}$$

We consider $N_i(\rho)$ defined by (17). Although this choice gives (20) and (22) indeterminate, following the same argument used in the 1-D subsection above we see that (21) and (23), the weak form of the integrals (20) and (22), are the same and are determinate.

(23) is also the integral used in the method to determine an approximation $\tilde{\phi}(\rho)$ for the wave envelope amplitude $\phi(\rho)$ from the standard Galerkin scheme, and so this method is equivalent to the Astley method.

**3-D space**

In spherical polar co-ordinates $(r, \Theta, \omega)$ we shall consider a solution for $\Phi$ independent of $\Theta$ and $\omega$ such that

$$\Phi(r) = \phi(r)e^{ikr} \tag{24}$$

in the 3-D domain $r_0 \leq r \leq r_{n+1}$. The Astley approximation for $\Phi$ is therefore

$$\int_{r_0}^{r_{n+1}} r^2 w_j^*(r)(D^*[\tilde{\Phi}(r)] - \gamma(r)e^{ikr})dr = 0 \tag{25}$$

where $\tilde{\Phi}$, $w_j^*$ and $D^*$ are defined by (4), (5) and (6). The integrand (25) differs from that for the Astley approximation for $\Phi$ in 1-D space by the multiplication of the factor $r^2$. Integrating by parts gives

$$\int_{r_0}^{r_{n+1}} \{-\frac{d(r^2 w_j^*)}{dr}\frac{d\tilde{\Phi}}{dr} + r^2 w_j^*(\alpha^*\frac{d\tilde{\Phi}}{dr} + \beta^*\tilde{\Phi} - \gamma e^{ikr})\}dr + [r^2 w_j^* \frac{d\tilde{\Phi}}{dr}]_{r_0}^{r_{n+1}} = 0 \tag{26}$$

Substituting in for $D^*[\Phi(r)] = e^{ikr}D[\phi(r)]$, $\tilde{\phi}(r) = \tilde{\Phi}(r)e^{-ikr}$ and $w_j^* = w_j e^{ikr}$, (25) becomes

$$\int_{r_0}^{r_{n+1}} r^2 w_j(D[\tilde{\phi}(r)] - \gamma(r))dr = 0 \tag{27}$$

and integrating (27) by parts gives

$$\int_{r_0}^{r_{n+1}}\{-\frac{d(r^2w_j)}{dr}\frac{d\tilde{\phi}}{dx}+r^2w_j(\alpha\frac{d\tilde{\phi}}{dr}+\beta\tilde{\phi}-\gamma)\}dr+[r^2w_j\frac{d\tilde{\phi}}{dr}]_{r_0}^{r_{n+1}}=0 \qquad (28)$$

We consider $N_i(r)$ defined by (17). Although this choice gives (25) and (27) indeterminate, following the same argument used in the 1-D subsection above we see that (26) and (28), the weak form of the integrals (25) and (27), are the same and are determinate.

(28) is also the integral used in the method to determine an approximation $\tilde{\phi}(r)$ for the wave envelope amplitude $\phi(r)$ using the standard Galerkin scheme.

Thus the Astley approach is equivalent to the approach using the standard Galerkin finite element scheme in order to find an approximation for the wave envelope amplitude for progressive radial waves in 1-D, 2-D and 3-D space. The Astley aproach is tested on a number of examples. These test examples are given next.

# 3 Examples and Method

We test for the five progressive wave examples which satisfy the second order differential equations dependent upon one variable and given below. For each example, the wave envelope amplitude $\phi$ is approximated by a piecewise linear function $\tilde{\phi}$ using the standard Galerkin finite element method (**4** p218-220). $\tilde{\phi}$ is then compared with the progressive wave solution for $\phi$ and the results are given in the next section.

The five progressive wave examples and the differential equations that they satisfy are given next. Example (a) has a progressive wave solution

$$A_1e^{ikx} \qquad (29)$$

which satisfies the 1-D Helmholtz equation

$$(\nabla^2+k^2)\Phi(x)=\frac{d^2\Phi}{dx^2}+k^2\Phi=0 \qquad (30)$$

Example (b) has a progressive wave solution

$$xe^{ikx} \qquad (31)$$

which satisfies the 1-D Helmholtz equation with a forcing term such that

$$(\nabla^2 + k^2)\Phi(x) = \frac{d^2\Phi}{dx^2} + k^2\Phi = 2ike^{ikx} \tag{32}$$

Example (c) has a progressive wave solution

$$A_3 x e^{ikx} \tag{33}$$

which satisfies the 1-D second order differential equation

$$(\nabla^2 + k^2)\Phi(x) = \frac{d^2\Phi}{dx^2} + k^2\Phi = \frac{2ik}{x}\Phi \tag{34}$$

Example (d) has a progressive wave solution given by Abramowitz and Stegun (**7**, p364 9.2.1,9.2.2)

$$A_4 J_0(k\rho) + B_4 Y_0(k\rho) \sim \frac{C_4}{\sqrt{\rho}} e^{ik\rho} \tag{35}$$

for large $k\rho$ which satisfies the 2-D Helmholtz equation

$$(\nabla^2 + k^2)\Phi(\rho) = \frac{d^2\Phi}{d\rho^2} + \frac{1}{\rho}\frac{d\Phi}{d\rho} + k^2\Phi = 0 \tag{36}$$

Hence (36) is the Bessel equation of zero order (**7**, p358 9.1) for the variable $k\rho$.

Example (e) has a progressive wave solution

$$\frac{A_5}{r} e^{ikr} \tag{37}$$

which satisfies the 3-D Helmholtz equation

$$(\nabla^2 + k^2)\Phi(r) = \frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} + k^2\Phi = 0 \tag{38}$$

$A_4$ and $B_4$ are constants, and we let $A_1$, $A_3$, $C_4$ and $A_5$ be real constants. $\nabla^2$ is the Laplacian operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

$\phi$ is then modelled using the standard Galerkin 1-D finite element method (**4**, p218-220): the domain is divided into $n+1$ finite elements which give a system of $n$ integral equations in $n$ unknowns $\phi_i$, where the approximation $\tilde{\phi}$ for $\phi$ is given by $\tilde{\phi} = \sum_i \phi_i N_i$, $1 \leq i \leq n$.

The resulting system of $n$ equations in $n$ unknowns $\phi_i$ is represented in matrix form by

$$K_{ij}\phi_i = y_j \tag{39}$$

where $K_{ij}$ is the stiffness matrix of the particular example and is tridiagonal, complex and non-symmetric. $\phi_i$ is then determined from (39) numerically for chosen values of the wave number $k$, number of nodes $n$ and boundary values; short waves are considered where $\lambda = \frac{2\pi}{k}$ is much less than the element length, and we use double precision complex numbers in a tridiagonal equation solver.

We shall find that the results are such that the piecewise linear function $\tilde{\phi}$ is given by a simple real expression in each example (a), (b) and (c). However, we shall find that for examples (d) and (e) the piecewise linear function is complex. For the examples (d) and (e) the real and complex parts of $\tilde{\phi}$ are shown graphically. $\tilde{\phi}$ is then compared with the exact progressive wave solution, except for example (d) which is compared with the asymptotic solution for large $k\rho$ given by (35). These results are given in the next section. Each example is now considered in turn and the progressive wave solution modelled.

# 4 Results

For examples (a), (b) and (c) the progressive wave solution is modelled by the standard Galerkin finite element method for the wave envelope amplitude $\phi$. This approach has been shown in section 2 to be equivalent to the Astley approach. However, for examples (d) and (e) it is found that the calculations using the weak integrals in the Astley method are more straight forward and therefore this method is prefered. Each example (a) to (e) is considered in turn.

Example (a).

We model the propagating wave solution $\Phi = A_1 e^{ikx}$ from the 1-D Helmholtz equation $\frac{d^2\Phi}{dx^2} + k^2\Phi = 0$ for the wave envelope amplitude $\phi =$

$\Phi e^{-ikx}$. $\phi$ satisfies the equation

$$\frac{d^2\phi}{dx^2} + 2ik\frac{d\phi}{dx} = 0 \tag{40}$$

The analytic solution is $\phi = A + Be^{-2ikx}$. Thus if the boundary values for $\phi$ for the propagating wave solution are given, then the solution is indeterminate for $2k(x_{n+1} - x_0) = \pi m$ for some integer constant $m$. In order to avoid this difficulty, we consider the boundary conditions $\phi(x_0) = A_1$ and at $x = x_{n+1}$ we consider $\frac{d\tilde{\phi}}{dx} = 0$ locally which gives $\phi_{n+1}\phi_n$.Considering (40) in the form $\frac{d^2\phi}{dx^2} + \alpha\frac{d\phi}{dx} + \beta\phi - \gamma = 0$, then $\alpha = 2ik$, $\beta = \gamma = 0$, and the standard Galerkin weak integral (16) is

$$(\frac{1}{L} - ik)\phi_{j-1} - \frac{2}{L}\phi_j + (\frac{1}{L} + ik)\phi_{j+1} = 0 \tag{41}$$

after performing the integration. Given in matrix form $K_{jq}\phi_q = y_j$,

$$\begin{aligned} K_{jq} &= 1/L - ik \ , \quad q = j - 1 \\ &= -2/L \ , \quad q = j \neq n \\ &= -1/L + ik \ , \quad q = j = n \\ &= 1/L + ik \ , \quad q = j + 1 \\ &= 0 \ \ \text{otherwise} \end{aligned} \tag{42}$$

and the vector $y_j$ is given by

$$\begin{aligned} y_j &= -(1/L - ik)\phi_0 \ , \quad j = 1 \\ &= 0 \ , \quad 1 < j \leq n \end{aligned} \tag{43}$$

The matrix equation is solved numerically for $\phi_i$ for choices of the number of nodes $n$, element length $L$, wavenumber $k$ and constant $A_1$. For the all choices considered, $\phi_i = A_1$, and thus $\tilde{\phi}(x) = A_1$.

Therefore the Astley approximation for the progressive wave solution of the 1-D Helmholtz equation is

$$\tilde{\Phi} = A_1 e^{ikx} \tag{44}$$

for all choices of the variables $n$, $k$ and $A_1$ considered. This is the exact progressive wave solution (29) for $\Phi$ example (a). We shall now consider example (b).

Example (b).

We model the propagating wave solution $\Phi = xe^{ikx}$ from the 1-D Helmholtz equation with a forcing term, $\frac{d^2\Phi}{dx^2} + k^2\Phi = 2ike^{ikx}$, for the wave envelope amplitude $\phi = \Phi e^{-ikx}$. $\phi$ satisfies the equation

$$\frac{d^2\phi}{dx^2} + 2ik\frac{d\phi}{dx} = 2ik \tag{45}$$

As in example (a), we shall avoid indeterminate solutions for specific values of $k$ by considering the following boundary conditions: at $x = x_0 = 1$, then $\phi = \phi_0 = 1$ and at $x = x_{n+1} = 10$, then $\frac{d\tilde{\phi}}{dx} = 1$ locally and so $\phi_{n+1} = \phi_n + L$.

Considering (45) in the form $\frac{d^2\phi}{dx^2} + \alpha\frac{d\phi}{dx} + \beta\phi - \gamma = 0$, then $\alpha = 2ik$, $\beta = 0$ and $\gamma = 2ik$, and the standard Galerkin weak integral (16) is

$$(\frac{1}{L} - ik)\phi_{j-1} - \frac{2}{L}\phi_j + (\frac{1}{L} + ik)\phi_{j+1} = 2ikL \tag{46}$$

after performing the integration. Given in matrix form $K_{jq}\phi_q = y_j$,

$$\begin{aligned} K_{jq} &= 1/L - ik\ , \quad q = j-1 \\ &= -2/L\ , \quad q = j \neq n \\ &= -1/L + ik\ , \quad q = j = n \\ &= 1/L + ik\ , \quad q = j+1 \\ &= 0 \ \ \text{otherwise} \end{aligned} \tag{47}$$

and the vector $y_j$ is given by

$$\begin{aligned} y_j &= -(1/L - ik)\phi_0 + 2ikL\ , \quad j = 1 \\ &= 2ikL\ , \quad 1 < j < n \\ &= -L + 2ikL\ , \quad j = n \\ &= -(1/L - ik)\phi_0 - L + 2ikL\ , \quad j = n = 1 \end{aligned} \tag{48}$$

The matrix equation is solved numerically for $\phi_i$ for choices of the number of nodes $n$, element length $L$, wavenumber $k$. For the all choices considered, $\phi_i = \tilde{\phi}(x_i) = x_i$ and thus $\tilde{\phi}(x) = x$

Therefore the Astley approximation for the progressive wave $\Phi = xe^{ikx}$ of example (b) is

$$\tilde{\Phi} = xe^{ikx} \tag{49}$$

for all choices of the variables $n$ and $k$ considered. This is the exact progressive wave solution (31) for $\Phi$ example (b). We shall now consider example (c).

Example (c).

We model the propagating wave solution $\Phi = A_3 x e^{ikx}$ from the 1-D second order differential equation $\frac{d^2\Phi}{dx^2} + k^2\Phi = 2ik\frac{\Phi}{x}$ for the wave envelope amplitude $\phi = \Phi e^{-ikx}$. $\phi$ satisfies the equation

$$\frac{d^2\phi}{dx^2} + 2ik\frac{d\phi}{dx} = \frac{2ik}{x}\phi \tag{50}$$

and we look for the solution satisfying the boundary conditions $\tilde{\phi}(x_0) = \phi_0 = A_3 x_0$, $x_0 = 1$ and $\tilde{\phi}(x_{n+1}) = \phi_{n+1} = A_3 x_{n+1}$, $x_{n+1} = 10$. Considering (50) in the form $\frac{d^2\phi}{dx^2} + \alpha\frac{d\phi}{dx} + \beta\phi - \gamma = 0$, then $\alpha = 2ik$, $\beta = -\frac{2ik}{x}$ and $\gamma = 0$, and the standard Galerkin weak integral (16) is

$$\begin{aligned} [ & \frac{1}{L} - ik - 2ikI^{(1)}(j,L)]\phi_{j-1} + [-\frac{2}{L} - 2ik(I^{(2)}(j,L) + I^{(3)}(j,L)]\phi_j \\ + & [\frac{1}{L} + ik - 2ikI^{(4)}(j,L))]\phi_{j+1} = 0 \end{aligned} \tag{51}$$

such that

$$\begin{aligned} I^{(1)}(j,L) &= j - \frac{1}{2} - \frac{1}{L} + j(j-1)\log(\frac{1+Lj-L}{1+Lj}) \\ I^{(2)}(j,L) &= -j + \frac{3}{2} + \frac{1}{L} - (1-j)^2\log(\frac{1+Lj-L}{1+Lj}) \\ I^{(3)}(j,L) &= -j - \frac{3}{2} + \frac{1}{L} + (1+j)^2\log(\frac{1+Lj+L}{1+Lj}) \\ I^{(4)}(j,L) &= j + \frac{1}{2} - \frac{1}{L} - j(j+1)\log(\frac{1+Lj+L}{1+Lj}) \end{aligned} \tag{52}$$

which were evaluated using maple. Given in matrix form $K_{jq}\phi_q = y_j$,

$$K_{jq} = 1/L - ik - 2ikI^{(1)}(j,L)\ , \quad q = j-1$$

$$
\begin{aligned}
&= -2/L - 2ik[I^{(2)}(j,L) + I^{(3)}(j,L)]\ , \quad q = j \\
&= 1/L + ik - 2ikI^{(4)}(j,L)\ , \quad q = j+1 \\
&= 0 \ \text{ otherwise}
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
y_j &= -[1/L - ik - 2ikI^{(1)}(j,L)]\phi_0\ , \quad j = 1 \\
&= 0\ , \quad 1 < j < n \\
&= -[1/L + ik - 2ikI^{(4)}(j,L)]\phi_{n+1}\ , \quad j = n \\
&= -[1/L - ik - 2ikI^{(1)}(j,L)]\phi_0 \\
&\quad -[1/L + ik - 2ikI^{(4)}(j,L)]\phi_{n+1}\ , \quad j = n = 1.
\end{aligned} \tag{54}
$$

The matrix equation is solved numerically with $A_3 = 1$ for particular choices of the number of nodes $n$ and wavenumber $k$. For all the choices considered, $\phi_i = \tilde{\phi}(x_i) = x_i$ and thus $\tilde{\phi}(x) = x$.

Therefore the Astley approximation for the progressive wave $\Phi = A_3 x e^{ikx}$ is

$$\tilde{\Phi} = A_3 x e^{ikx} \tag{55}$$

for all the choices of the variables $n$, $L$, and $k$ that are considered. This is the exact progressive wave solution. For example, for $n = 8$ and $k = 100$, the following values for $\phi_i$ were obtained:

| $x$ | Re($\tilde{\phi}$) | Im($\tilde{\phi}$) |
|---|---|---|
| 1 | 1.0 | |
| 2 | 2.00000 | 0.00000 |
| 3 | 3.00000 | 0.00000 |
| 4 | 4.00000 | 0.00000 |
| 5 | 5.00000 | 0.00000 |
| 6 | 6.00000 | 0.00000 |
| 7 | 7.00000 | 0.00000 |
| 8 | 8.00000 | 0.00000 |
| 9 | 9.00000 | 0.00000 |
| 10 | 10.0 | |

Table 1: Results for example (c) with $n = 8$ and $k = 100$.

We shall now consider example (d).

Example (d).

We model the propagating wave solution $\Phi = A_4 J_0(k\rho) + B_4 Y_0(k\rho) \sim \frac{C_4}{\sqrt{\rho}} e^{ik\rho}$ for large $k\rho$ (**7**, p364 9.2.1-9.2.4) from the 2-D Helmholtz equation $\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\Phi}{d\rho}\right) + k^2\Phi = 0$. In particular, we consider the boundary conditions $\Phi(1) = e^{ik}$ and $\Phi(10) = \frac{1}{\sqrt{10}} e^{10ik}$. Following the Astley method, (21) gives

$$\int_{\rho_0}^{\rho_{n+1}} \left\{ -\frac{dw_j^*}{d\rho}\rho\frac{d\tilde{\Phi}}{d\rho} + k^2 w_j^* \tilde{\Phi}\rho \right\} d\rho + \left[ w_j^* \rho \frac{d\tilde{\Phi}}{d\rho} \right]_{\rho_0}^{\rho_{n+1}} = 0_j \qquad (56)$$

where $w_j^* = N_j e^{-ik\rho}$ is the Astley weighting function. In section (2) we show by considering (23) that (56) is equivalent to

$$\int_{\rho_0}^{\rho_{n+1}} \left\{ -\frac{dw_j}{d\rho}\rho\frac{d\tilde{\phi}}{d\rho} + 2ikw_j\rho\frac{d\tilde{\phi}}{d\rho} + ikw_j\tilde{\phi} \right\} d\rho + \left[ w_j \rho \frac{d\tilde{\phi}}{d\rho} \right]_{\rho_0}^{\rho_{n+1}} = 0_j \qquad (57)$$

where $w_j = N_j$ and

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\phi}{d\rho}\right) + 2ik\frac{d\phi}{d\rho} + ik\frac{\phi}{\rho} = 0 \qquad (58)$$

In order to determine the Astley approximation $\tilde{\Phi} = \sum_i \phi_i N_i e^{ik\rho}$ for $\Phi$, we may use either (56) or (57). However, it is seen that the integration (56) simplifies greatly and is therefore a better choice. Both (56) and (57) give the equation set

$$\begin{aligned} [1 - ikL]\alpha^*(j,L)\phi_{j-1} &- [\alpha^*(j,L) + \alpha^\dagger(j,L)]\phi_j \\ +[1 + ikL]\alpha^\dagger(j,L)\phi_{j+1} &= 0_j \end{aligned} \qquad (59)$$

$\alpha^* = \int_0^L \frac{\rho^*}{L^2} d\rho$, $\rho^* = \rho_0 + L(j-1) + \rho$
$\alpha^\dagger = \int_0^L \frac{\rho^\dagger}{L^2} d\rho$, $\rho^\dagger = \rho_0 + Lj + \rho$

Given in matrix form $K_{jq}\phi_q = y_j$, then from (59) the stiffness matrix is

$$\begin{aligned} K_{jq} &= [1 - ikL]\alpha^*(j,L)\ , \quad q = j-1 \\ &= -\alpha^*(j,L) - \alpha^\dagger(j,L)\ , \quad q = j \\ &= [1 + ikL]\alpha^\dagger(j,L)\ , \quad q = j+1 \\ &= 0 \ \text{ otherwise} \end{aligned} \qquad (60)$$

and the vector $y_j$ is given by

$$\begin{aligned} y_j &= -[1 - ikL]\alpha^*(j, L)\phi_0 \ , \ j = 1 \\ &= -[1 + ikL]\alpha^\dagger(j, L)\phi_{n+1} \ , \ j = n \\ &= -[1 - ikL]\alpha^*(j, L)\phi_0 \\ &\quad -[1 + ikL]\alpha^\dagger(j, L)\phi_{n+1} \ , \ j = 1 = n \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{61}$$

In particular, we consider $k = 10,000$, and thus the progressive wave solution tends to the asymptotic value $\Phi = A_4 J_0(k\rho) + B_4 Y_0(k\rho) \sim \frac{C_4}{\sqrt{\rho}} e^{ik\rho}$, and let $n = 1$, 10, and 50. The matrix equation is solved numerically and the results are given below plotted against the asymptotic solution $\phi \sim \frac{1}{\sqrt{\rho}}$.

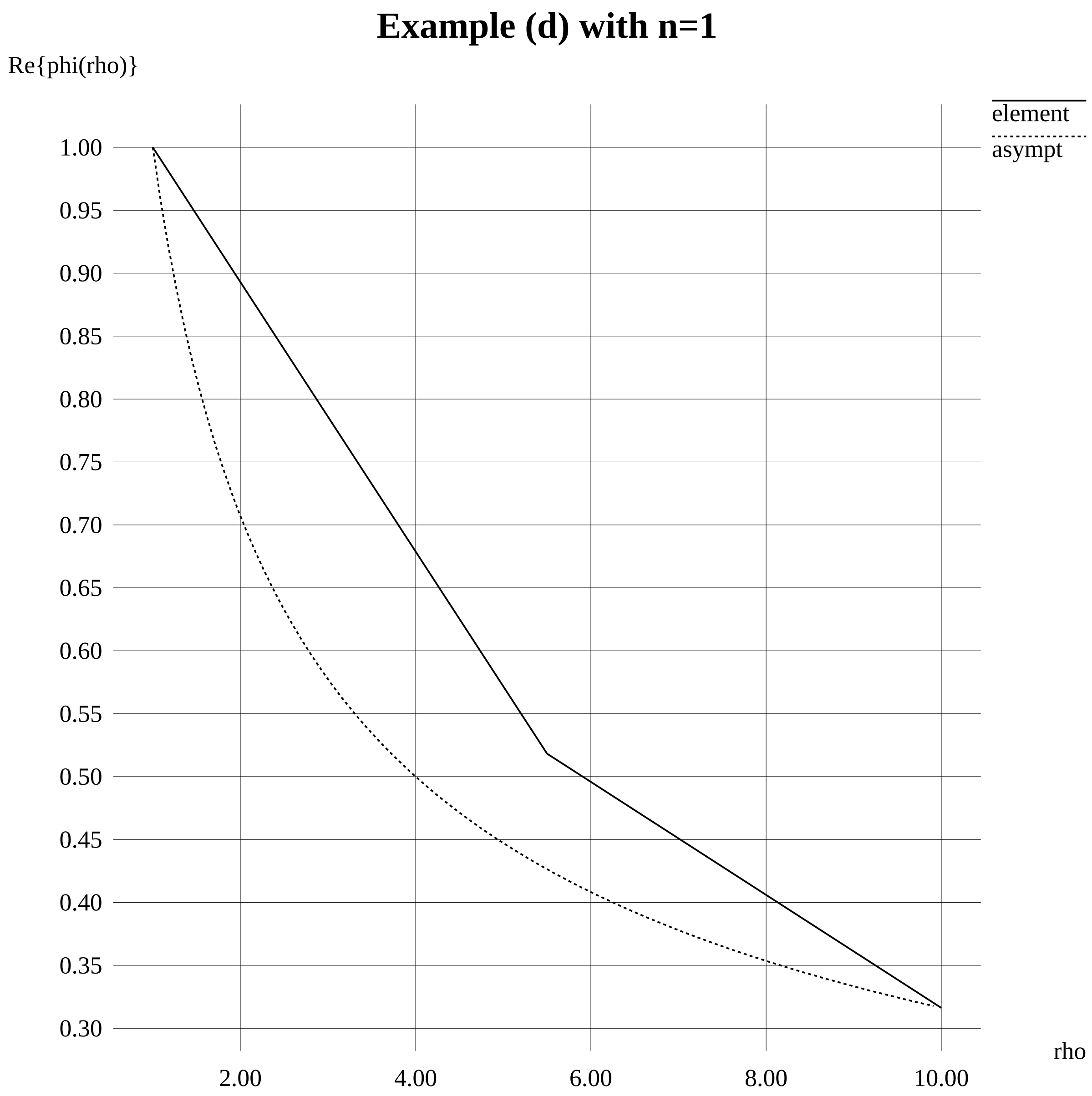


Figure 1. Example (d) $\mathrm{Re}[\tilde{\phi}(\rho)]$ vs. $\rho$ with $n = 1$ and $k = 10000$.

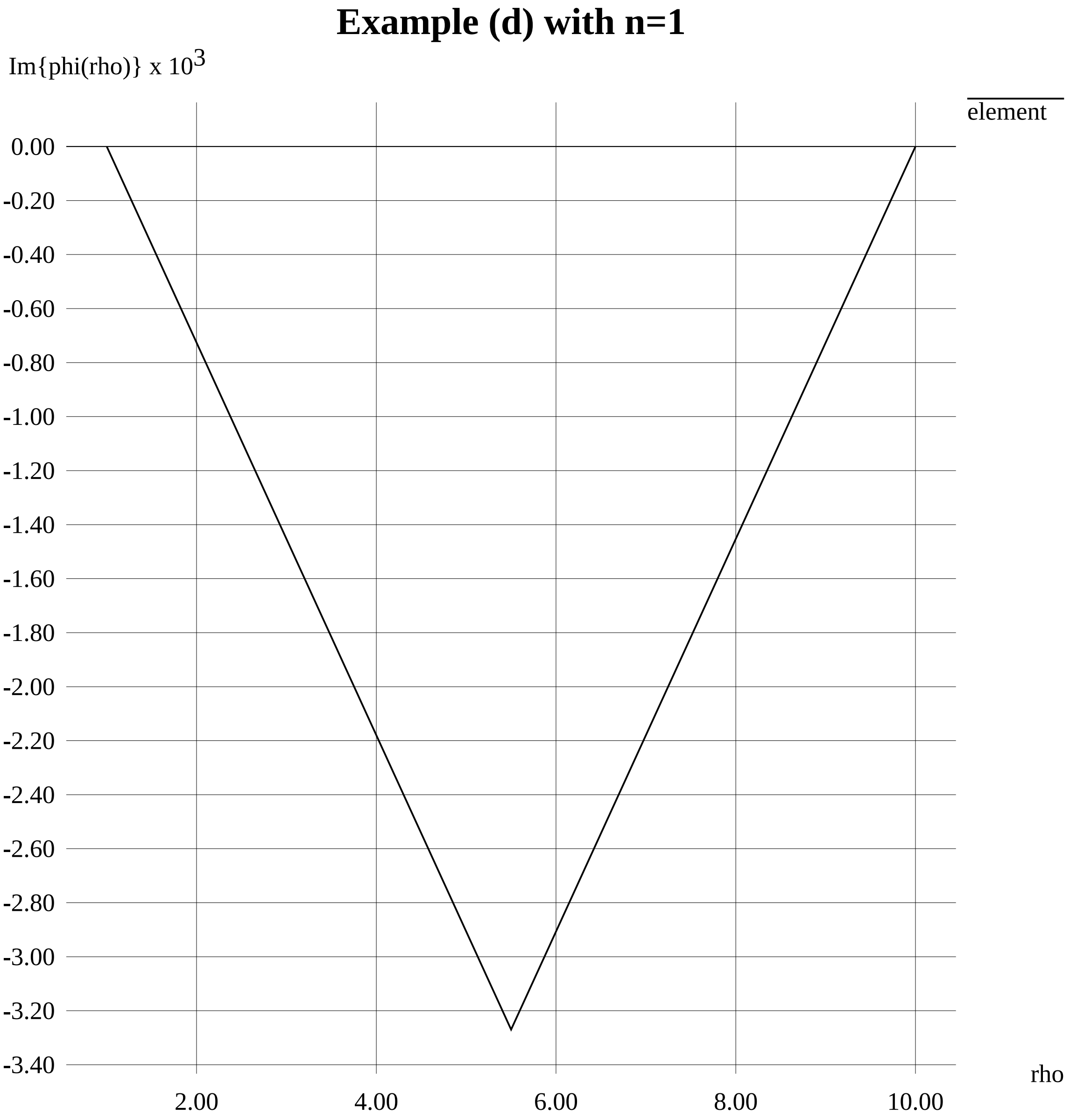


Figure 2. Example (d) $\mathrm{Im}[\tilde{\phi}(\rho)]$ vs. $\rho$ with $n = 1$ and $k = 10000$.

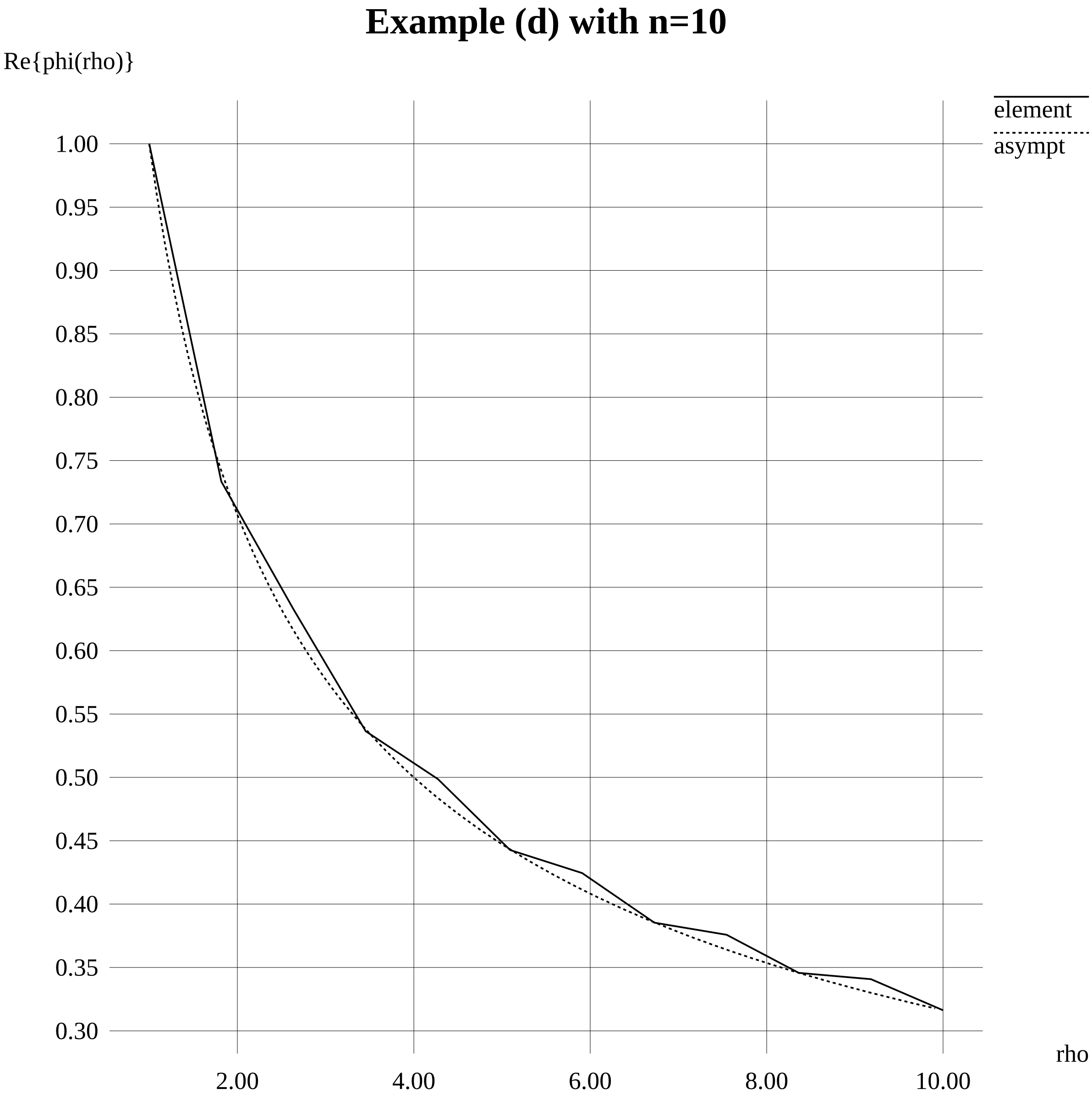


Figure 3. Example (d) $\mathrm{Re}[\tilde{\phi}(\rho)]$ vs. $\rho$ with $n = 10$ and $k = 10000$.

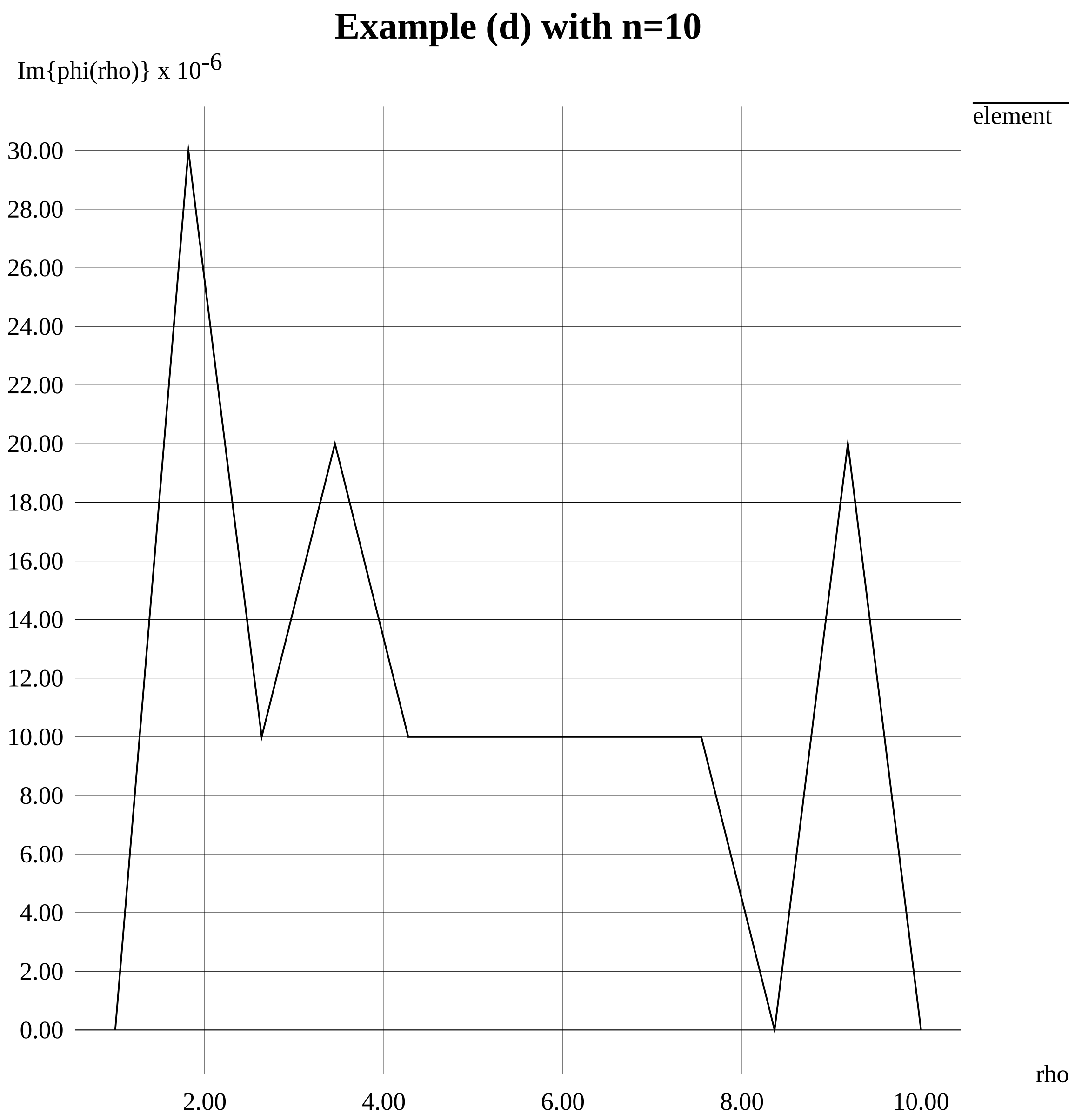


Figure 4. Example (d) $\mathrm{Im}[\tilde{\phi}(\rho)]$ vs. $\rho$ with $n = 10$ and $k = 10000$.

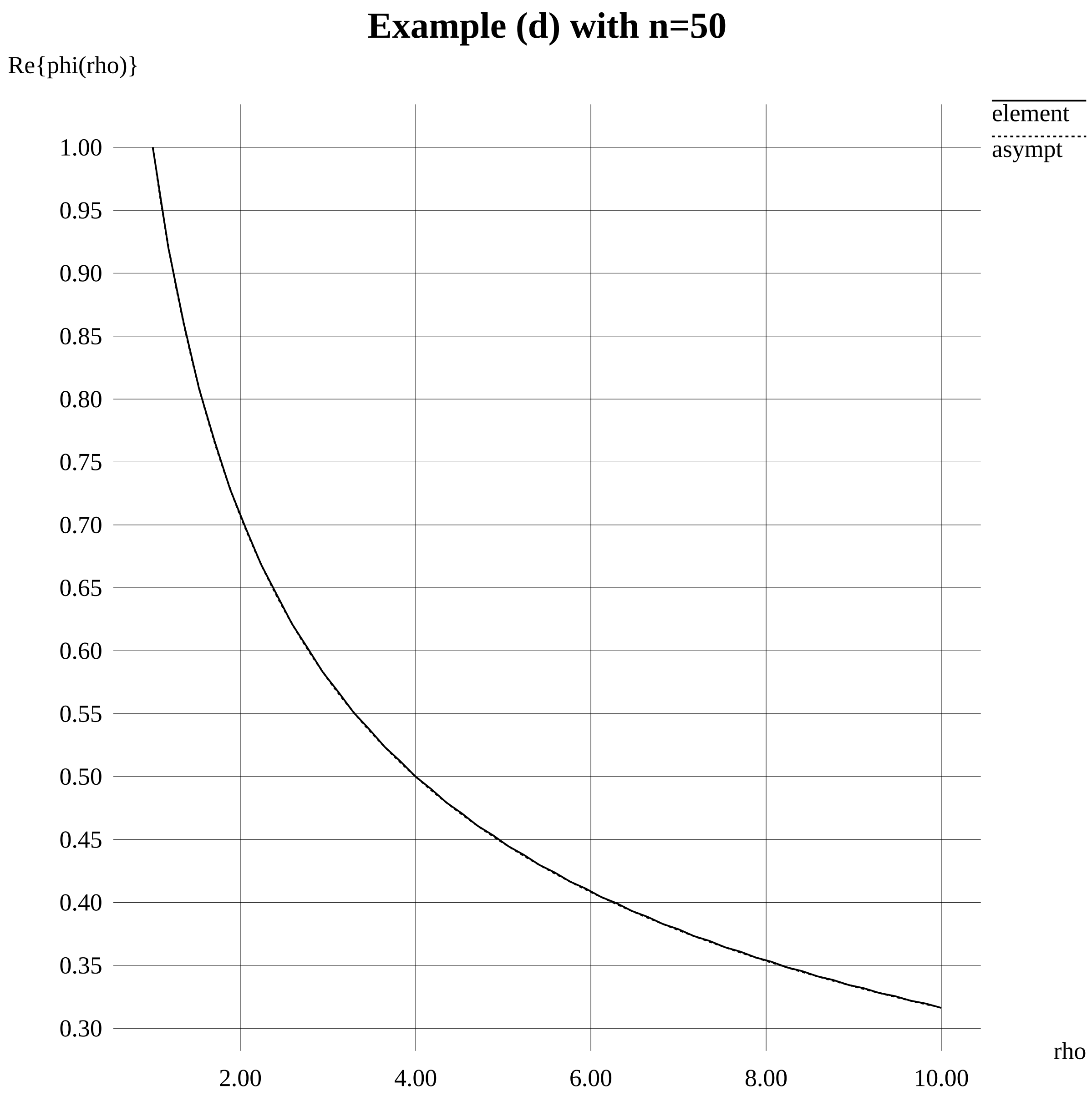


Figure 5. Example (d) $\mathrm{Re}[\tilde{\phi}(\rho)]$ vs. $\rho$ with $n = 50$ and $k = 10000$.

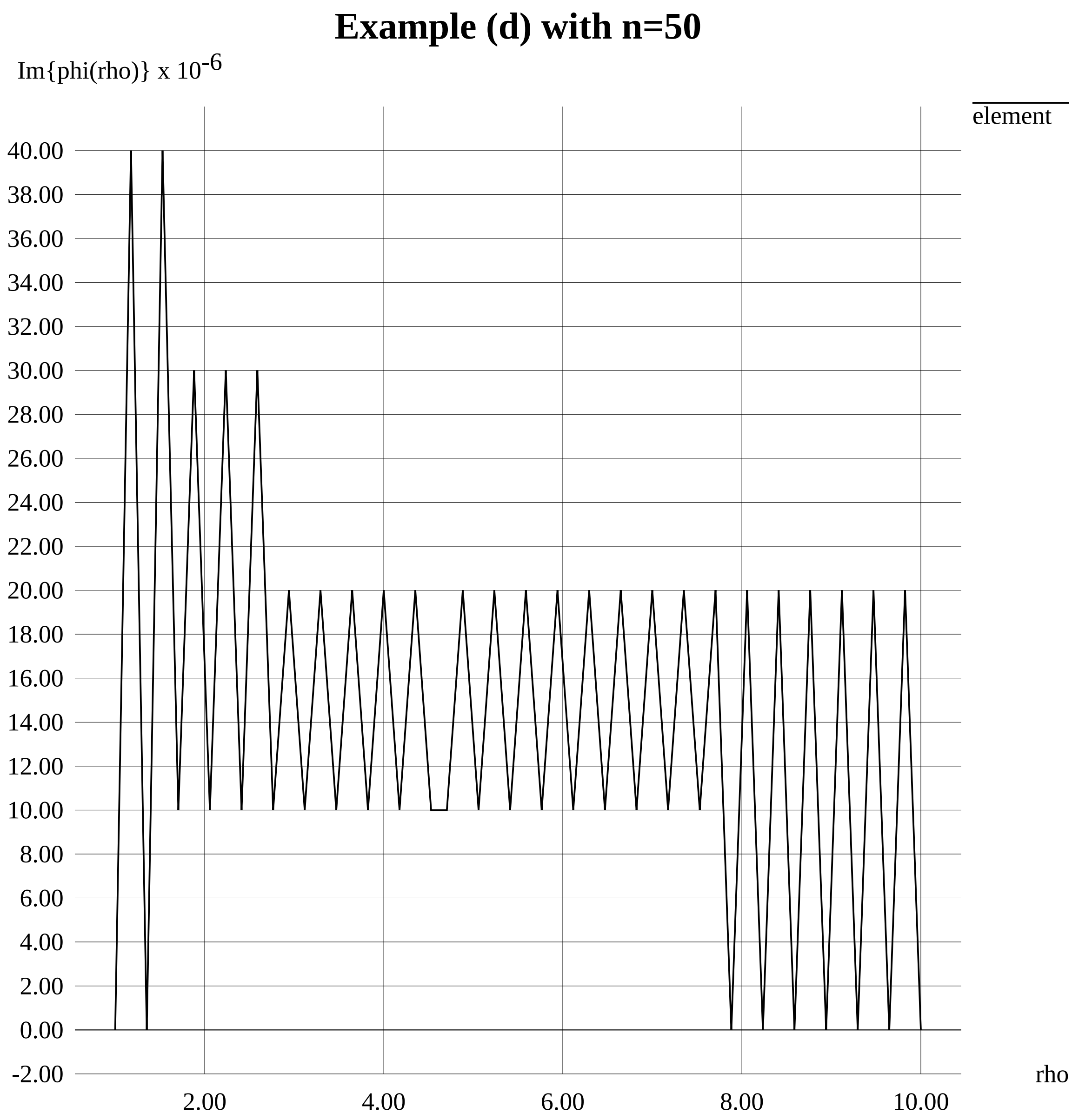


Figure 6. Example (d) $\mathrm{Im}[\tilde{\phi}(\rho)]$ vs. $\rho$ with $n = 50$ and $k = 10000$.

Thus the finite element approximation $\tilde{\phi}$ is complex, but as $n$ increases the imaginary part of $\tilde{\phi}$ approaches zero and the real part of $\tilde{\phi}$ approaches $\frac{1}{\sqrt{\rho}}$.

There is an imaginary component to $\phi_j$. However, as $n$ increases we see that the imaginary component of the finite element solution tends to zero.

We now consider example (e)

Example (e).

We model the propagating wave solution $\Phi = \frac{A_5}{r}e^{ikr}$ from the 3-D Helmholtz equation $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) + k^2\Phi = 0$. We let $A_5 = 1$ and look for the solutions satisfying the boundary conditions $\Phi(1) = e^{ik}$, $\Phi(10) = \frac{1}{10}e^{10ik}$. Following the Astley method, (26) gives

$$\begin{aligned}\{ & \int_{r_0}^{r_{n+1}} \left\{-\frac{d}{dr}(N_j e^{-ikr}) r^2 \frac{d}{dr}(N_i e^{ikr}) + k^2 N_j N_i r^2\right\} dr \\ + & \left[N_j e^{-ikr} r^2 \frac{d}{dr}(N_i e^{ikr}\right]_{r_0}^{r_{n+1}} \}\phi_i = 0_j \qquad (62)\end{aligned}$$

where $N_j$ is the piecewise linear triangle function defined by (17). Evaluating (62) gives

$$\begin{aligned}[1-ikL]\beta^*(j,L)\phi_{j-1} \quad &- \quad [\beta^*(j,L)+\beta^\dagger(j,L)]\phi_j \\ +[1+ikL]\beta^\dagger(j,L)\phi_{j+1} \quad &= \quad 0_j \qquad (63)\end{aligned}$$

$\beta^* = \int_0^L \frac{r^{*2}}{L^2}dr$, $r^* = r_0 + L(j-1) + r$
$\beta^\dagger = \int_0^L \frac{r^{\dagger 2}}{L^2}dr$, $r^\dagger = r_0 + Lj + r$

Given in matrix form $K_{jq}\phi_q = y_j$, then from (63) the stiffness matrix is

$$\begin{aligned}K_{jq} \quad &= \quad [1-ikL]\beta^*(j,L)\ , \quad q = j-1 \\ &= \quad -\beta^*(j,L) - \beta^\dagger(j,L)\ , \quad q = j \\ &= \quad [1+ikL]\beta^\dagger(j,L)\ , \quad q = j+1 \\ &= \quad 0 \ \text{ otherwise} \qquad (64)\end{aligned}$$

and the vector $y_j$ is given by

$$y_j \quad = \quad -[1-ikL]\beta^*(j,L)\phi_0\ ,\ j = 1$$

$$
\begin{aligned}
&= -[1+ikL]\beta^\dagger(j,L)\phi_{n+1}\ ,\ j=n \\
&= -[1-ikL]\beta^*(j,L)\phi_0 \\
&\quad -[1+ikL]\beta^\dagger(j,L)\phi_{n+1}\ ,\ j=1=n \\
&= 0\ ,\ \ \text{otherwise}
\end{aligned}
\tag{65}
$$

In particular, we consider $k = 100$ and let $n = 1$, 10 and 50. The matrix equation is solved numerically and the results plotted against the exact solution $\phi = \frac{1}{r}$. (We shall also consider a smaller wave number $k = 3$ and let $n = 10$. This enables us to plot the results against the exact wave solution $\Phi = \frac{1}{r}e^{ikr}$.)

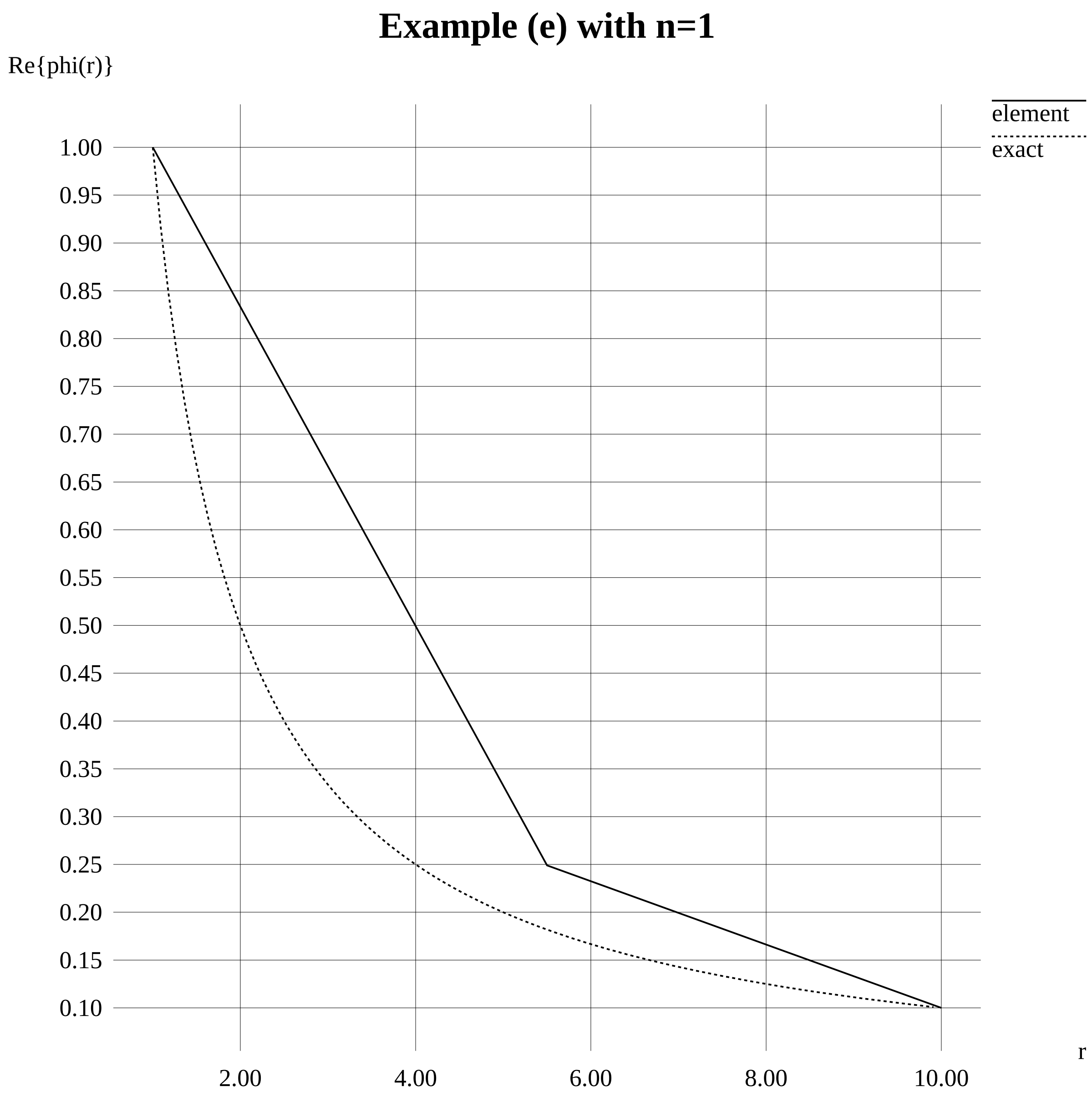


Figure 7. Example (e) $\mathrm{Re}[\tilde{\phi}(r)]$ vs. $r$ with $n = 1$ and $k = 100$.

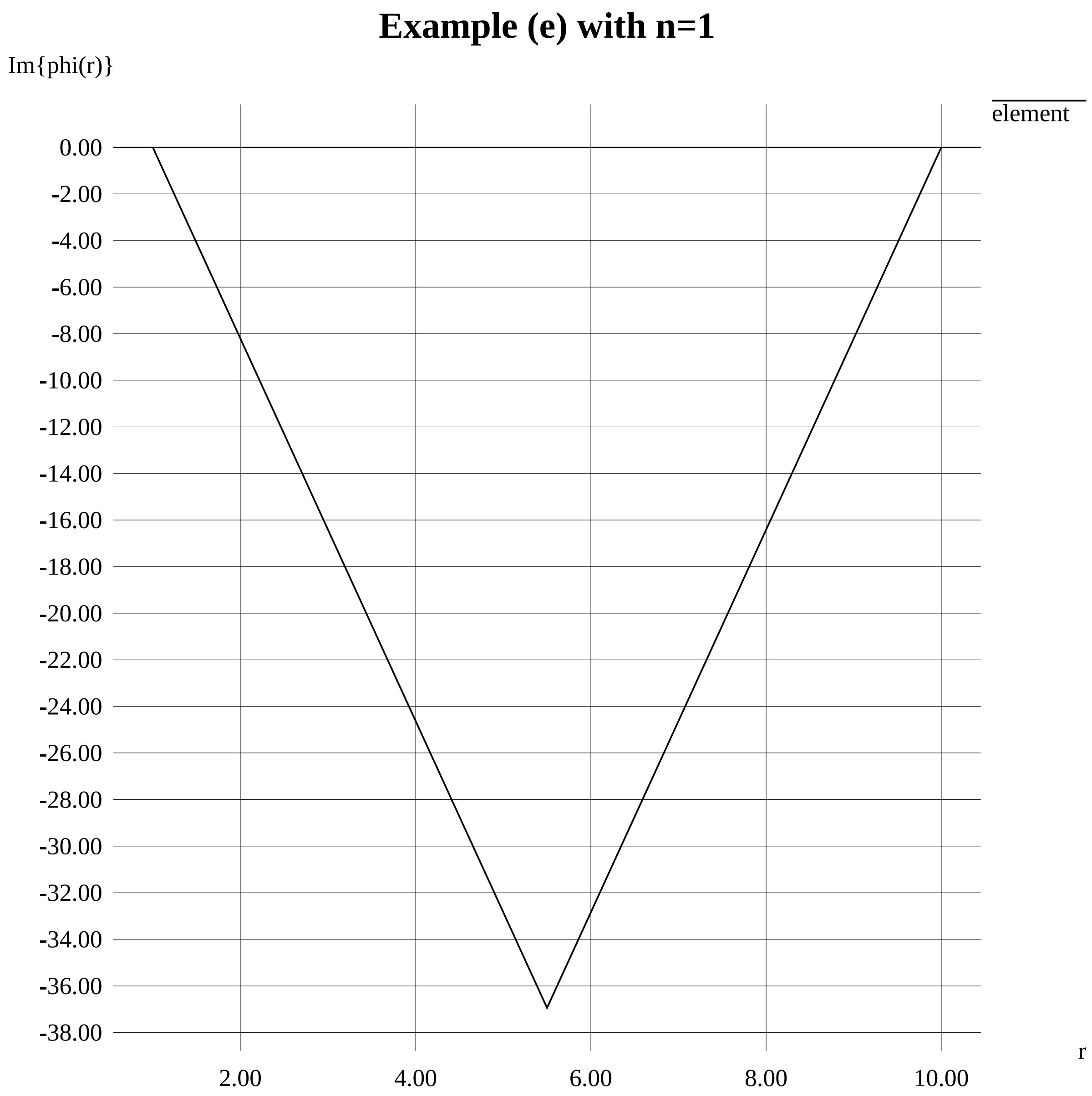


Figure 8. Example (e) $\mathrm{Im}[\tilde{\phi}(r)]$ vs. $r$ with $n = 1$ and $k = 100$.

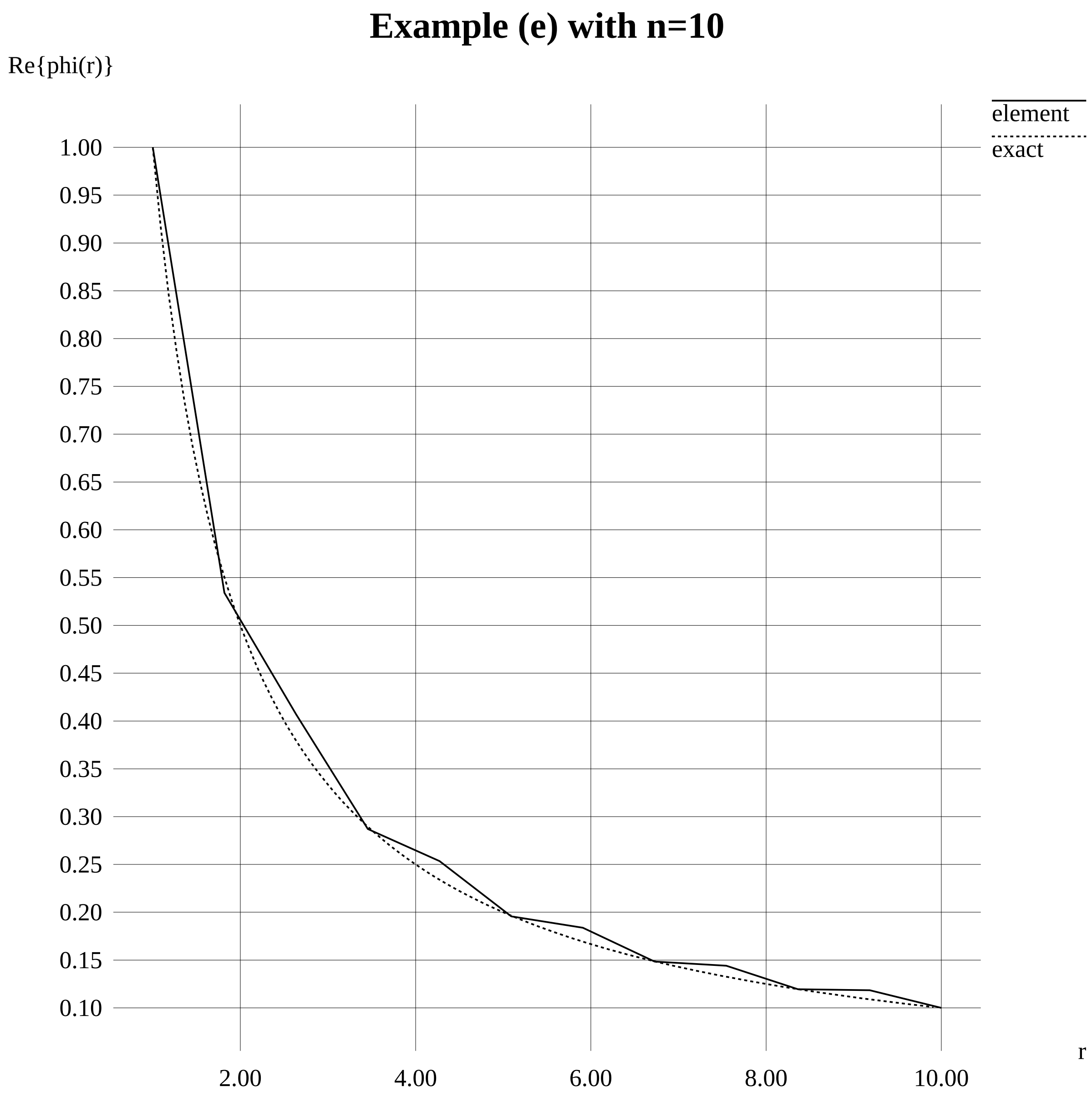


Figure 9. Example (e) $\mathrm{Re}[\tilde{\phi}(r)]$ vs. $r$ with $n = 10$ and $k = 100$.

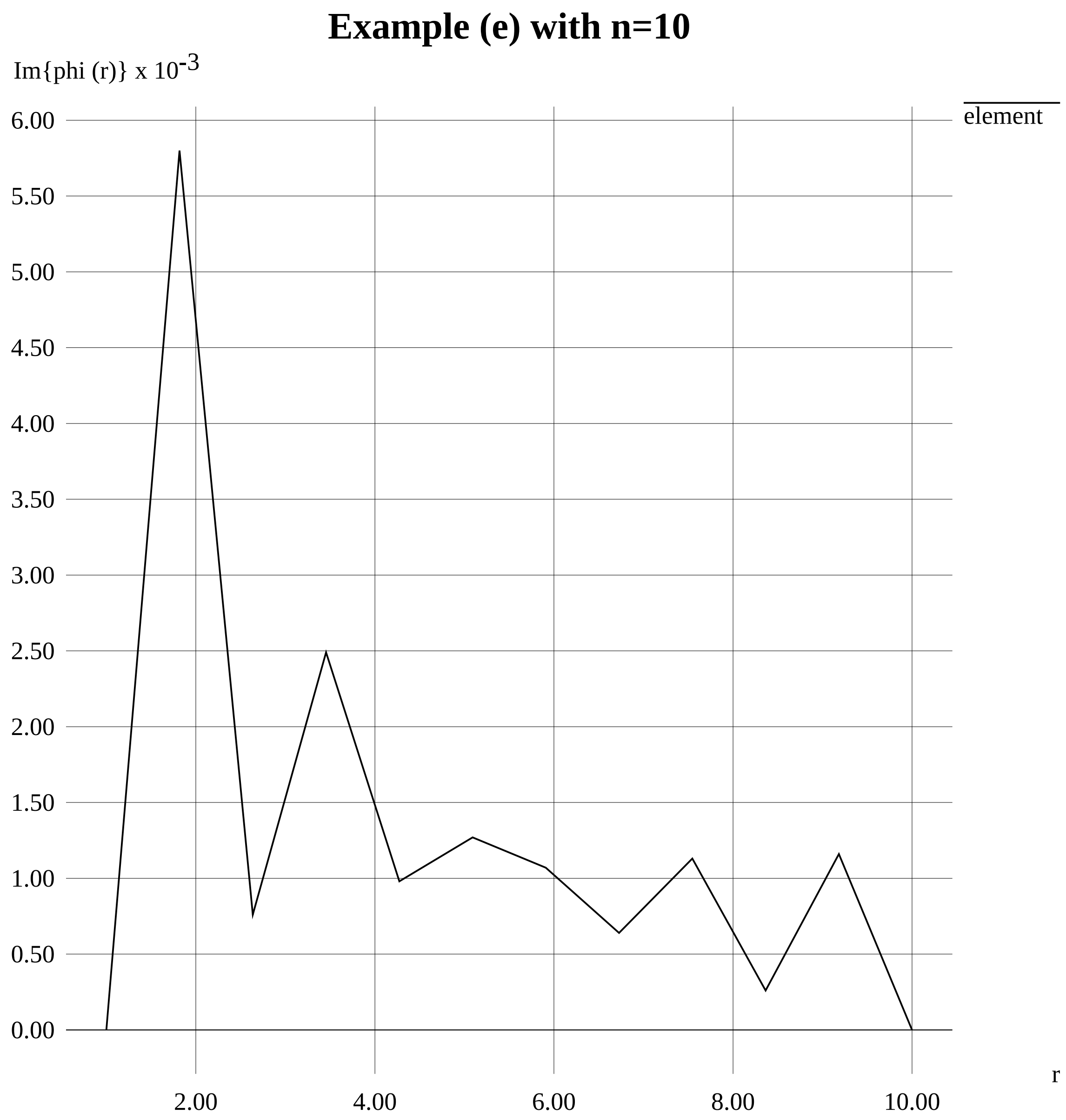


Figure 10. Example (e) $\mathrm{Im}[\tilde{\phi}(r)]$ vs. $r$ with $n = 10$ and $k = 100$.

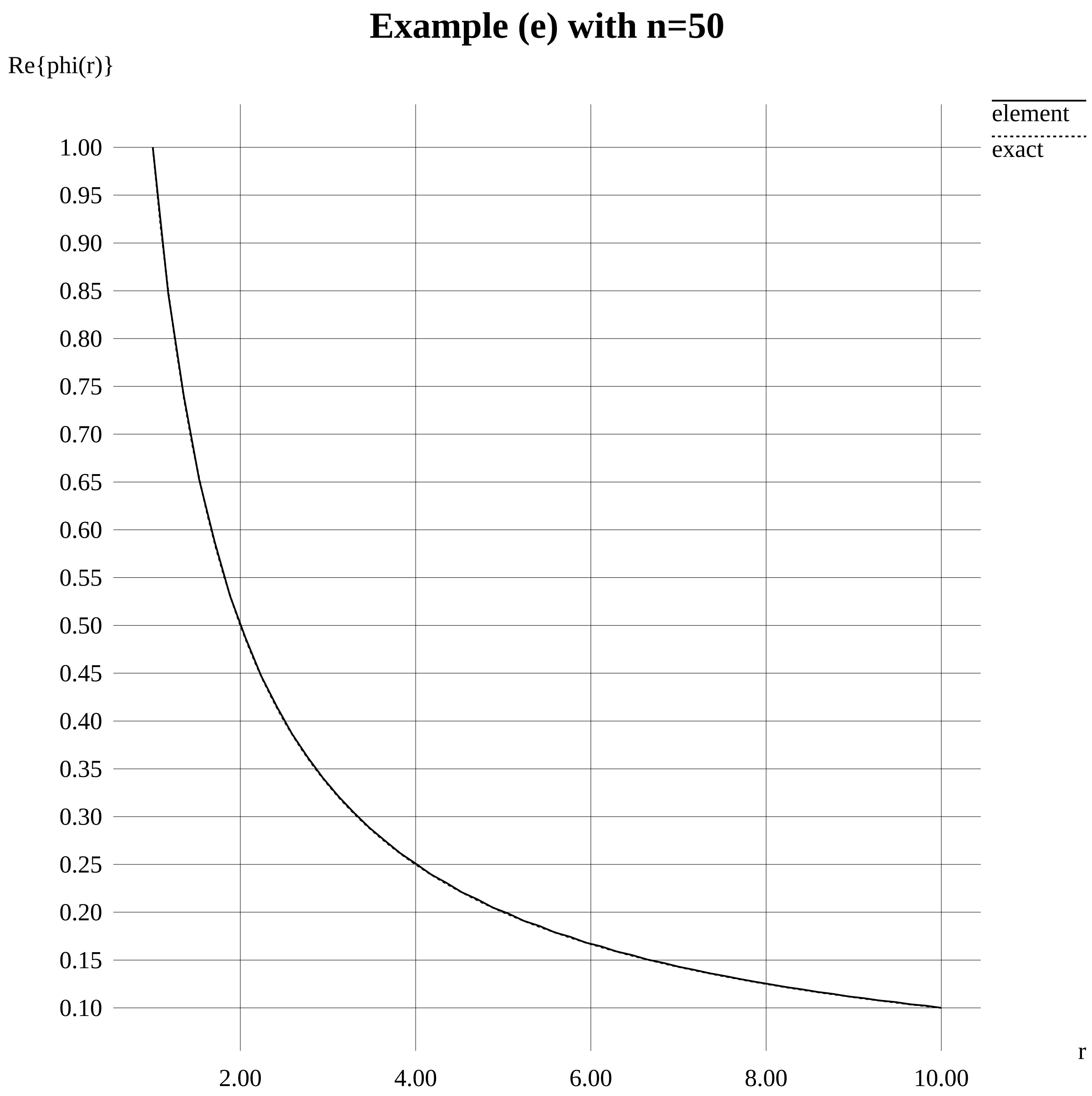


Figure 11. Example (e) $\mathrm{Re}[\tilde{\phi}(r)]$ vs. $r$ with $n = 50$ and $k = 100$.

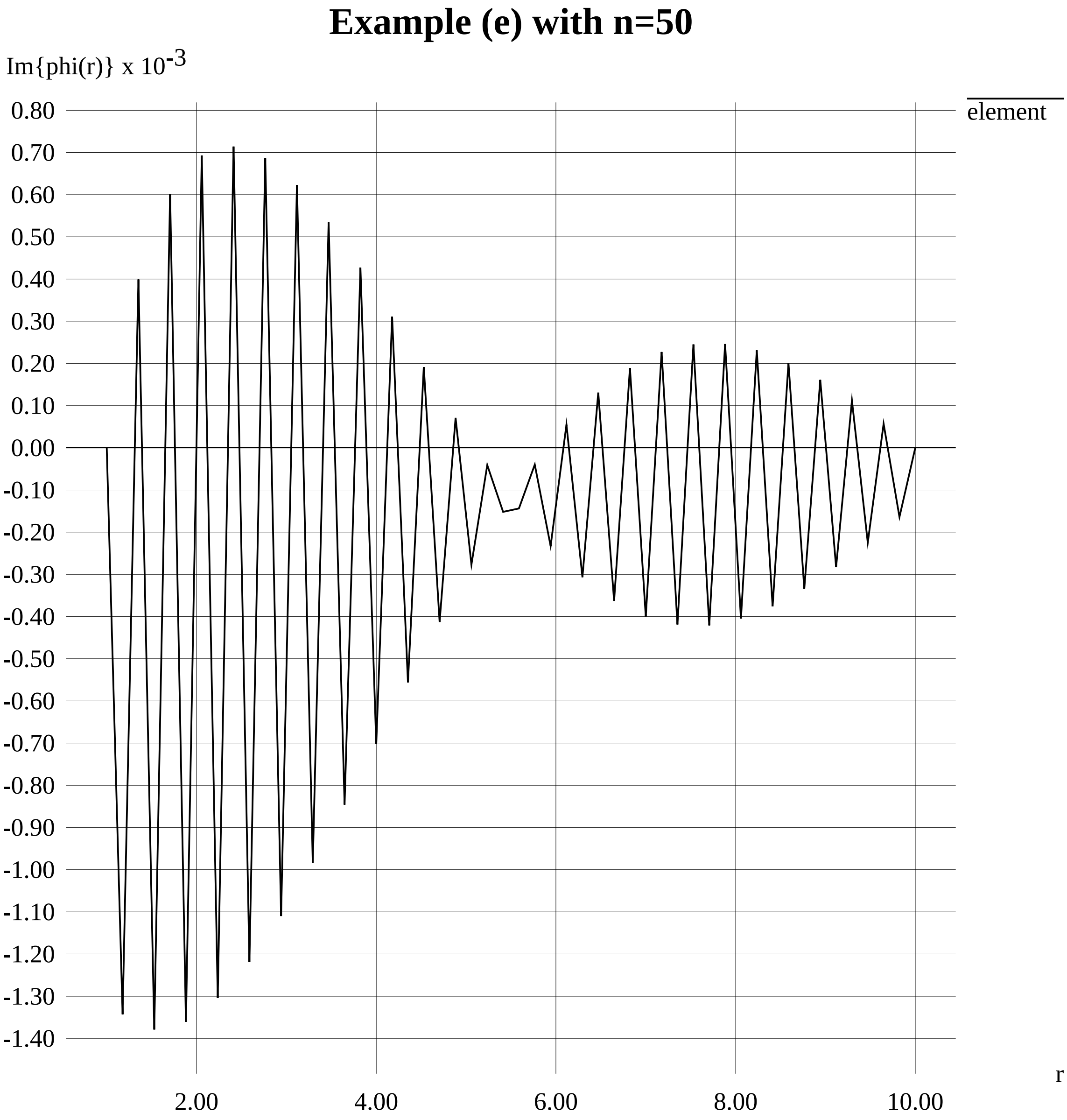


Figure 12. Example (e) $\mathrm{Im}[\tilde{\phi}(r)]$ vs. $r$ with $n = 50$ and $k = 100$.

Thus the approximation $\tilde{\phi}$ is complex, but as $n$ increases the imaginary part of $\tilde{\phi}$ approaches zero and the real part of $\tilde{\phi}$ approaches $\frac{1}{r}$.

We also consider a smaller wave number $k = 3$ and let $n = 10$ which enables us to plot the results against the exact wave solution $\Phi = \frac{1}{r}e^{ikr}$.

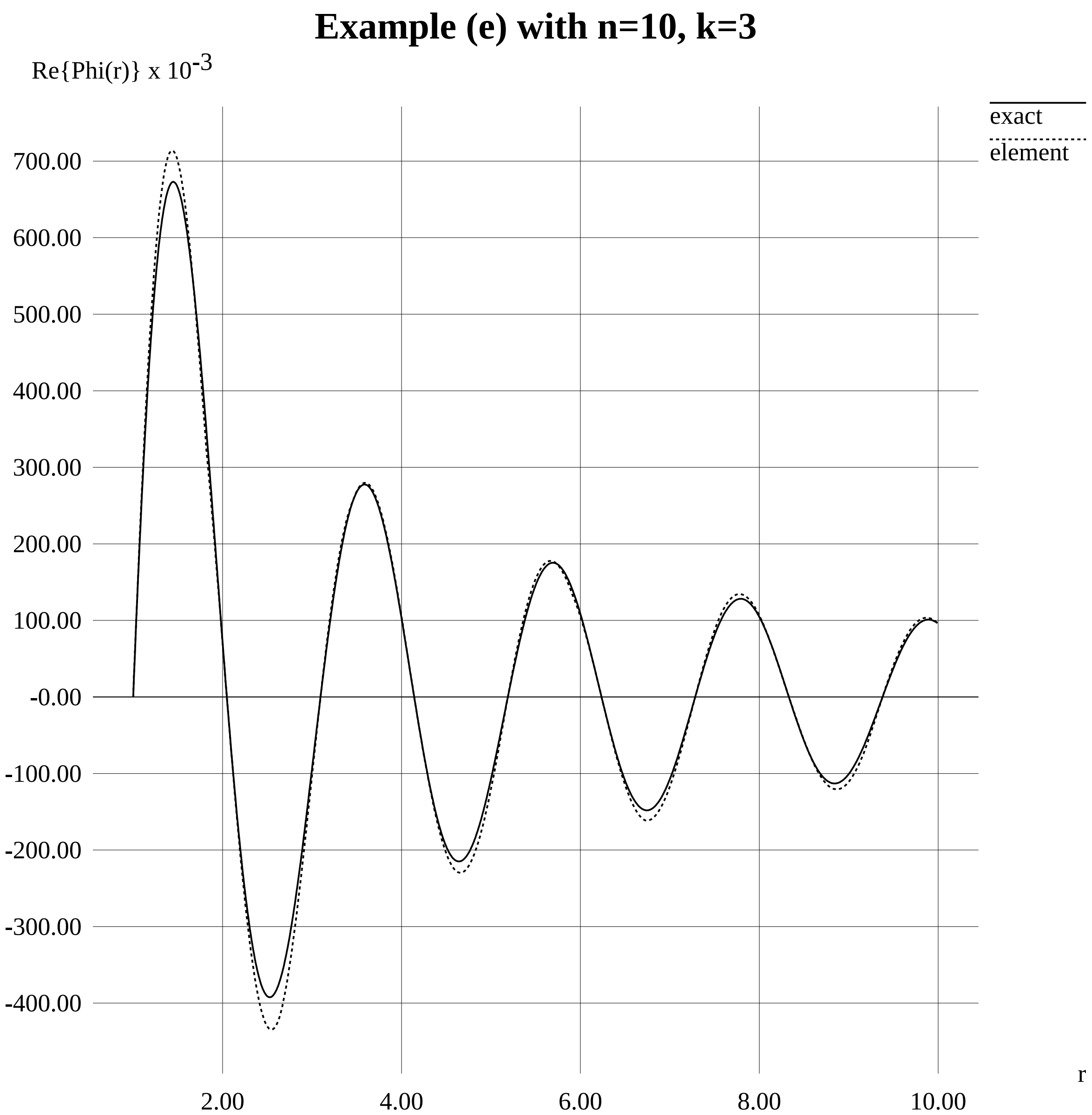


Figure 13. Example (e) $\mathrm{Re}[\tilde{\phi}(r)]$ vs. $r$ with $n = 10$ and $k = 3$.

We see that a good approximation to the progressive wave is achieved with only 10 elements, rather than about 50 (10 nodes per wavelength ) which would be needed using the standard finite element method to determine $\Phi$.

The results obtained for equations (a) to (e) are discussed next in the conclusion.

# 5 Conclusion

Astley's method (**1**) for finding a finite element wave potential approximation has been shown to be equivalent to the standard Galerkin finite element method used to find the wave envelope amplitude. Further, the method has been successfully tested for some simple progressive wave examples; the progressive short wave solutions are modelled with elements of length much greater than the wavelength.

The progressive wave solutions for the 1-D Helmholtz equation (example (a)) and the 1-D Helmholtz equation with a forcing term (given by example (b)) were modelled exactly.

Similarly, the progressive wave solution $xe^{ikx}$ satisfying the second order differential equation $\frac{d^2\Phi}{dx^2} + k^2\Phi = \frac{2ik}{x}$ was modelled exactly.

However, the progressive wave solutions dependent upon the 2-D and 3-D radial variable for the 2-D and 3-D Helmholtz equations respectively are complex and approximate; the choices $k = 10000$ for 2-D and $k = 100$ for 3-D were considered with the number of nodes $n = 1$, 10 and 50. (We also show that a good appproximation to the progressive wave solution in 3-D space dependent upon the 3-D radial variable for $k = 3$ can be successfully achieved by using only ten finite elements: using the standard Galerkin finite element method to model the progressive wave solution would require about 50 elements (ten nodes per wavelength).) We have shown that as the number of elements within the domain increases, the approximation approaches the progressive wave solution. For $n = 50$ in examples (d) and (e), the finite element solution approximates the exact solution to within the accuracy allowed by the computer program. (We use double precision complex numbers in a tridiagonal equation solver.)

The real test for the method, however, is if economical modelling of short waves can be achieved such that the progressive wave is dependent upon two and three variables. There is then the additional problem that the wave direction is also an unknown. This shall be considered in future work.

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