The far field Oseen velocity expansion

Edmund Chadwick

Department of Mathematics, University of Manchester,

Oxford Road, Manchester M13 9PL, England

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ABSTRACT

Consider uniform, steady, incompressible fluid flow in an unbounded domain past a fixed, closed body. In the far field the Oseen equations [1, pp30-38] approximately hold. We give the Oseen velocity and pressure expansions in the far field for two- and three-dimensional flow by using two approaches: the first decomposes the velocity into a potential velocity and a wake velocity, and was introduced by Lamb [2, pp611, 615] and Goldstein [3, 4]; the second approach uses the Oseen representation of the velocity and pressure by a Green's integral distribution of singularities called 'Oseenlets' [1, pp30-38]. These are the force (drag and lift) singular solutions.

We show that, in general, the Lamb-Goldstein approach will not model the Oseen velocity in the far field wake. In contrast, by using Oseen's representation to expand each Oseenlet in a Taylor series, we obtain velocity and pressure expansions everywhere in the far field.

Nevertheless, there are exceptions for which the Lamb-Goldstein approach can be used, notably: (i) the singular drag and lift solutions; (ii) axisymmetric flow, such that the axis of rotational symmetry is parallel to the uniform stream velocity direction; (iii) low Reynolds number flow, and (iv) two-dimensional flow. For these special cases, the two approaches are shown to be equivalent.

1 Introduction

The motivation for the research arises from the field of ship manoeuvrability for which the hydrodynamic forces and moments on the ship need to be determined. In the linearisation of the Navier-Stokes velocity to a uniform stream the highest order perturbation is the Oseen velocity and is enough to give the hydrodynamic forces. However, we note that the moments depend on terms of lower order which do not satisfy the Oseen equations (see [5] and [6]). In the present paper, though, we shall restrict our aim to obtaining the Oseen velocity and pressure expansions in the far field which could then be matched to the velocity and pressure of a near field flow in order to model the hydrodynamic forces on the ship.

Goldstein [4] gives a far field Oseen velocity expansion; he obtains expansions for the potential velocity and wake velocity in the far field by assuming that they are regular everywhere except along the infinite half line $x > 0$, $y = z = 0$; the cartesian co-ordinates (x, y, z) are chosen such that the x direction is in the direction of the uniform stream velocity U, and also the origin lies within the near field. However, we show that Goldstein's assumption of regularity is wrong (see section 6.1) and as a result there are terms in his expansion which have been inadvertently omitted. In fact, we give an example for which the potential velocity and wake velocity are indeterminate in the shadow region $r = \{x^2 + y^2 + z^2\}^{1/2} > r_0$ of the far field wake; r_0 represents the inner radial limit of the outer flow region over which the Oseen solution is assumed to be valid. Furthermore, a specific example of a singular lift located away from the $x-$axis is considered in section 6.2, for which it is shown that, by using the Lamb-Goldstein approach, the lift is evaluated incorrectly. This means that matching the velocity potential and wake velocity on the boundary $r = r_0$, which is common to both the near and far fields, will not work. A far field expansion which could be used instead is given in terms of the Oseen drag and lift singularities and is given in section 7.

However, for low Reynolds number flow the Lamb-Goldstein velocity decomposition has been widely used to model the far field Oseen equations [7, pp44-48]. As the Reynolds number tends to zero, then the radius of the far field inner boundary r_0' also tends to zero (see [8, p240] for example); $r_0' = \frac{\rho U}{\mu} r_0$, and so r_0' is dimensionless; ρ is the fluid density and μ is the dynamical coefficient of viscosity. This means that the shadow region will lie on the semi-infinite half line $(x > 0, 0, 0)$, and so the Goldstein potential expansion [4] for three-dimensional flow can be used to determine the far field velocity expansion. The Oseen outer flow is then usually asymptotically matched onto a Stokes near field flow ([9] and [10]). However, in ship manoeuvrability problems the Reynolds number is not small. So, even though the Lamb-Goldstein velocity decompostion has been successfully used to model low Reynolds number flows, unfortunately this does not mean it can be used to model problems in ship manoeuvrability. Even so, the Oseen approximation can still be used to model the far field: Oseen flow is applicable in the far field for all flows independent of the Reynolds number since the linearisation to a uniform stream is still valid.

2 Statement of the problem

Let the Oseen approximation $|\underline{u}^\dagger - U\hat{\underline{x}}| << U$ hold in the far field $r = \{x^2+y^2+z^2\}^{1/2} > r_0$; $\underline{u}^\dagger$ is the Navier-Stokes fluid velocity and $\hat{\underline{x}}$ is the unit vector in the x-direction which is taken to be the uniform stream direction; the uniform stream velocity is taken to be $U\hat{\underline{x}}$. The Oseen equations and the conditions satisfied by the Oseen velocity and pressure in the far field $r > r_0$ are now given.

Steady incompressible fluid flow is governed by the Navier-Stokes equation [2, p 577]:

$$\rho(\underline{u}^\dagger.\underline{\nabla})\underline{u}^\dagger = -\underline{\nabla}p^\dagger + \mu\nabla^2\underline{u}^\dagger \tag{1}$$

and the continuity equation

$$\underline{\nabla}.\underline{u}^\dagger = 0. \tag{2}$$

$p^\dagger$ is the Navier-Stokes pressure, ρ is the fluid density and is assumed to be constant, and μ is the dynamical coefficient of viscosity and is also assumed to be constant. $\underline{\nabla}$ is the gradient operator and ∇^2 is the Laplacian operator. We shall linearise the Navier-Stokes velocity to a uniform stream. Hence we consider

$$\begin{aligned} \underline{u}^\dagger &= U\hat{\underline{x}} + \underline{u} + O(\underline{\delta}) \\ p^\dagger &= p + O(\delta) \end{aligned} \tag{3}$$

where 'O' means 'of the order of', $U >> |\underline{u}| >> |\underline{\delta}|$ and $|p| >> \delta$. It is known that in determining the moment on the body, terms of order δ must also be considered [5], [6]. Substituting (3) into the Navier-Stokes equations (1) yields the Oseen equations [1, pp30-38], to first order, given by

$$\begin{aligned} \rho U\frac{\partial \underline{u}}{\partial x} &= -\underline{\nabla}p + (\mu\nabla^2)\underline{u} \\ \underline{\nabla}.\underline{u} &= 0. \end{aligned} \tag{4}$$

Also, as

$$r \to \infty, \quad \underline{u} \to 0. \tag{5}$$

$\underline{u}$ and p are the Oseen velocity and pressure respectively given in the region $r > r_0$.

We give the differential equation satisfied by the pressure.

Taking the divergence of the Oseen equations (4) gives

$$\nabla^2 p = 0. \tag{6}$$

Substituting the condition (5), that $\underline{u} \to 0$ as $r \to \infty$, into the Oseen equations (4) gives $\underline{\nabla p} \to 0$ as $r \to \infty$. Thus, we may choose

$$p \to 0 \ \text{ as } \ r \to \infty. \tag{7}$$

We also must satisfy the condition of no fluid outflow from any surface S_0 enclosing the body, and so

$$\int\int_{S_0} \underline{u}.\underline{n} ds = 0 \tag{8}$$

where ds is an element of surface S_0 in the far field region $r > r_0$. We now give the force on a body enclosed within the near field $r < r_0$ as a far field integral.

2.1 The force on the body

We use tensor notation such that a_i represents the vector $\underline{a} = (a_1, a_2, a_3)$ in cartesian co-ordinates $(x, y, z) = (x_1, x_2, x_3)$, $1 \leq i \leq 3$. The force on the body F_i can be expressed as an integral over a surface S_0 in the far field but enclosing the body. By using Green's integral theorem,

$$\begin{aligned} F_i &= \int\int_{S_B} \tau_{ij} n_j dS \\ &= \int\int_{S_B} \left\{ -p^\dagger \delta_{ij} + \mu \frac{\partial u_i^\dagger}{\partial y_j} - \rho u_i^\dagger u_j^\dagger \right\} n_j dS \\ &= \int\int_{S_0} \left\{ -p^\dagger \delta_{ij} + \mu \frac{\partial u_i^\dagger}{\partial y_j} - \rho u_i^\dagger u_j^\dagger \right\} n_j dS \end{aligned} \tag{9}$$

since the volume integral is zero from (1); τ_{ij} is the stress tensor, $\underline{u}^\dagger$ and $p^\dagger$ are the Navier-Stokes fluid velocity and pressure respectively; S_B is the body surface; $1 \leq i, j, \leq 3$. Substituting the Oseen linearisation (3) into (9) and then neglecting the terms that do not contribute to the force and taking S_0 to infinity

$$F_i = \int\int_{S_0} \left\{ -p\delta_{ij} + \mu \frac{\partial u_i}{\partial y_j} - \rho U u_i \delta_{j1} \right\} n_j dS \tag{10}$$

since, from (8) $\int\int_{S_0} u_j n_j ds = 0$. Similarly, the moment is given by

$$\begin{aligned} M_i &= \int\int_{S_B} \epsilon_{ijk} y_j \tau_{kl} n_l dS \\ &= \epsilon_{ijk} \int\int_{S_0} \left\{ y_j(-p^\dagger \delta_{kl} + \mu \frac{\partial u_k^\dagger}{\partial y_l} - \rho u_k^\dagger u_l^\dagger) - \mu u_k^\dagger \delta_{jl} \right\} n_l dS. \end{aligned} \tag{11}$$

Thus, higher order terms in the Oseen approximation of the pressure and velocity (3) must also be considered in the moment calculation. However, we shall determine the contribution to the moment M_i^0 from the terms in the Oseen velocity and pressure expansions only. Neglecting terms that do not contribute to the moment gives

$$M_i^0 = \epsilon_{ijk} \int\int_{S_0} \left\{ y_j(-p\delta_{kl} + \mu \frac{\partial u_k}{\partial y_l} - \rho U u_l \delta_{k1} - \rho U u_k \delta_{l1}) - \mu u_k \delta_{jl} \right\} n_l dS \tag{12}$$

since $\int\int_{S_0} y_j n_l dS = \delta_{jl}$ for $j \neq l$ and $\epsilon_{ijk} = -\epsilon_{ikj}$. (Thus $\epsilon_{ijk} b_{jk} = -\epsilon_{ikj} b_{jk}$ for $b_{jk} = \int\int_{S_0} -\rho U y_j u_l \delta_{k1} n_l dS$.)

We now consider the Lamb-Goldstein velocity decomposition.

2.2 The Lamb-Goldstein velocity decomposition

We consider the Lamb-Goldstein velocity decomposition

$$\underline{u} = \underline{\nabla}\phi + \underline{w} \tag{13}$$

replacing Goldstein's definition $\frac{\partial \phi}{\partial x} = -\frac{p}{\rho U}$ by

$$\phi = -\frac{1}{\rho U} \int_{-\infty}^{x} p(x', y, z) dx'. \tag{14}$$

We note that initially $\phi(\xi, \eta, \zeta)$ is only defined if the line $x = \xi + l$, $y = \eta$, $z = \zeta$, where $-\infty < l < \xi$, does not intersect the near field (where Oseen flow is not valid). We then continue analytically (if possible) into the shadow region $\{y^2 + z^2\}^{1/2} \leq r_0$, $x > [r_0^2 - y^2 - z^2]^{1/2}$. The near field, far field and shadow region are shown in the figure below.

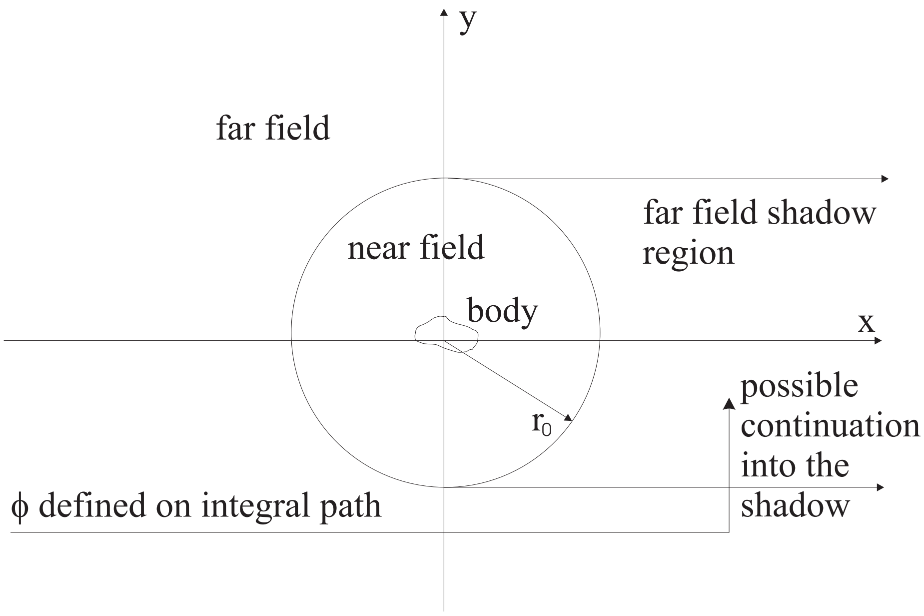

Figure 1. The different regions in the flow for general values of the Reynolds number.

Thus ϕ is undefined by (14) in the shadow region. We shall find later (section 6) that, in general, ϕ is undefined in the shadow, and so (14) is a more precise definition than Goldstein's.

Substituting (14) and (13) into (4) gives the equation for the wake velocity

$$(\nabla^2 - 2k\frac{\partial}{\partial x})\underline{w} = \underline{0} \tag{15}$$

where $k = \frac{\rho U}{2\mu}$. The potential velocity $\underline{\nabla}\phi$, and therefore the velocity $\underline{w}$ need not be single-valued. Further, we shall find that they may not even be able to be determined within the shadow region.

For the cases where the Lamb-Goldstein representation is applicable, we can ask whether the representation is unique; equivalently, for the far field solution $\underline{u} = \underline{0}$, $p = 0$, we can ask whether the solution for ϕ and w is $\phi = 0$, $\underline{w} = \underline{0}$. From (14), continuing ϕ into the shadow we see that $\phi = 0$ everywhere in the far field, and hence from (13) $\underline{w} = \underline{0}$.

3 Singular force solutions

The drag solution has been given by Lamb [2, p616] and the drag and lift solutions by Oseen [1, pp30-38]. Lamb's theory, however, gives little explanation of the properties of the function χ which he introduces. We expand upon this in the present section. Also, we give the two singular force solutions in the form of the Lamb-Goldstein velocity representation $\underline{u} = \underline{\nabla}\phi + \underline{w}$. This is important for showing the equivalence of the expansions obtained from the Lamb-Goldstein theory and the Oseen representation given in section 7.

3.1 Drag

Lamb [2, p616] represents the wake velocity $\underline{w}$ by the vorticity potential χ. We expand upon Lamb's definition, and then give the drag singular solution. The direction of the vorticity $\underline{\Omega}$ for axisymmetric flow is along the tangent to the circle of circumference r^* where $y = r^* \cos\beta$, $z = r^* \sin\beta$. (r, β) are polar co-ordinates in the $x-$plane. (The vorticity is defined as $\underline{\Omega} = \underline{\nabla} \times \underline{u}$ where $\underline{\nabla}\times$ is the curl operator.) Hence $\underline{\Omega} = \Omega\underline{\hat{\beta}}$ where $\hat{\beta}$ is the unit normal in the angle β direction. We shall define χ as a potential of the function Ω by

$$\chi(r^*, x) = \int_{\infty}^{r^*} \Omega(r^{*'}, x) dr^{*'}. \tag{16}$$

Thus χ is finite everywhere in the far field if we assume that for $r^* \to \infty$, then $\Omega = o(1/r^*)$, where 'o' means 'of lower order than'.

Also, since Ω and all its derivatives with respect to x and y are defined uniquely at each point, then from (16), χ and all its derivatives are also defined uniquely and are finite at each point in the far field. Thus, from the Taylor series expansion, χ and its derivatives are continuous.

Lamb [2, p611] gives the singular drag solution

$$\begin{aligned} p &= \frac{\partial}{\partial x}\left(\frac{1}{4\pi r}\right) \\ \phi &= -\frac{1}{\rho U}\left(\frac{1}{4\pi r}\right) \\ \chi &= \frac{k}{2\pi\rho U}\frac{e^{-k(r-x)}}{r} \end{aligned} \tag{17}$$

such that the wake velocity is given by

$$\begin{aligned} 2kw_1 &= \frac{\partial\chi}{\partial x} - 2k\chi \\ 2kw_2 &= \frac{\partial\chi}{\partial y} \\ 2kw_3 &= \frac{\partial\chi}{\partial z} \end{aligned} \tag{18}$$

where $\underline{w} = (w_1, w_2, w_3)$. Applying the far field force integral (10) gives a unit drag force due to the action of this singular solution.

3.2 Lift

Oseen [1, pp30-38] gives the unit lift solution. We give this in the Lamb-Goldstein velocity form $\underline{u} = \underline{\nabla}\phi + \underline{w}$. This is

$$\begin{aligned} p &= \frac{\partial}{\partial y}\left(\frac{1}{4\pi r}\right) \\ \phi &= \frac{1}{4\pi\rho U}\frac{\partial}{\partial y}\left[\log(r-x)\right] \\ \chi^* &= -\frac{1}{4\pi\rho U}e^{-k(r-x)}\frac{\partial}{\partial y}\left[\log(r-x)\right] \\ w_2^* &= \left(0, -\frac{1}{4\pi\mu}\frac{e^{-k(r-x)}}{r}, 0\right) \end{aligned} \tag{19}$$

where the wake velocity is given by

$$\underline{w} = \underline{\nabla}\chi^* + \underline{w}^*. \tag{20}$$

Applying the far field force integral (10) to this solution gives unit lift in the $y-$direction.

4 Axisymmetric flow

We give an expansion for axisymmetric Oseen flow by considering the two potentials ϕ (velocity potential) and χ (vorticity potential) defined in 3.2.

We shall first show that ϕ is determined everywhere in the Oseen flow region $r > r_0$ for axisymmetric flow and then obtain Lamb's [2, p611] expansion for ϕ which he gives for low Reynolds number flow past a sphere. We define χ (which Lamb uses to determine the wake velocity) as a vorticity potential by a line integral of the vorticity in the radial cross-section direction $r^* = (y^2 + z^2)^{1/2}$. We shall give an expansion for χ.

4.1 The velocity potential ϕ expansion

We determine the expansion of ϕ for axisymmetric flow. From (6) and (7) and using the result in appendix A, the pressure expansion is

$$p = \sum_{n=1}^{\infty}\frac{S_n(\Theta)}{r^{n+1}} = \sum_{n=1}^{\infty}E_{n-1}\frac{\partial^n}{\partial x^n}\left(\frac{1}{r}\right) \tag{21}$$

where $S_n(\Theta)$ are spherical surface harmonics of degree n; (r, Θ, β) are spherical polar coordinates such that $x = r\cos\Theta$, $y = r\sin\Theta\cos\beta$, $z = r\sin\Theta\sin\beta$;

E_n are constant coefficients. Hence the potential is given by

$$\phi = -\frac{1}{\rho U} \sum_{n=0}^{\infty} E_n \frac{\partial^n}{\partial x^n} \left(\frac{1}{r} \right) \tag{22}$$

We see that (22) is convergent in the shadow region and so an analytic continuation for ϕ into the shadow is given by (22).

4.2 The vorticity potential χ expansion

We shall now find the differential equation satisfied by χ which is defined by (16) and from this its far field expansion. Taking the curl of (15), $(\nabla^2 - 2k\frac{\partial}{\partial x})\underline{w} = \underline{0}$, gives $(\nabla^2 - 2k\frac{\partial}{\partial x})\Omega = 0$, and so substituting into (16) we get

$$\left(\nabla^2 - 2k \frac{\partial}{\partial x} \right) \chi = 0. \tag{23}$$

Thus

$$(\nabla^2 - k^2)(\chi e^{-kx}) = 0. \tag{24}$$

Considering spherical polars, (r, Θ, β) where $x = r\cos\Theta$, $y = r\sin\Theta\cos\beta$ and $z = r\sin\Theta\sin\beta$, we see that $\chi(R, \Theta)$ and applying Fourier's theorem (see for example [11]) to the variable Θ gives us an expansion for χ with harmonic terms. The resulting function in r from separation of variables can then be found by considering the expansion in the form $\chi = e^{kx} \sum_{n=0}^{\infty} F_n(kr) S_n(\Theta)$, where $S_n(\Theta)$ are spherical surface harmonics. (This expansion form is also considered, for example, by Carlsaw and Jaegar [12, p142] for the Helmholtz equation.) Letting $F_n(kr) = f_n(kr)\frac{e^{-kr}}{r}$, then $f_n = \frac{b_n^*}{r^n}$ since $\chi \to 0$ as $r \to \infty$; b_n^* are constant coefficients. The expansion for χ is therefore

$$\chi = e^{-k(r-x)} \sum_{n=0}^{\infty} \frac{1}{r} \left(b_n^* \frac{S_n(\Theta)}{r^n} \right). \tag{25}$$

An equivalent expansion for χ to (25) is given by

$$\chi = \sum_{n=0}^{\infty} C_n^* \frac{\partial^n}{\partial x_n} \left(\frac{e^{-k(r-x)}}{r} \right) \tag{26}$$

with $-\frac{E_0}{\rho U} = \frac{F_0}{2k} = \frac{D}{4\pi\rho U}$, where C_n^* are constants.

5 Low Reynolds number flow

For low Reynolds number flow, the near field boundary lies on the surface $r_0' = \frac{\rho U}{\mu} r_0 \to 0$ (see [8, p240] for example), and so Oseen's approximation is valid throughout the entire flow region. This means that ϕ is defined from the line integral of the pressure p given by (14) everywhere in the far field except along the positive $x-$axis, and that the Lamb-Goldstein approach can be used in low Reynolds number flow: the pressure expansion is expressed as a summation of spherical harmonics p_n [2, p110]; ϕ_n is obtained by integration of the spherical harmonic p_n, and ϕ is given by $\phi = \sum_{n=1}^{n=\infty} \phi_n$.

The pressure is expanded as

$$p = \sum_1^\infty \frac{S_n(\Theta, \beta)}{r^{n+1}} = \sum_1^\infty \frac{p_n(x, y, z)}{r^{2n+1}} \tag{27}$$

where $S_n(\Theta, \beta)$ are spherical surface harmonics of degree n, $p_n(x, y, z)$ are solid spherical harmonics of degree n; In the appendix A we show that (27) can be expressed as

$$p = \sum_{n=1}^\infty C_n p_n(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(\frac{1}{r}) \tag{28}$$

where

$$C_n = \frac{(-1)^n}{1.3.5...(2n-1)}.$$

Using the definition of ϕ by (14), we consider ϕ_n such that

$$\begin{aligned} \phi_n &= -\frac{1}{\rho U} \int_{-\infty}^x \frac{p_n(x', y, z)}{r^{2n+1}} dx' \\ &= \frac{1}{\rho U} c_n p_n(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \log(r - x). \end{aligned} \tag{29}$$

In the far field upstream from the body the expansion for ϕ

$$\phi = \sum_{n=1}^\infty \phi_n = \frac{1}{\rho U} \sum_{n=1}^\infty C_n p_n(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \log(r - x) \tag{30}$$

is convergent since as $x \to -\infty$, $\frac{\partial}{\partial y} \log(r - x) = \frac{y}{r(r-x)} = O(x^{-2})$ and $\frac{\partial}{\partial z} \log(r - x) = \frac{z}{r(r-x)} = O(x^{-2})$. The terms in (30) are given by Goldstein [4] and the expansion is only valid for low Reynolds number flow where the shadow region is along the infinite half line $x > 0$, $y = z = 0$. The difficulties with the Lamb-Goldstein approach arise when this analysis is extended to the general case.

For general flow, we use the right hand side of (30) as a continuation of ϕ into the shadow region. However, the expansion is not necessarily convergent in the shadow region. This may be seen more clearly if we consider the function $\Phi(y,z) = \int_{-\infty}^{\infty} p(x',y,z)dx'$ which satisfies the 2-D Laplace equation outside the circle C for which $\{y^2+z^2\}^{1/2} = r_0$, and $\Phi \to 0$ as $r \to \infty$. Thus the problem of continuing ϕ into the shadow region is equivalent to that of continuing Φ into the interior of C. In general, Φ will not vanish identically outside C, and so we expect Φ to be singular somewhere inside C. Hence ϕ is not convergent in the shadow region. For general flow this means ϕ cannot be found in the shadow region. Such a case is given in the next section. An example is also given whereby the inconsistencies which arise when using the Lamb-Goldstein approach lead to an incorrect evaluation of the lift.

6 Failure of the Lamb-Goldstein approach

In this section we give two cases which exhibit the failure of the Lamb-Goldstein approach to correctly model far field Oseen flow. The first case has a potential which is indeterminate everywhere in the shadow, counter to Goldstein's assertion [4] that the potential is regular in the shadow except along the positive x-axis. The second case describes a matching procedure whereby, with the application of the Lamb-Goldstein approach, the potential velocity and wake velocity are matched on a near/far field boundary. The resulting evaluation of the lift by a far field integral contour (10) is shown to be incorrect.

6.1 Counter example

We will now give a counter example in which ϕ is not convergent in the shadow. First consider the flow given by the Oseen pressure

$$p_1^* = \frac{r_0}{\sqrt{x^2+(y+r_0)^2+z^2}} - \frac{r_0}{r}.$$

This gives an Oseen potential $\rho U \phi_1^* = \log\{[x^2+(y-r_0)^2+z^2]^{1/2} - x\} - \log(r - x)$ which is singular along $x>0, y=0, z=0$ and $x>0, y=r_0, z=0$. We shall now determine the Goldstein expansion for the Oseen potential in the far field for this example. From (28), the Goldstein expansion for the pressure is given by

$$p_1^* = \sum_{n=1}^{\infty} \frac{r_0^{n+1}}{n!} \frac{\partial^n}{\partial y^n} \left(\frac{1}{r} \right) \tag{31}$$

which is absolutely convergent in the Oseen flow region $r > r_0$. We may determine the Goldstein expansion for ϕ_1^* from the expansion of the pressure given by (31). From (30)

$$\phi_1^* = \frac{1}{\rho U} \sum_{n=1}^{\infty} \frac{r_0^{n+1}}{n!} \frac{\partial^n}{\partial y^n} \left[\log(r - x) \right]. \tag{32}$$

For large x

$$\phi_1^* = \frac{1}{\rho U} \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n-1} r_0^{n+1} \frac{\cos n\beta}{r^{*n}} \left(1 + O(1/x)\right)$$

where $y = r^* \cos\beta$ and $z = r^* \sin\beta$. ϕ_1^* is divergent for large positive x along $(x, r_0, 0)$ because $\sum_{n=1}^{\infty} 1/n$ is divergent.

Thus, ϕ_1^* expressed by its Lamb-Goldstein potential expansion, (32), has a discontinuity along the infinite half line $x > 0, y = r_0, z = 0$. And so, the Lamb-Goldstein potential expansion has half line spikes in the shadow along which the potential is singular. Therefore, we can also construct a counter example in which ϕ is indeterminate everywhere in the shadow. Consider the pressure field given by

$$p_2^* = \int_0^{2\pi} \int_0^{r_0} \left\{ \frac{r_0^*}{\sqrt{x^2 + r_0^{*2}(1 - \cos\beta^*) + z^2}} - \frac{r_0^*}{r} \right\} f(r_0^*) dr_0^* g(\beta^*) d\beta^*$$

for some regular functions $f(x)$ and $g(x)$. Generally, the resulting potential ϕ_2^* will be indeterminate in the shadow.

6.2 An application which uses matching

The inconsistencies which arise with the use of the Lamb-Goldstein approach are evident in the example which follows. Consider a numerical model which matches an outer Oseen flow region to an inner region on the boundary $r = r_0$. Using the Lamb-Goldstein approach, the potential velocity and wake velocity are matched on this boundary. We can ask whether such a scheme accurately models a lift Oseen singularity located at $(0, r_0/2, 0)$.

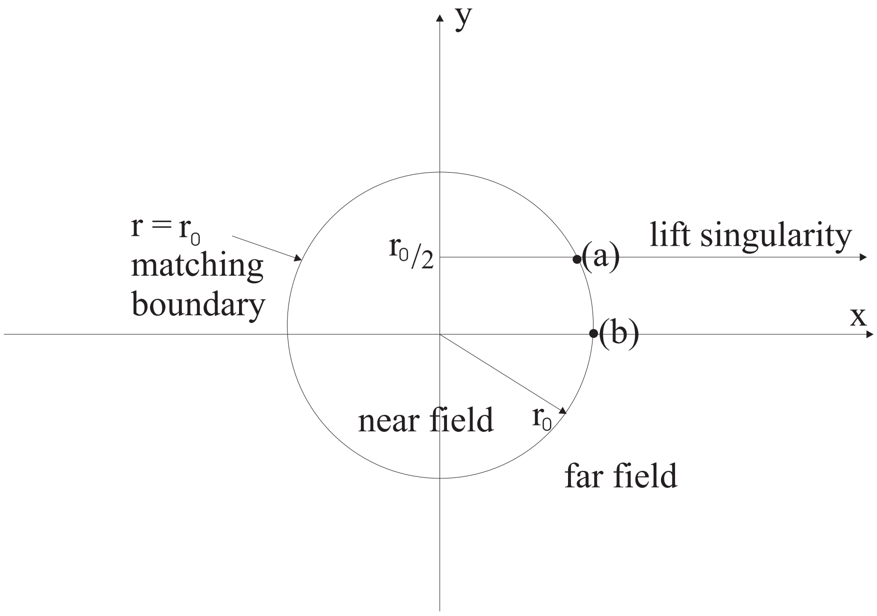

Figure 2. The matching boundary.

Consider matching at two points on the boundary:
(i) Near the point (a) at $x > 0, y = r_0/2, z = 0, r = r_0$, the potential velocity is asymptotically large. This means the Oseen wake velocity is also asymptotically large near (a). Since the wake velocity is (wrongly) assumed to be regular by Goldstein near this point, then matching near (a) is liable to give very large coefficients in the Goldstein wake velocity, and this leads to an over-estimation of the lift, by several orders of magnitude.
(ii) Near the point (b) at $x = r_0, y = 0, z = 0$, the potential velocity is finite. Correspondingly, the wake velocity is also finite. Since Goldstein's wake velocity term for the lift is singular at this point, then matching near (b) is liable to give an under-estimation of the lift, by several orders of magnitude.

Consequently, a matching procedure using the Lamb-Goldstein approach is likely to give the wrong value for the lift. In preference, the far field Oseen flow may be modelled by a near field distribution of Oseenlets (the singular lift and drag solutions, section 3.1, 3.2). This representation follows.

7 General flow

We give the far field Oseen velocity and pressure expansions in terms of the force singularity solutions (3.1) and (3.2). These are obtained from Oseen's representation of the Oseen velocity and pressure by substituting the Taylor series expansions for the Oseenlets into the integrals. The coefficients in the expansions are related to the hydrodynamic forces on the body.

For clarity, we shall use tensor suffix notation in which the velocity is given by $\underline{u} = (u_1, u_2, u_3)$ and the cartesian coordinate system is given by $\underline{x} = (x_1, x_2, x_3) = (x, y, z)$.

Consider four solutions to the Oseen equations given by $(\underline{u}, p)$ and $(\underline{u}^{(m)}, p^{(m)})$, $1 \leq m \leq 3$, for the Oseen velocity and pressure respectively. From (4) we find that

$$\begin{aligned}\frac{\partial}{\partial x_1}\left\{\rho U u_i(\underline{x}) u_i^{(m)}(\underline{a}-\underline{x})\right\} &= -\tfrac{\partial}{\partial x_i}\left\{p(\underline{x}) u_i^{(m)}(\underline{a}-\underline{x}) + u_i(\underline{x}) p^{(m)}(\underline{a}-\underline{x})\right\} \\ &+\mu\tfrac{\partial}{\partial x_j}\left\{\tfrac{\partial u_i}{\partial x_j}(\underline{a}-\underline{x}) - u_i(\underline{x})\tfrac{\partial u_i^{(m)}}{\partial x_j}(\underline{a}-\underline{x})\right\} \quad (33)\end{aligned}$$

for a point $\underline{x} = \underline{a}$ within the fluid. A closed surface S in the far field domain is considered, and so from (33)

$$\begin{aligned}\int\int_S \Big[p(\underline{y}) u_j^{(m)}(\underline{a}-\underline{y}) &+ u_j(\underline{y}) p^{(m)}(\underline{a}-\underline{y}) \\ &+ \mu\left\{u_i(\underline{y})\frac{\partial u_i^{(m)}}{\partial y_j}(\underline{a}-\underline{y}) - \frac{\partial u_i^{(m)}}{\partial y_j}(\underline{a}-\underline{y}) u_i^{(m)}(\underline{a}-\underline{y})\right\} \\ &+\rho U u_i(\underline{y}) u_i^{(m)}(\underline{a}-\underline{y})\delta_{1j}\Big] n_j dS = 0^{(m)} \qquad (34)\end{aligned}$$

by applying the divergence theorem. $\underline{x} = \underline{y}$ are points on the surface S and (33) holds for all $\underline{x} = \underline{a}$ in $r > r_0$. $\underline{a}$ can be considered as variable and so

$$\begin{aligned}\int\int_S \Big[p(\underline{y}) u_j^{(m)}(\underline{x}-\underline{y}) &+ u_j(\underline{y}) p^{(m)}(\underline{x}-\underline{y}) \\ &+ \mu\left\{u_i(\underline{y})\frac{\partial u_i^{(m)}}{\partial y_j}(\underline{x}-\underline{y}) - \frac{\partial u_i^{(m)}}{\partial y_j}(\underline{x}-\underline{y}) u_i^{(m)}(\underline{x}-\underline{y})\right\} \\ &+\rho U u_i(\underline{y}) u_i^{(m)}(\underline{x}-\underline{y})\delta_{1j}\Big] n_j dS = 0^{(m)}. \qquad (35)\end{aligned}$$

Let S consist of three surfaces: S_δ which encloses the point at $\underline{x}$ with a spherical surface of radius $\delta \rightarrow 0$; S_C which is the boundary $r = r_0$ lying between the near and far fields; S_R which encloses S_C and S_δ with a spherical surface of radius $R \rightarrow \infty$.

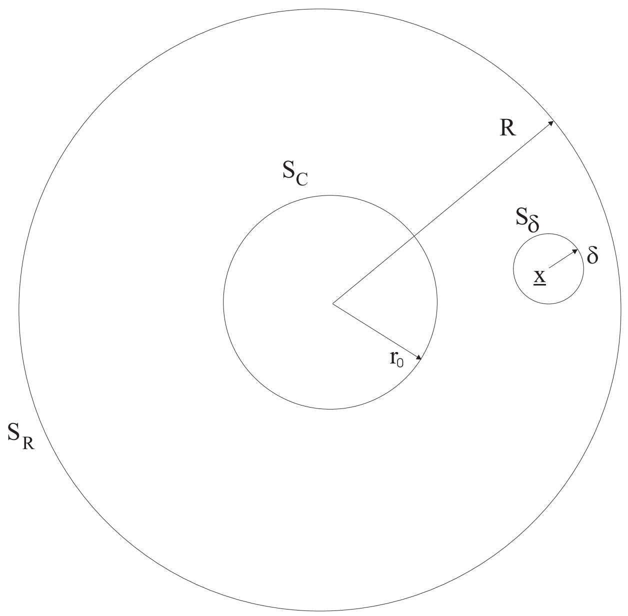

Figure 3. The surface S.

We consider the contributions of the surfaces S_δ and S_R to (35) where $u_i^{(m)}$ $p^{(m)}$ are the 'Oseenlet' singular force solutions chosen to give integral representations for the Oseen velocity and pressure.

7.0.1 Oseen velocity

In order to give an integral representation for the Oseen velocity, Oseen considers the singular force solutions (3.1) and (3.2) given in suffix notation as

$$\begin{aligned} u_k^{(1)}(\underline{z}) &= -\frac{1}{4\pi\rho U}\frac{\partial}{\partial z_k}\left\{\frac{1}{R} - \frac{e^{-k(R-z_1)}}{R}\right\} - \frac{1}{4\pi\mu}\frac{e^{-k(R-z_1)}}{R}\delta_{k1} \\ u_k^{(2)}(\underline{z}) &= \frac{1}{4\pi\rho U}\frac{\partial}{\partial z_k}\left\{(1-e^{-k(R-z_1)})\frac{\partial}{\partial z_2}\ln(R-z_1)\right\} - \frac{1}{4\pi\mu}\frac{e^{-k(R-z_1)}}{R}\delta_{k2} \\ u_k^{(3)}(\underline{z}) &= \frac{1}{4\pi\rho U}\frac{\partial}{\partial z_k}\left\{(1-e^{-k(R-z_1)})\frac{\partial}{\partial z_3}\ln(R-z_1)\right\} - \frac{1}{4\pi\mu}\frac{e^{-k(R-z_1)}}{R}\delta_{k3} \end{aligned} \tag{36}$$

which have pressure expansions

$$p^{(m)}(\underline{z}) = \frac{\partial}{\partial z_m}\left(\frac{1}{4\pi R}\right) \tag{37}$$

where $R = |\underline{z}|$ and $\underline{z} = \underline{x} - \underline{y}$. $k = \frac{U}{2\nu}$, $\nu = \frac{\mu}{\rho}$. We can now calculate the contributions over the surface S_δ and S_R.

Integral contribution from S_δ.
As $R \to 0$

$$u_k^{(j)} \sim -\frac{1}{8\pi\mu}\left(\frac{\delta_{kj}}{R} + \frac{z_k z_j}{R^3} + O(1)\right) \ , \ p^{(j)} \sim -\frac{1}{4\pi R^2}\left(\frac{z_j}{R}\right)$$

where "$\sim$" means "asymptotically tends to". Thus there are contributions to the integral S_δ of $-\frac{u_m}{3}$ from the second term in the integrand and $-\frac{2u_m}{3}$ from the third term in the integrand.

Integral contribution from S_R.
For large R, outside the wake $R - y_1$ is large and so the term $e^{-k(R-y_1)}$ is exponentially small. Thus, outside the wake the velocity and pressure fundamental solutions are of order $\frac{1}{R^2}$. Since the velocity $\underline{u}$ and the pressure p tend to zero for large R, the integral contribution from S_R outside the wake is zero.

Inside the wake, the largest term in the velocity fundamental solution is $\frac{1}{y_1}e^{-k\frac{y_2^2+y_3^2}{2y_1}}(1 + O(\frac{1}{y_1}))$. Thus, since both $\underline{u}$ and $p \to 0$ as $r \to \infty$ the integral wake cross-section contribution from S_R inside the wake is therefore also zero.

Hence, the Oseen velocity is given by

$$\begin{aligned} u_m &= \int\int_{S_0} \Big[p(\underline{y})u_j^{(m)}(\underline{x}-\underline{y}) + u_j(\underline{y})p^{(m)}(\underline{x}-\underline{y}) \\ &+ \mu\left\{u_i(\underline{y})\frac{\partial u_i^{(m)}}{\partial y_j}(\underline{x}-\underline{y}) - \frac{\partial u_i^{(m)}}{\partial y_j}(\underline{x}-\underline{y})u_i^{(m)}(\underline{x}-\underline{y})\right\} \\ &\quad +\rho U u_i(\underline{y})u_i^{(m)}(\underline{x}-\underline{y})\delta_{1j}\Big]\, n_j dS \end{aligned} \tag{38}$$

7.0.2 Oseen pressure

Oseen gives an integral representation of Oseen pressure by considering the fundamental solutions

$$u_i^{(m)}(\underline{z}) = q_i\delta_{m1} = \frac{1}{4\pi}\frac{\partial}{\partial z_m}\left(\frac{1}{R}\right)\delta_{1m} \ , p^{(m)} = -\rho U q_1\delta_{m1} = -\rho U\frac{1}{4\pi}\frac{\partial}{\partial z_1}\left(\frac{1}{R}\right)\delta_{1m} \tag{39}$$

in the integral equation (35).

As $R \to 0$, there are contributions to S_δ of $-p$ from the first term in the integrand, $\frac{\rho U}{3}u_1$ from the second term in the integrand and $-\frac{\rho U}{3}u_1$ from the third term in the integrand. As $R \to \infty$, there are no contributions to S_R. Thus, the pressure is given by

$$
\begin{aligned}
p &= \int\int_{S_0} \Big[p(\underline{y})q_j(\underline{x}-\underline{y}) + u_j(\underline{y})q_1(\underline{x}-\underline{y}) \\
&+ \mu\left\{u_i(\underline{y})\frac{\partial q_i}{\partial y_j}(\underline{x}-\underline{y}) - \frac{\partial q_i}{\partial y_j}(\underline{x}-\underline{y})q_i(\underline{x}-\underline{y})\right\} \\
&\quad +\rho U u_i(\underline{y})q_i(\underline{x}-\underline{y})\delta_{1j}\Big] n_j dS
\end{aligned} \tag{40}
$$

7.1 The Oseen velocity and pressure expansions

We consider the far field vector $\underline{x}$ such that $|\underline{y}| << |\underline{x}|$ and expand the fundamental solutions as a Taylor series

$$
\begin{aligned}
u_k^{(i)}(\underline{x}-\underline{y}) &= u_k^{(i)}(\underline{x}) - y_l\frac{\partial}{\partial x_l}w_k^{(i)}(\underline{x}) + 1/2 y_p y_l \frac{\partial^2}{\partial x_p \partial x_l}u_k^{(i)}(\underline{x}) + ... \quad (41) \\
q_k(\underline{x}-\underline{y}) &= q_k(\underline{x}) - y_l\frac{\partial}{\partial x_l}q_k(\underline{x}) + 1/2 y_p y_l \frac{\partial^2}{\partial x_p \partial x_l}q_k(\underline{x}) + ...
\end{aligned}
$$

where we note that $p^{(k)} = q_k$. Using the relation $\frac{\partial q_k}{\partial x_l} = \nabla^2 u_k^{(l)} - 2k\frac{\partial u_k^{(l)}}{\partial x_1}$ and the condition that there is no fluid outflow from the surface S_0, we write

$$
\begin{aligned}
u_k(\underline{x}) &= A_i u_k^{(i)} + B_{il}\frac{\partial u_k^{(i)}}{\partial x_l} + ... \qquad (42) \\
A_i &= \int\int_{S_0}\left\{-p\delta_{ij} + \mu\frac{\partial u_i}{\partial y_j} - \rho U u_i \delta_{j1}\right\} n_j dS \\
&= F_i
\end{aligned}
$$

where F_i is the force vector from (10).

Similarly,

$$
B_{il} = \int\int_{S_0}\left\{y_l(p\delta_{ij} - \mu\frac{\partial u_i}{\partial y_j} + \rho U u_i \delta_{j1}) - \rho U y_i u_j \delta_{l1} + \mu u_i \delta_{jl}\right\} n_j dS
$$

where M_i^0, which is the contribution to the moment from the Oseen terms defined by (12), is given by

$$
M_i^0 = \epsilon_{ijk}B_{jk}. \tag{43}
$$

Similarly, the Oseen representation of the Oseen pressure is given by [1, pp30-38]

$$
\begin{aligned}
p(\underline{x}) = -\int\int_{S_0}\{p(\underline{y})q_j(\underline{x}-\underline{y}) - \rho U u_j(\underline{y})q_1(\underline{x}-\underline{y}) \\
+\mu u_i(\underline{y})\frac{\partial}{\partial y_j}q_i(\underline{x}-\underline{y}) - \mu\frac{\partial u_i(\underline{y})}{\partial y_j}q_i(\underline{x}-\underline{y}) + \rho U u_i(\underline{y})q_i(\underline{x}-\underline{y})\delta_{j1}\}n_j dS
\end{aligned} \tag{44}
$$

and so

$$p(\underline{x}) = A_i q_i(\underline{x}) + B_{il} \frac{\partial q_i(\underline{x})}{\partial x_l} + ... \qquad (45)$$

since, from (39) $\frac{\partial q_i}{\partial x_l} = \frac{\partial q_l}{\partial x_i}$.

We see that the nine second order terms in the expansions are linearly dependent. However, we may choose a linearly independent set of eight terms (see Appendix B).

8 Two-dimensional flow

In this section, we give far field Oseen velocity and pressure expansions by using two different approaches; the first, by using the Lamb-Goldstein velocity representation, and the other by expanding the Oseenlets in Oseen's integral representation of the velocity and pressure. We also show that the two approaches give the same expansions.

8.1 Lamb-Goldstein representation

Lamb [2, p615] gives the Oseen velocity and pressure expansions for low Reynolds number flow past a circular cylinder. We shall use, and also modify, his approach in order to find the far field velocity expansion for two-dimensional flow in general. By following Lamb, we obtain a far field expansion for the velocity potential ϕ and also the vorticity potential χ. However, unlike the special case Lamb considers, we find that ϕ is a multi-valued function and that χ is discontinuous on the infinite half line $(x < 0, 0)$. From this expansion, we determine the wake velocity. (Alternatively, the wake velocity can be determined by considering the wake stream function ψ which is discontinuous on the infinite half line $(x > 0, 0)$.)

8.1.1 The two-dimensional potential ϕ.

In two-dimensional flow, by applying Fourier's theorem [11] to the θ variable (where $x = r_{2D}\cos\theta$, $y = r_{2D}\sin\theta$) the pressure expansion is given by

$$p = -\rho U \sum_{n=1}^{\infty} \left\{ a_n^* \frac{\cos n\theta}{r_{2D}^n} + b_n^* \frac{\sin n\theta}{r_{2D}^n} \right\} \qquad (46)$$

since $\nabla^2 p = 0$ and $p \to 0$ as $r_{2D} \to \infty$; a_n^* and b_n^* are constants.

The two-dimensional velocity potential is defined as

$$\frac{\partial\phi}{\partial x} = -\frac{1}{\rho U}p\,, \quad \nabla^2\phi = 0. \tag{47}$$

Let us first consider the potential ϕ^* which is obtained from term by term integration of the expansion (46). Thus

$$\phi^* = \frac{D\log r_{2D}}{2\pi\rho U} - \frac{L(\theta-\pi)}{2\pi\rho U} + \sum_{n=1}^{\infty}\left\{\alpha_n\frac{\cos n\theta}{r_{2D}^n} + \beta_n\frac{\sin n\theta}{r_{2D}^n}\right\} \tag{48}$$

where $a_1^* = \frac{D}{2\pi\rho U}$, $b_1^* = \frac{L}{2\pi\rho U}$, and for $n \geq 2$, $a_n^* = -(n-1)\alpha_{n-1}$ and $b_n^* = -(n-1)\beta_{n-1}$. Let

$$\phi = \phi^* + \Phi^*.$$

Thus, from (5) and (47) $\nabla^2\Phi = 0$, $\frac{\partial\Phi^*}{\partial x} = 0$ and $\nabla^2\Phi^* \to 0$ as $r_{2D} \to \infty$. Therefore $\Phi^* = A$, an arbitrary constant, and so ϕ is given by

$$\phi = \frac{D\log r_{2D}}{2\pi\rho U} - \frac{L(\theta-\pi)}{2\pi\rho U} + \sum_{n=1}^{\infty}\left\{\alpha_n\frac{\cos n\theta}{r_{2D}^n} + \beta_n\frac{\sin n\theta}{r_{2D}^n}\right\}. \tag{49}$$

8.1.2 The two-dimensional vorticity potential χ.

The vorticity satisfies the differential equation

$$(\nabla^2 - 2k\frac{\partial}{\partial x})\Omega = 0 \tag{50}$$

and so $(\nabla^2 - k^2)e^{-kx}\Omega = 0$. We consider the cylindrical polar coordinates (r_{2D},θ) and apply Fourier's Theorem to the θ variable. Following the method of Carslaw and Jaegar [12, p120] (who consider the Helmholtz equation), we get

$$\begin{aligned}\Omega e^{-kx} = \;& 2k \sum_{n=0}^{\infty}\{[A_n^*K_n(kr_{2D}) + C_n^*I_n(kr_{2D})]\cos n\theta \\ &+ [B_n^*K_n(kr_{2D}) + D_n^*I_n(kr_{2D})]\sin n\theta\}\end{aligned} \tag{51}$$

where $K_n(kr_{2D})$ and $I_n(kr_{2D})$ are modified Bessel functions defined in Abramowitz and Stegun [13, p374]. Also, by taking a Fourier component we get the condition

$$\begin{aligned}|A_n^*K_n(kr_{2D}) + C_n^*I_n(kr_{2D})| &= \left|\frac{1}{\pi}\int_{\pi}^{\pi}\Omega e^{-kx}\cos n\theta d\theta\right| \\ &\leq O(e^{kr_{2D}})|\Omega|\end{aligned}$$

as $r_{2D} \to \infty$. We must satisfy the condition (5) which is equivalent to $\Omega \to 0$ as $r_{2D} \to \infty$. However, we shall assume the stricter condition that as $r_{2D} \to \infty$, then $\Omega = o(\frac{1}{\sqrt{kr_{2D}}})$. From Abramowitz and Stegun [13, p377 9.7.1, p378 9.7.2], as $r_{2D} \to \infty$ then

$$K_n(kr_{2D}) \sim \sqrt{\frac{\pi}{2kr_{2D}}} e^{-kr_{2D}} \ , I_n(kr_{2D}) \sim \sqrt{\frac{1}{2\pi kr_{2D}}} e^{kr_{2D}}.$$

Thus $C_n = 0$ and similarly $D_n = 0$. Therefore

$$\Omega e^{-kx} = 2k \sum_{n=0}^{\infty} \{A_n^* K_n(kr_{2D}) \cos n\theta + B_n^* K_n(kr_{2D}) \sin n\theta\}. \qquad (52)$$

For two-dimensional flow, we shall define the vorticity potential χ by the equations

$$\frac{\partial \chi}{\partial y} = \Omega \,, \quad (\nabla^2 - 2k\frac{\partial}{\partial x})\chi = 0. \qquad (53)$$

We consider the function χ^* obtained from term by term integration of the Oseen vorticity expansion (52)

$$\begin{aligned}
\chi^* &= \chi_{D1} + \chi_{D2} + 2ke^{kx} \sum_{n=0}^{\infty} \{a_n K_n(kr_{2D}) \cos n\theta + b_n K_n(kr_{2D}) \sin n\theta\} \\
\chi_{D1} &= 2ka \int_0^{\infty} e^{k(x-\xi)} \frac{\partial}{\partial y} K_0(kr_\xi) d\xi \\
\chi_{D2} &= 2kb \int_0^{-\infty} e^{k(x+\xi)} \frac{\partial}{\partial y} K_0(kr_\xi) d\xi
\end{aligned} \qquad (54)$$

where $r_\xi = [(x-\xi)^2 + y^2 + z^2]^{1/2}$; a, b, a_n and b_n are constants.

Letting $\chi = \chi^* + \chi^\dagger$, we find that $\chi^\dagger = 0$ and

$$\begin{aligned}
\chi &= \chi_{D2} + 2ke^{kx} \sum_{n=0}^{\infty} \{a_n K_n(kr_{2D}) \cos n\theta + b_n K_n(kr_{2D}) \sin n\theta\} \\
\chi_{D2} &= 2kb \int_0^{-\infty} e^{k(x+\xi)} \frac{\partial}{\partial y} K_0(kr_\xi) d\xi.
\end{aligned} \qquad (55)$$

From the appendix C, using Ursell's method [14], the discontinuity term may be expressed as

$$\begin{aligned}
\chi_{D2} &= 2kbe^{kx}(\theta - \pi) I_0(kr_{2D}) \\
&- 4kbe^{kx} \sum_1^{\infty} \left\{ \frac{\partial}{\partial \nu}[I_\nu(kr_{2D})] \bigg|_{\nu=m} \sin m\theta - (\theta - \pi) I_m(kr_{2D}) \cos m\theta \right\}.
\end{aligned} \qquad (56)$$

We can give the wake velocity in terms of the function χ by using the same method used for three-dimensional axisymmetric flow (section 4). The wake velocity is then given by

$$\begin{aligned} w_1 &= \frac{1}{2k}\frac{\partial \chi}{\partial x} - \chi \\ w_2 &= \frac{1}{2k}\frac{\partial \chi}{\partial y} \end{aligned} \tag{57}$$

Applying the condition (8), of no total outflow from a contour enclosing the body, gives $a_0 = \frac{D}{\pi \rho U}$. Also, applying the force integral (10), the drag on the body is D and the lift is L in the $y-$direction.

We now consider Oseen's representation for two-dimensional flow.

8.2 Oseen's representation

We follow the method for three-dimensional general flow (section 7) and consider the contour integral

$$\begin{aligned} \int_C \Big[p(\underline{y})u_j^{(m)}(\underline{x}-\underline{y}) &+ u_j(\underline{y})p^{(m)}(\underline{x}-\underline{y}) \\ &+ \mu\left\{u_i(\underline{y})\frac{\partial u_i^{(m)}}{\partial y_j}(\underline{x}-\underline{y}) - \frac{\partial u_i^{(m)}}{\partial y_j}(\underline{x}-\underline{y})u_i^{(m)}(\underline{x}-\underline{y})\right\} \\ &+\rho U u_i(\underline{y})u_i^{(m)}(\underline{x}-\underline{y})\delta_{1j}\Big] n_j dl = 0^{(m)} \end{aligned} \tag{58}$$

where the suffixes now run from 1 to 2. C is a contour enclosing a domain in the far field consisting of three contours: C_δ, a circle circumference of radius δ centre $\underline{x} = (x_1, x_2)$; C_0, enclosing the near field $r_{2D} = (x_1^2 + x_2^2)^{1/2} > r_0$; C_R, a contour enclosing C_0 and C_δ of radius $R \to \infty$. We consider the contributions of the surfaces C_δ and C_R, where $u_i^{(m)}$ and $p^{(m)}$ are now the Oseen fundamental solutions in two-dimensions, in order to obtain integral representations for the Oseen velocity and pressure.

8.2.1 Two-dimensional Oseen velocity

The two-dimensional Oseen fundamental solutions, giving unit drag and force respectively, are

$$\underline{u}^{(1)}(\underline{z}) = \frac{1}{2\pi\rho U}\left(\frac{\partial g}{\partial z_1} + 2kK_0(kr_{2D})e^{-kz_1}, \frac{\partial g}{\partial z_2}\right) \;, \quad p^{(1)} = \frac{1}{2\pi}\frac{\partial}{\partial z_1} log r_{2D}$$

$$\underline{u}^{(2)}(\underline{z}) = \frac{1}{2\pi\rho U}\left(\frac{\partial g}{\partial z_2}, -\frac{\partial g}{\partial z_1}\right) \; , \; p^{(2)} = \frac{1}{2\pi}\frac{\partial}{\partial z_2}\log r_{2D}$$
$$g = \log r_{2D} + K_0(kr_{2D}) \quad |\underline{z}| = r_{2D} \quad k = \rho U/2\mu \quad \rho\nu = \mu \tag{59}$$

where $\underline{z} = \underline{x} - \underline{y}$ and $K_0(kr_{2D})$ is a modified Bessel function of the second kind order zero defined in Abramowitz and Stegun [13, p374].

Integral contribution from C_δ

As $r_{2D} \to 0$, $u_i^{(j)} \sim -\frac{1}{4\pi\mu}\delta_{ij}\log r_{2D}$. There is a contribution from the second term in the integrand of $-u_m/2$ and also from the third term of $-u_m/2$. (We note that $n_j = \frac{\partial r_{2D}}{\partial y_j} = -\frac{z_j}{r_{2D}}$.)

Integral contribution from C_R

As $R \to \infty$, from Abramowitz and Stegun [13, p377] $K_0(kR) \sim \sqrt{\frac{\pi}{2kR}}e^{-kR}$.

Thus $u_j^{(i)} \sim \left[f_{ij}\frac{\sin\alpha}{R} + g_{ij}\frac{\cos\alpha}{R} + h_{ij}\frac{e^{-k(R-y_1)}}{\sqrt{R}}\right](1 + O(1/R))$ and $p^{(i)} \sim \left[F_i\frac{\sin\alpha}{R} + G_i\cos\alpha\right](1 + O(1/R))$, where the matrices f, g, h, F and G are regular functions independent of R. Therefore, the integral contributions over the contour are zero as $R \to \infty$.

The integral representation of the Oseen velocity is then given by

$$\begin{aligned} u_m = \int_{C_0} \Big[p(\underline{y})u_j^{(m)}(\underline{x}-\underline{y}) \; &+ \; u_j(\underline{y})p^{(m)}(\underline{x}-\underline{y}) \\ &+ \; \mu\left\{u_i(\underline{y})\frac{\partial u_i^{(m)}}{\partial y_j}(\underline{x}-\underline{y}) - \frac{\partial u_i^{(m)}}{\partial y_j}(\underline{x}-\underline{y})u_i^{(m)}(\underline{x}-\underline{y})\right\} \\ &+\rho U u_i(\underline{y})u_i^{(m)}(\underline{x}-\underline{y})\delta_{1j}\Big] n_j dl \end{aligned} \tag{60}$$

8.2.2 The two-dimensional Oseen pressure

Oseen considers the two-dimensional fundamental solutions given by

$$u_i^{(m)} = q_i\delta_{m1} = -2\pi\frac{\partial}{\partial x_i}\log r_{2D}\delta_{m1} \; , \; p^{(m)} = 2\pi\rho U q_1\delta_{m1}. \tag{61}$$

These give an integral contribution from C_δ of -1 from the first term, $\rho U u_1$ from the second term , and $-\rho U u_1$ from the third term. The integral contribution from C_R is zero. Hence, the pressure is given by

$$\begin{aligned} p &= \int_{C_0}\Big[p(\underline{y})q_j(\underline{x}-\underline{y}) + u_j(\underline{y})q_1(\underline{x}-\underline{y}) \\ &+ \; \mu\left\{u_i(\underline{y})\frac{\partial q_i}{\partial y_j}(\underline{x}-\underline{y}) - \frac{\partial q_i}{\partial y_j}(\underline{x}-\underline{y})q_i(\underline{x}-\underline{y})\right\} \\ &+\rho U u_i(\underline{y})q_i(\underline{x}-\underline{y})\delta_{1j}\Big] n_j dl \end{aligned} \tag{62}$$

8.2.3 The two-dimensional Oseen velocity and pressure expansions

Following the method in three-dimensions for the two-dimensional expansion we obtain

$$
\begin{aligned}
u_k(\underline{x}) &= A_i u_k^{(i)} + B_{il}\frac{\partial u_k^{(i)}}{\partial x_l} + \ldots \\
p(\underline{x}) &= A_i q_i(\underline{x}) + B_{il}\frac{\partial q_i(\underline{x})}{\partial x_l} + \ldots \\
A_i &= \int_{C_0}\left\{-p\delta_{ij} + \mu\frac{\partial u_i}{\partial y_j} - \rho U u_i \delta_{j1}\right\} n_j dl \\
B_{il} &= \int_{C_0}\left\{y_l(p\delta_{ij} - \mu\frac{\partial u_i}{\partial y_j} + \rho U u_i \delta_{j1}) - \rho U y_i u_j \delta_{l1} + \mu u_i \delta_{jl}\right\} n_j dl
\end{aligned}
\tag{63}
$$

where all the suffixes in this instance run from 1 to 2. The body force is again found to be $F_i = A_i$.

8.3 Equivalence of the two-dimensional expansions

By inspection, we see that the Oseen velocity 'Oseenlet' expansion and the Lamb-Goldstein velocity expansion contain potential velocity terms $\log r_{2D}$, θ, and derivatives with respect to x and y of these terms. Similarly, the wake velocity contains the terms $K_0(kr_{2D})e^{kx}$ and its derivatives with respect to x and y. However, the lift Oseenlet is given by (59) as

$$\underline{u}^{(2)} = \frac{1}{2\pi\rho U}\left(\frac{\partial g}{\partial y}, -\frac{\partial g}{\partial x}\right), \quad g = \log r_{2D} + K_0(kr_{2D})e^{-kx}$$

which is the second term in the Oseen velocity 'Oseenlet' expansion.

Thus, at first sight it appears that because the coefficients in front of the potential velocity and wake velocity are the same for the lift Oseenlet, the expansion will not be sufficiently general; for the Lamb-Goldstein expansion, the coefficients for these two terms are different. However, the drag vorticity is given by $\Omega^{(1)} = \frac{\partial w^{(1)}}{\partial y} - \frac{\partial w^{(1)}}{\partial x} = 2k\underline{w}_2^{(1)} = 2k\underline{w}_1^{(2)}$ and the lift vorticity is given by $\Omega^{(2)} = \frac{\partial w^{(2)}}{\partial y} - \frac{\partial w^{(2)}}{\partial x} = 2k\underline{w}_2^{(2)}$. Consequently, the wake velocity for the singular lift solution is also given by the expression $2k\underline{w}^{(2)} = (\Omega^{(1)}, \Omega^{(2)})$.

Contribution to the lift term in the wake velocity therefore also comes from lower order terms in the expansion. Hence, the expansions obtained from the two different methods are equivalent.

[Alternatively, a lift Oseenlet of the form

$$\underline{u}^{(2)*} = \frac{1}{2\pi\rho U}\left(\frac{\partial g^*}{\partial y}, -\frac{\partial g^*}{\partial x}\right) \quad , \quad g^* = \log r_{2D} + AK_0(kr_{2D})e^{-kx}$$

could also be considered, for some constant A. Evaluating the contour integral (58) with this alternative Oseenlet would then give the integral representation (58) for $u_m - \frac{1-A}{2k}\Omega\delta_{1m}$ where Ω is the Oseen vorticity.]

We note, however, that for low Reynolds number flow, we expect that the rotation around the inner domain will remain finite, and then the coefficients for the two terms in the expansion will be the same.

9 Discussion

When modelling the far field Oseen flow region, the potential velocity and wake velocity cannot, in general, be treated independently from each other (the Lamb-Goldstein approach). In preference, modelling the far field by a distribution of Oseenlets (in the near field) is suggested.

The Lamb-Goldstein approach fails when Goldstein's assumption [4], that the potential velocity and wake velocity are regular everywhere except along the positive x-axis, does not hold. An example of a flow for which the assumption is invalid is given in section 6.1. Furthermore, we give another example in section 6.2 in which it is shown that using this assumption leads to an incorrect evaluation of the lift.

Instead, we suggest using a different approach to modelling the Oseen equations in the far field: Oseen represents the Oseen velocity by a Green's surface integral distribution of Oseenlet (drag and lift) singularities. By giving the Taylor series expansion of the Oseenlets in the integrals, the far field velocity and pressure expansions are obtained. The coefficients in the expansions are related to the hydrodynamic forces and the 'Oseen moment' on the body.

For the special cases where the Lamb-Goldstein representation holds, the two approaches are shown to be equivalent. Finally, we discuss another interesting application for future investigation.

Future work: slender body theory

Consider slender body theory for manoeuvring problems. An important factor in the hydrodynamic force calculation is the vorticity which is shed from the hull. Comparisons can be made with aerodynamic wing theory in which determining the shed vortex strength by using the Kutta-Joukowski condition gives the lift on the wing: for example, Gadd [15] determines the side force on a hull by considering trailing vortex sheets from the stern. Also, in laminar wake theory [16, pp71-76], determining the wake cross sections of components of vorticity far downstream from the body give the drag and lift. Any viable slender body theory should therefore model the vortex wake. In the present paper, we have considered uniform flow past a fixed body in an unbounded domain and modelled a steady vortex wake region behind the body outside of which there is almost potential flow.

In slender body potential theory [17, pp331-337], the axial source distribution term in the potential is related to the two-dimensional source outflow from cross-sections (or strips) along the length of the slender body. We note that in Oseen flow there is an 'inflow' in the wake which can be determined from the 'drag' Oseenlet; this should also be taken into account when developing a more realistic slender body theory for manoeuvring problems.

Future work on this topic would be to develop slender body theory for Oseen flow by considering an axial distribution of Oseenlets (lift and drag singular solutions). The results will be applied to the determination of manoeuvring criteria for various hull shapes at general Reynolds number flow, and compared to the corresponding experimental results.

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Appendix A

We want to show that, if $p_n(x,y,z)$ is a solid spherical harmonic of degree n [2, p110], that is $\nabla^2 p_n = 0$ and p_n is a homogeneous polynomial of degree n, then

$$p_n(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(1/r) = (-1)^n 1.3.5....(2n-1)\frac{p_n(x,y,z)}{r^{2n+1}}. \tag{64}$$

The proof is by induction; (64) holds for $n = 0, 1$ and we shall assume it holds for degree n. By Euler's theorem on homogeneous functions [18, p334]

$$(n+1)p_{n+1} = x\frac{\partial p_{n+1}}{\partial x} + y\frac{\partial p_{n+1}}{\partial y} + z\frac{\partial p_{n+1}}{\partial z} \tag{65}$$

where p_{n+1} is a solid harmonic of degree $n+1$. Hence

$$p_{n+1} = xF_n + yG_n + zH_n \tag{66}$$

where F_n, G_n and H_n are solid harmonics of degree n. Then from (64)

$$\begin{aligned} p_{n+1}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(1/r) &= \frac{\partial}{\partial x}\left\{F_n(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(1/r)\right\} + \frac{\partial}{\partial y}\left\{G_n(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(1/r)\right\} \\ &\quad + \frac{\partial}{\partial z}\left\{H_n(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(1/r)\right\} \\ &= (-1)^n 1.3.....(2n-1)\left\{\frac{\partial}{\partial x}\left(\frac{F_n(x,y,z)}{r^{2n+1}}\right) + \frac{\partial}{\partial y}\left(\frac{G_n(x,y,z)}{r^{2n+1}}\right) + \frac{\partial}{\partial Z}\left(\frac{H_n(x,y,z)}{r^{2n+1}}\right)\right\} \\ &= (-1)^n 1.3.....(2n-1)\left\{\frac{1}{r^{2n+1}}\left(\frac{\partial F_n}{\partial x} + \frac{\partial G_n}{\partial y} + \frac{\partial H_n}{\partial z}\right) - \frac{(2n+1)}{r^{2n+3}}(xF_n + yG_n + zH_n)\right\} \end{aligned} \tag{67}$$

But from (66)

$$\nabla^2 p_{n+1} = x\nabla^2 F_n + y\nabla^2 G_n + z\nabla^2 H_n + 2\left(\frac{\partial F_n}{\partial x} + \frac{\partial G_n}{\partial y} + \frac{\partial H_n}{\partial z}\right).$$

Thus

$$p_{n+1}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})(1/r) = (-1)^{n+1} 1.3....(2n-1)\frac{(2n+1)}{r^{2n+3}} p_{n+1}(x,y,z)$$

by (67). Assuming (64) holds for degree n implies it also holds for degree $n+1$. Hence the induction is established.

Appendix B

We want to give the velocity and pressure expansions with linearly independent terms. However, the expansions given for the velocity and pressure are not linearly independent, since

$$\frac{\partial u_k^{(i)}}{\partial x_i} = \frac{\partial u_i^{(k)}}{\partial x_i}\,, \quad \frac{\partial p^{(i)}}{\partial x_i} = \frac{\partial^2}{\partial x_i \partial x_i}\left(\frac{1}{4\pi r}\right) = 0$$

where $u_k^{(i)}$ and $p^{(i)}$ are defined by (36) and (37) respectively.

Thus of the nine solutions $\frac{\partial \underline{u}^{(i)}}{\partial x_j}$, $\frac{\partial p^{(i)}}{\partial x_j}$, only eight are independent. We now give eight linearly independent solutions for the velocity and pressure by $\underline{u}_m$ and p_m respectively where $1 \leq m \leq 8$.

Letting $x_1 = r\cos\Theta$, $x_2 = r\sin\Theta\cos\beta$, $x_3 = r\sin\Theta\sin\beta$, these solutions may be taken, with appropriate velocity fields, to be

$$\begin{aligned}
p_1 &= \frac{1}{4\pi r^3}P_2(\cos\Theta) = \frac{1}{2}\frac{\partial p^{(1)}}{\partial x_1}\,, \quad \underline{u}_1 = \frac{1}{2}\frac{\partial \underline{u}^{(1)}}{\partial x_1} \\
p_2 &= \frac{1}{4\pi r^3}P_2'(\cos\Theta)\sin\Theta\cos\beta = \frac{\partial p^{(1)}}{\partial x_2} = \frac{\partial p^{(2)}}{\partial x_1}\,, \quad \underline{u}_2 = \frac{1}{2}\left(\frac{\partial \underline{u}^{(1)}}{\partial x_2} + \frac{\partial \underline{u}^{(2)}}{\partial x_1}\right) \\
p_3 &= \frac{1}{4\pi r^3}P_2'(\cos\Theta)\sin\Theta\sin\beta = \frac{\partial p^{(1)}}{\partial x_3} = \frac{\partial p^{(3)}}{\partial x_1}\,, \quad \underline{u}_3 = \frac{1}{2}\left(\frac{\partial \underline{u}^{(1)}}{\partial x_3} + \frac{\partial \underline{u}^{(3)}}{\partial x_1}\right) \\
p_4 &= \frac{1}{4\pi r^3}P_2''(\cos\Theta)\sin^2\Theta\cos 2\beta = \frac{\partial p^{(2)}}{\partial x_2} - \frac{\partial p^{(3)}}{\partial x_3}\,, \quad \underline{u}_4 = \frac{\partial \underline{u}^{(2)}}{\partial x_2} - \frac{\partial \underline{u}^{(3)}}{\partial x_3} \\
p_5 &= \frac{1}{4\pi r^3}P_2''(\cos\Theta)\sin^2\Theta\sin 2\beta = 2\frac{\partial p^{(2)}}{\partial x_3} = 2\frac{\partial p^{(3)}}{\partial x_2}\,, \quad \underline{u}_5 = \frac{\partial \underline{u}^{(2)}}{\partial x_3} + \frac{\partial \underline{u}^{(3)}}{\partial x_2}.
\end{aligned}$$

P_2 is a spherical zonal harmonic of degree 2, and P_2' and P_2'' are the first and second derivatives of the spherical zonal harmonic of degree 2 respectively, with respect to $\cos\Theta$. Each of these solutions gives zero moment $M_i = 0_i$. The remaining three solutions of the Oseen equations have $p_m = 0$ and $\underline{u}_m$ anti-symmetric to give

$$\begin{aligned}
p_6 = 0\,,\ \underline{u}_6 = \mathrm{curl}\underline{u}^{(1)} = \frac{\partial \underline{u}^{(2)}}{\partial x_3} - \frac{\partial \underline{u}^{(3)}}{\partial x_2}\,, \quad M_i = 2\delta_{i1} \\
p_7 = 0\,,\ \underline{u}_7 = \mathrm{curl}\underline{u}^{(2)} = \frac{\partial \underline{u}^{(3)}}{\partial x_1} - \frac{\partial \underline{u}^{(1)}}{\partial x_3}\,, \quad M_i = 2\delta_{i2} \\
p_8 = 0\,,\ \underline{u}_8 = \mathrm{curl}\underline{u}^{(3)} = \frac{\partial \underline{u}^{(1)}}{\partial x_2} - \frac{\partial \underline{u}^{(2)}}{\partial x_1}\,, \quad M_i = 2\delta_{i3}.
\end{aligned}$$

Appendix C

We represent the discontinuity in the wake velocity stream function ψ and the vorticity potential χ by expansions involving modified Bessel functions using Ursell's result [14]. This means that the Helmholtz dipole line

$$I = \int_0^\infty e^{-ky} \frac{\partial}{\partial y} K_0(kr_\xi) d\xi \ , \tag{68}$$

where $r_\xi = \{(x-\xi)^2 + y^2\}^{1/2}$, is represented by an expansion of modified Bessel functions. The procedure used to obtain this representation follows.

We first give an integral representation for $K_0(kr_{2D})$ and then change the path of integration. From Abramowitz and Stegun [13, p376 9.6.24]

$$K_0(kr_{2D}) = \frac{1}{2} \int_{-\infty}^\infty e^{-kr_{2D}} \cosh t dt. \tag{69}$$

We substitute (69) into (68) and consider separately the cases $y > 0$ and $y < 0$. For $y > 0$, we lower the line of integration by $i\alpha$, where $r_{2D} \cosh(t - i\alpha) = y \cosh t - x \sinh t$, and we get

$$\begin{aligned} I &= -\frac{1}{2} k \int_{-\infty}^\infty \int_0^\infty \cosh t e^{-k\xi} e^{-ky\cosh t + ik(x-\xi)\sinh t} d\xi dt \\ &= -\frac{1}{2} \int_{-\infty}^\infty \frac{\cosh t}{1 + i\sinh t} e^{-ky\cosh t + ikx\sinh t} dt \\ &= \frac{1}{4\pi i} \left\{ \int + \int + \int \right\} \frac{t\cosh t}{1 + i\sinh t} e^{-ky\cosh t + ikx\sinh t} dt \\ &= I_c + I_a + I_b \end{aligned} \tag{70}$$

for the integral paths

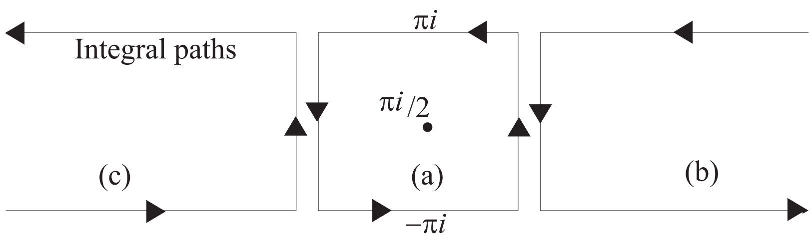

Figure 4. Integral paths.

We now evaluate the integral paths in turn:

(a) There is a singularity at $t = \frac{1}{2}\pi i$ which gives a residue contribution to the integral of $I_a = \frac{1}{2}\pi e^{-kx}$. (Near $t = \frac{1}{2}\pi i$, $\frac{t\cosh t}{1+i\sinh t} = \frac{\pi}{t-\frac{1}{2}\pi i}(1+O(t-\frac{1}{2}\pi i))$.)

(b) Along this integral path, the real part of t is large and positive, and so we consider the asymptotic expansion

$$\frac{\cosh t}{1+i\sinh t} = \sum_{m=0}^{\infty} a_m e^{-mt} \quad , \quad a_0 = -i \quad , \quad a_m = 2e^{\frac{1}{2}\pi i(m-1)} \quad , \quad m \geq 1.$$

Thus

$$I_b = \frac{1}{4\pi i}\sum_{m=0}^{\infty} a_m \int_{\infty-\pi i}^{\infty+\pi i} e^{-mt} t e^{-kr_{2D}\cosh(t+i\alpha)} dt \tag{71}$$

This integral is expressed in terms of modified Bessel functions by using the integral representation of the modified Bessel function I_ν given by Watson [19, p181]

$$\begin{aligned} I_\nu &= \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i} e^{-\nu t} e^{kr_{2D}\cosh t} dt \\ &= \frac{1}{2\pi i}\int_{\infty}^{\infty+2\pi i} e^{-\nu(t-\pi i)} e^{-kr_{2D}\cosh t} dt \\ &= \frac{1}{2\pi i}\int_{\infty}^{\infty+2\pi i} e^{-\nu(t-i\alpha-i\pi)} e^{-kr_{2D}\cosh(t-i\alpha)} dt \end{aligned} \tag{72}$$

where first the variable of integration was changed to $t+\pi i$ and then the line of integration was lowered by $i\alpha$; this is allowed since the contribution along the integral path $X - i\alpha$ to X, for example, is of order $|\alpha| e^{-|\nu|X} e^{-kr_{2D}\cosh X} \to 0$ as $X \to \infty$.

Thus

$$I_b = \frac{1}{2}\sum_{m=0}^{\infty} a_m \frac{\partial}{\partial\nu}\left\{I_\nu(kr_{2D})e^{-i\nu(\alpha+\pi)}\right\}\Big|_{\nu=m}$$

where $a_0 = -i$, $a_m = 2e^{\frac{1}{2}i\pi(m-1)}$, $m \geq 1$.

(c) A similar method is used to calculate I_c, and is expressed in real form

$$\begin{aligned} I &= \frac{1}{2}\pi e^{-kx} + \frac{1}{2}(2\theta - 3\pi)I_0(kr_{2D}) \\ &\quad - \sum_{m=1}^{\infty} 2\frac{\partial}{\partial\nu}\left\{I_\nu(kr_{2D})\right\}\big|_{\nu=m} \sin m(\theta-\pi) - (2\theta-2\pi)I_m(kr_{2D})\cos m(\theta-\pi) \\ &= \frac{1}{2}\pi e^{-kx} + f(r_{2D}, \frac{1}{2}\pi - \theta) \end{aligned} \tag{73}$$

for some function f determined from (73). We note that using a similar method, (this time raising the line of integration by $i\alpha$), the same equation (73) is obtained for $y < 0$.

We now give a modified Bessel expansion for $\frac{1}{2}\pi e^{-kx}$; by raising the line of integration in (72) for $y > 0$ instead of lowering it (and for $y < 0$ lowering the line of integration instead of raising it) we get a different expansion for I which is given by

$$I = -\frac{1}{2}\pi e^{-kx} - f(r_{2D}, \theta - 3\pi/2). \tag{74}$$

Finally, combining (73) and (74) we get an expansion for I

$$\begin{aligned} I = I_0(kr_{2D}) \quad &+ \quad 2\sum_{m=1}^{\infty} (-1)^{m+1} \frac{\partial}{\partial \nu} I_\nu(kr_{2D}) \bigg|_{\nu=m} \sin m\theta \\ &+ \quad (-1)^m(\theta - \pi) I_m(kr_{2D}) \cos m\theta \end{aligned} \tag{75}$$

which is the desired result.