CE 222 HW #3 Solution

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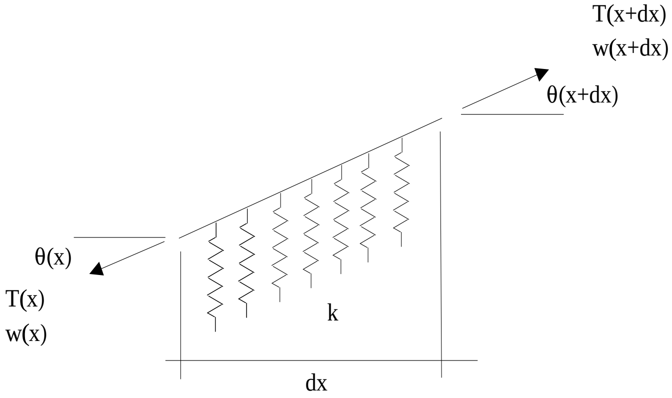
1 Problem 1

1.1 Derivation and Exact Solution of Governing Differential Equation

1.1.1 Derivation

Consider the following following free body diagram of the cable in its deformed configuration, exagerated for clarity:



We derive the differential equation by considering equilibrium in this configuration. By considering a small element of differential length, we can assume the displacement for the spring is $w(x)$. We could consider a 'better' approximation, but once we neglect higher order differentials we would end up with $w(x)$ anyway.

$\Sigma F_y = 0$

$$T(x+dx)\sin\left(\theta(x+dx)\right) - kw(x)dx - T(x)\sin\left(\theta(x)\right) = 0$$

Expand (linearize) about x:

$$T(x+dx) = T(x) + T'(x)dx \qquad \theta(x+dx) = \theta(x) + \theta'(x)dx$$

Subbing in

$$\left(T(x) + T'(x)dx\right)\sin\left(\theta(x) + \theta'(x)dx\right) - T(x)\sin\left(\theta(x)\right) - kw(x)dx = 0$$

Make the standard small angle approximation,

$$\sin(\theta) \approx \theta \qquad \cos(\theta) \approx 1$$

$$\left(T(x) + T'(x)dx\right)\left(\theta(x) + \theta'(x)dx\right) - T(x)\left(\theta(x)\right) - kw(x)dx = 0$$

$$\left(T(x)\theta'(x) + T'(x)\theta(x) - kw(x)\right)dx + T'(x)\theta'(x)(dx)^2 = 0$$

As usual, neglect higher order differentials, and note that $\theta = w'(x)$,

$$T(x)w''(x) + T'(x)w'(x) - kw(x) = 0$$

What to do about T'? Consider equilibrium in x direction:
$\Sigma F_x = 0$

$$T(x+dx)\cos(\theta(x+dx)) - T(x)\cos(\theta(x)) = 0$$

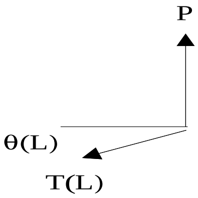
Utilizing small angle approximation, and linearizing,

$$T(x) + T'(x)dx - T(x) = 0 \implies T'(x) = 0$$

Finally,

$$\boxed{w'' - \frac{k}{T}w = 0}$$

Boundary Conditions: At the left hand side, we obviously have $w(0) = 0$. To determine right hand B.C, consider equilibrium of the right end node, as seen in the following figure:



$\Sigma F_y = 0$

$$P - T\sin(\theta(L)) = 0$$

$$w'(L) = \frac{P}{T}$$

$$\boxed{w'' - \frac{k}{T}w = 0 \quad ; \quad w(0) = 0 \quad ; \quad w'(L) = \frac{P}{T}}$$

1.1.2 Solution

This is a standard second order linear homogeneous differential equation with constant coefficients. The general solutions is:

$$w(x) = A\cosh(\gamma x) + B\sinh(\gamma x) \quad ; \quad \gamma \equiv \sqrt{\frac{k}{T}}$$

$$w(0) = 0 \quad \implies \quad A = 0$$

$$w'(L) = \frac{P}{T} \quad \implies \quad B = \frac{P}{T\gamma\cosh(\gamma L)}$$

$$\boxed{w(x) = \frac{P\sinh(\gamma x)}{T\gamma\cosh(\gamma L)}}$$

1.2 Weak Form

For this problem, the suitable function space is:

$$\mathfrak{V} = \left\{ v(x) \mid v(0) = 0 \quad ; \quad v \in C^1 \right\}$$

Construct the weak form as follows:
Let $v \in \mathfrak{V}$, then

$$\int_0^L vw'' - v\frac{k}{T}wdx = 0$$

Integrate by parts,

$$\boxed{-\int_0^L v'w' + v(L)\frac{P}{T} - \int_0^L v\frac{k}{T}wdx = 0}$$

The continuity requirements on $w(x)$ are that it be C^1, that is it must possess one derivative *and* the derivative must be continuous. This latter requirement is to eliminate any kinks in the cable.

1.3 FEM Solution

This problem is similar to the last problem in the previous problem set, in that the element stiffness matrix is modified by a term involving the shape functions themselves, not just their derivatives.

$$u(x) = \mathbf{N}\mathbf{u} \qquad v(x) = \mathbf{N}\mathbf{v}$$

For 2-node element,

$$\mathbf{N} = \begin{bmatrix} 1 - \frac{x}{L_e} & \frac{x}{L_e} \end{bmatrix}$$

$$\mathbf{k} = \int_0^{L_e} \mathbf{B}^\dagger \mathbf{B} + \int_0^{L_e} \frac{k}{T}\mathbf{N}^\dagger \mathbf{N}$$

$$\mathbf{k} = \frac{1}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{kL_e}{6T}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

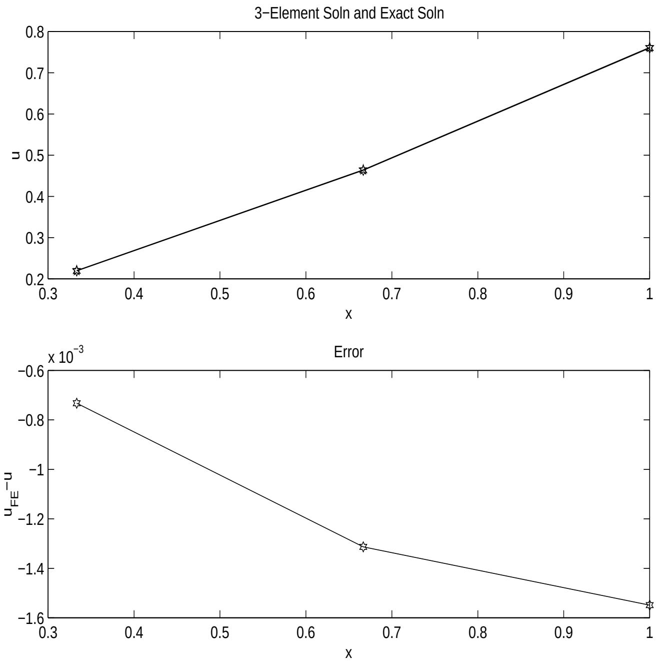
Using 3 elements and the definition $k = \frac{T}{L^2}$, the assembled global stiffness matrix and load vector are:

$$\mathbf{K} = \begin{bmatrix} 6.2222 & -2.9444 & 0 \\ -2.9444 & 6.2222 & -2.9444 \\ 0 & -2.9444 & 3.1111 \end{bmatrix} \qquad \mathbf{P} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad ; \quad (P = T = L = 1)$$

Solving, the nodal displacements are:

$$\mathbf{U} = \begin{bmatrix} 0.2193 \\ 0.4634 \\ 0.7600 \end{bmatrix}$$

The following graphs plot the two solutions and the error.



2 Problem 2

Construct a weak form for the following boundary value problem:

$$(kT,_i),_i + Q = 0 \quad ; \quad \begin{cases} T(x_i) = T_0 & \forall x \in S_t \\ -k\frac{\partial T}{\partial n} = q_0 & \forall x \in S_q \end{cases}$$

For this problem, the suitable function space is:

$$\mathfrak{V} = \{v(\mathbf{x}) \mid v \in C^1 \; ; \; v(\mathbf{x}) = 0 \; \forall \mathbf{x} \in S_t\}$$

Construct the weak form as follows:
Let $v \in \mathfrak{V}$, then

$$\int\limits_{\Omega} v(kT,_i),_i \, d\Omega + \int\limits_{\Omega} vQd\Omega = 0$$

Use the Gauss Theorem,

$$-\int\limits_{\Omega} v,_i (kT,_i) d\Omega + \int\limits_{\partial\Omega} vkT,_i n_i dS + \int\limits_{\Omega} vQd\Omega = 0$$

The boundary integral can be broken up into two disjoint pieces,

$$-\int\limits_{\Omega} v,_i (kT,_i) d\Omega + \int\limits_{S_t} vkT,_i n_i dS + \int\limits_{S_q} vkT,_i n_i dS + \int\limits_{\Omega} vQd\Omega = 0$$

The boundary term over S_t vanishes because the trial functions are zero there. We can substitute in the other boundary condition $-kT,_i n_i = q_0$ to obtain

$$\boxed{-\int\limits_{\Omega} v,_i (kT,_i) d\Omega - \int\limits_{S_q} q_0 v dS + \int\limits_{\Omega} vQd\Omega = 0}$$

Given the weak form above, show the strong form holds when all admissible variations are considered. Essentially, reverse the steps, with one key argument at the end.

$$-\int\limits_{\Omega} v_{,i}(kT_{,i})d\Omega - \int\limits_{S_q} q_0 v dS + \int\limits_{\Omega} vQd\Omega = 0$$

Since all admissible functions are zero at the essential boundary conditions, we can add in the following integral over S_t and using the natural boundary condition, convert the partial boundary integral into an integral over the complete boundary

$$-\int\limits_{\Omega} v_{,i}(kT_{,i})d\Omega + \int\limits_{S_t} vkT_{,i}n_i dS + \int\limits_{S_q} vkT_{,i}n_i dS + \int\limits_{\Omega} vQd\Omega = 0$$

Use the Gauss Theorem (in other direction), assuming T is sufficiently smooth that we can take another derivative,

$$\int\limits_{\Omega} v(kT_{,i})_{,i}\, d\Omega + \int\limits_{\Omega} vQd\Omega = 0$$

At this stage, all we have done is formally undo the steps we used to go from strong form to weak form. All we know at this point is that the integral is zero. How do we go from here to recovering the strong form? First, re-write the integral as

$$\int\limits_{\Omega} v\left\{(kT_{,i})_{,i} + Q\right\} d\Omega = 0$$

We now argue that, since v is *arbitrary* that in fact, the *only* way for the integral to *always* be zero is for the term in brackets to be identically zero. This is sometimes called the Localization Theorem. Thus,

$$(kT_{,i})_{,i} + Q = 0$$

which is the strong form.

3 Problem 3

I chose as my degrees of freedom the vertical displacements of each end, u_1 at the left, u_2 at the right. Zero displacement when the bar is unloaded. The displacement anywhere along the bar can be written as

$$u(x) = u_1 + \frac{u_2 - u_1}{40}x$$

$$\mathbf{\Pi}(u_1, u_2) = \frac{1}{2}\int\limits_{20}^{40} ku^2 dx + 10u_1$$

Evaluating,

$$\mathbf{\Pi}(u_1, u_2) = 50\left[20u_1^2 + 30u_1(u_2 - u_1) + \frac{35}{3}(u_2 - u_1)^2\right] + 10u_1$$

Minimize the potential,

$$\delta\mathbf{\Pi} = 0 = \frac{\partial \mathbf{\Pi}}{\partial u_1}\delta u_1 + \frac{\partial \mathbf{\Pi}}{\partial u_2}\delta u_2$$

Each coefficient of the variation must separately vanish, so

$$\frac{\partial \mathbf{\Pi}}{\partial u_1} = 0 = 50\left[40u_1 + 30(u_2 - 2u_1) - \frac{70}{3}(u_2 - u_1)\right] + 10$$

$$\frac{\partial \mathbf{\Pi}}{\partial u_2} = 0 = 30u_1 + \frac{70}{3}(u_2 - u_1)$$

These reduce to the simultaneous equations

$$2u_1 + 7u_2 = 0 \quad ; \quad 500u_1 + 1000u_2 = -30$$

Solving,

$$u_1 = -0.14 \quad ; \quad u_2 = 0.04$$

These can be verified by using statics on the bar. The spring force is

$$f(x) = 14 - \frac{9}{20}x$$

Equilibrium requires

$$\int_{20}^{40} f(x)dx = 10$$

and (moments about left end)

$$\int_{20}^{40} f(x)xdx = 0$$

these are easily verified.