CE 222 HW #2 Solution

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1 Problem 1

The boundary value problem (BVP) to be solved is:

$$\frac{\partial}{\partial x}\left[EA\frac{\partial u}{\partial x}\right] + p_0 = 0 \qquad u(0) = 0 \qquad EAu'(L) = \frac{P_0 L}{2} \tag{1}$$

1.1 Exact Solution

The cross sectional area varies along the length, so we must take this into account. The area as a function of x is:

$$A(x) = A\left(2 - \frac{x}{L}\right)$$

This is a second-order linear differential equation, we can solve it by direct integration. The solution is:

$$\boxed{u(x) = \frac{p_0}{EA}\left(xL + \frac{1}{2}L^2 \ln\left(2 - \frac{x}{L}\right) - \frac{L^2}{2}\ln 2\right)}$$

The axial force is $P(x) = EAu'$,

$$\boxed{P(x) = p_0\left(\frac{3L}{2} - x\right)}$$

1.2 Rayleigh-Ritz Soluton

The Rayleigh-Ritz method is similar to the Rayleigh method, but we can choose a linear combination of trial functions. The set of functions that we take our linear combination from is the same as for the Rayleigh method, we just take more of them. It is convenient to use a matrix notation for these calculations. Let's denote by $\mathbf{\Phi}(x)$ a row-vector of trial functions, and $\mathbf{c}$ the column-vector of unknown coefficients. Then we can write:

$$\bar{u}(x) = c_1\phi_1 + c_2\phi_2 + c_3\phi_3$$

as

$$\bar{u}(x) = \mathbf{\Phi}(x)\mathbf{c}$$

with

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad \text{and} \quad \mathbf{\Phi}(x) = \begin{bmatrix} \phi_1(x) & \phi_2(x) & \phi_3(x) \end{bmatrix}$$

For this problem, choose

$$\mathbf{\Phi}(x) = \begin{bmatrix} x & x^2 & x^3 \end{bmatrix}$$

$$\bar{u}(x) = \mathbf{\Phi c} \qquad \bar{\epsilon} = \mathbf{\Phi' c} \qquad \delta u = \mathbf{\Phi} \delta \mathbf{c} \qquad \delta \epsilon = \mathbf{\Phi'} \delta \mathbf{c}$$

Plug into the virtual work expression, after integrating over the cross section,

$$W_i = \int_0^L \delta \mathbf{c}^\dagger \mathbf{\Phi}'^\dagger EA(x) \mathbf{\Phi}' \mathbf{c} dx$$

$$W_i = \delta \mathbf{c}^\dagger \left[\int_0^L \mathbf{\Phi}'^\dagger EA(x) \mathbf{\Phi}' dx \right] \mathbf{c} = \delta \mathbf{c}^\dagger \mathbf{K} \mathbf{c}$$

$$\mathbf{K} \equiv \int_0^L \mathbf{\Phi}'^\dagger EA(x) \mathbf{\Phi}' dx$$

In expanded form, the integrand is:

$$EA \begin{bmatrix} 2 - \frac{x}{L} & 2\left(2 - \frac{x}{L}\right) x & 3\left(2 - \frac{x}{L}\right) x^2 \\ 2\left(2 - \frac{x}{L}\right) x & 4\left(2 - \frac{x}{L}\right) x^2 & 6\left(2 - \frac{x}{L}\right) x^3 \\ 3\left(2 - \frac{x}{L}\right) x^2 & 6\left(2 - \frac{x}{L}\right) x^3 & 9\left(2 - \frac{x}{L}\right) x^4 \end{bmatrix}$$

after integrating,

$$\mathbf{K} = EA \begin{bmatrix} 3/2\, L & 4/3\, L^2 & 5/4\, L^3 \\ 4/3\, L^2 & 5/3\, L^3 & 9/5\, L^4 \\ 5/4\, L^3 & 9/5\, L^4 & \frac{21}{10}\, L^5 \end{bmatrix}$$

$$W_e = \int_0^L \delta \mathbf{c}^\dagger \mathbf{\Phi}^\dagger p_0 \mathbf{c} dx + \delta \mathbf{c}^\dagger \mathbf{\Phi}^\dagger (L) p_0 \frac{L}{2}$$

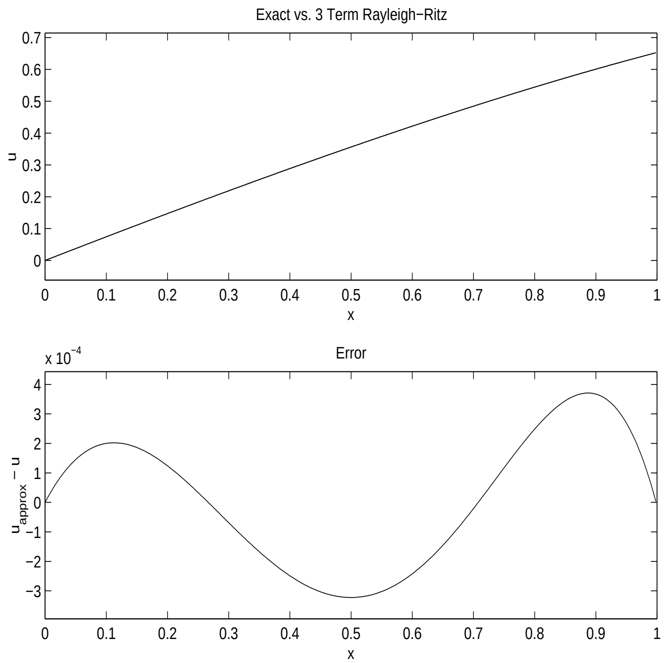
$$W_e = p_0 \begin{bmatrix} L^2 \\ 5/6\, L^3 \\ 3/4\, L^4 \end{bmatrix}$$

Solving for $\mathbf{c}$,

$$\mathbf{c} = \frac{p_0}{EA} \begin{bmatrix} \frac{47}{63}\, L \\ -\frac{5}{126} \\ -\frac{10}{189}\, L^{-1} \end{bmatrix}$$

$$\boxed{\bar{u} = \frac{p_0}{EA} \left(\frac{47}{63} Lx - \frac{5}{126} x^2 - \frac{10}{189} \frac{x^3}{L} \right) \qquad P(x) = \frac{p_0}{63} \left[94L - 57x - 15\frac{x^2}{L} + 10\frac{x^3}{L^2} \right]}$$

The following graphs show both the three term Rayleigh-Ritz and exact solution - they are so close you can't see the difference. The second plot shows the difference between them - note the scale.



2 Problem 2

You are asked to re-solve problem 1 using finite elements.

2.1 Part a)

For a two-node element, the shape functions and the strain-displacement matrix is:

$$\mathbf{N}(x) = \left[\begin{array}{cc} 1 - \frac{x}{L} & \frac{x}{L} \end{array} \right]$$

$$\mathbf{B}(x) = \left[\begin{array}{cc} \frac{-1}{L} & \frac{-1}{L} \end{array} \right]$$

Parameterize cross-sectional area:

$$A(x) = A_1 + (A_2 - A_1)\,\frac{x}{L_e} \qquad L_e = \text{length of element}$$

The element stiffness matrix is defined as:

$$\mathbf{k} \equiv \int_0^{L_e} \mathbf{B}^{\dagger} E A(x) \mathbf{B} dx$$

Performing the integration, we find,

$$\mathbf{k} = \left[\begin{array}{cc} \frac{1}{2}\frac{A_2 - A_1}{L_e} + \frac{A_1}{L_e} & \frac{-1}{2}\frac{A_2 - A_1}{L_e} - \frac{A_1}{L_e} \\ \frac{-1}{2}\frac{A_2 - A_1}{L_e} - \frac{A_1}{L_e} & \frac{1}{2}\frac{A_2 - A_1}{L_e} + \frac{A_1}{L_e} \end{array} \right]$$

Load vector:

$$\mathbf{p} = \int_0^{L_e} p(x) \mathbf{N} dx = L_e \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right]$$

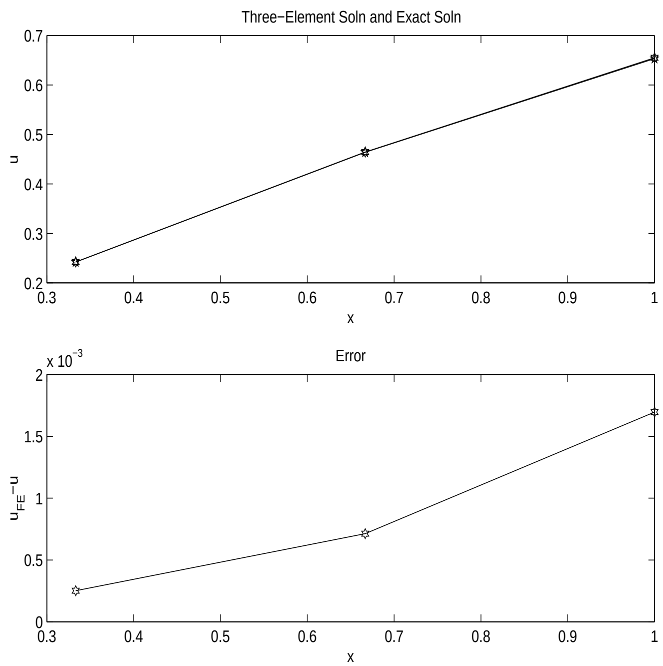
2.2 Part b)

The assembled global stiffness matrix and load vector are (using $L_e = \frac{L}{2}$):

$$\mathbf{K} = \frac{EA}{2L}\begin{bmatrix} 20 & -9 & 0 \\ -9 & 8 & -7 \\ 0 & -7 & 7 \end{bmatrix} \qquad \mathbf{P} = \frac{p_0 L}{3}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\mathbf{u} = \frac{p_0 L^2}{EA}\begin{bmatrix} 0.2424 \\ 0.4646 \\ 0.6551 \end{bmatrix}$$

See attached MATLAB script for details. Here are the plots for 3 elements:



Note that a 3 element solution is not as good as the 3-term Rayleigh-Ritz solution in part a), but it is still pretty good. This is due to the fact that our choice of finite element restricted us to piece-wise linear solutions.

2.3 Part c)

The axial force is computed by:

$$\mathbf{P} = EA(x)\mathbf{B}\mathbf{u}$$

$$\mathbf{P} = p_0 L \left[\ \frac{16}{11} - \frac{8}{11}x \quad \frac{10}{9} - \frac{2}{3}x \quad \frac{20}{21} - \frac{4}{7}x\ \right]$$

2.4 Part d)

The procedure for finding element equilibrium is analogous to that for find member forces in a structure by mapping the global DOF's to the local DOF's and using a basic element. This is covered in the lecture notes on page 8 in Lecture 4. See the accompanying MATLAB script for the details, here are the results:

$$\mathbf{p_e} = p_0 L \begin{bmatrix} -1.5000 & -1.1667 & -0.8333 \\ 1.1667 & 0.8333 & 0.5000 \end{bmatrix}$$

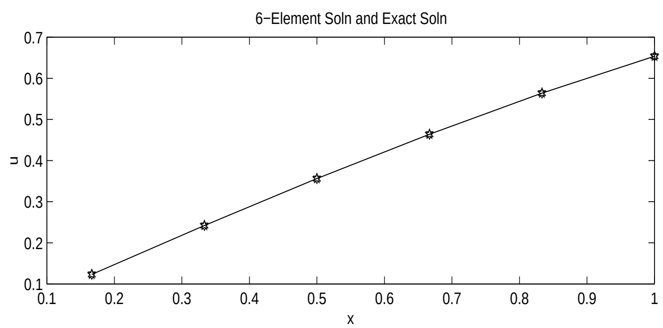
Each column represents an element, the first entry in each column is local DOF 1, second entry local DOF 2.

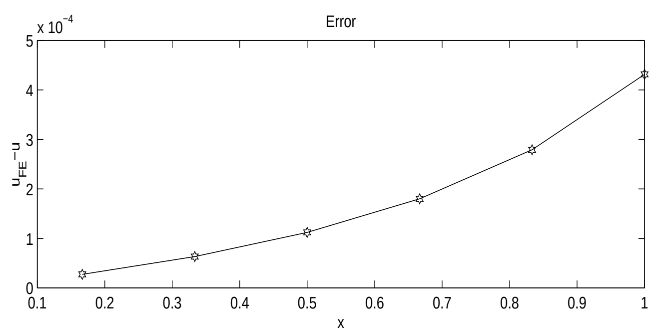
2.5 part e) repeat with 6 elements

$$\mathbf{u} = \frac{p_0 L^2}{EA} \begin{bmatrix} 0.1232 \\ 0.2422 \\ 0.3563 \\ 0.4641 \\ 0.5641 \\ 0.6539 \end{bmatrix}$$

$$\mathbf{P} = p_0 L \begin{bmatrix} -0.7391x + 1.4783 \\ -0.7143x + 1.3095 \\ -0.6842x + 1.2544 \\ -0.6471x + 1.1863 \\ -0.6000x + 1.1000 \\ -0.5385x + 0.9872 \end{bmatrix}$$

$$\mathbf{pe} = p_0 L \begin{bmatrix} -1.5000 & -1.3333 & -1.1667 & -1.0000 & -0.8333 & -0.6667 \\ 1.3333 & 1.1667 & 1.0000 & 0.8333 & 0.6667 & 0.5000 \end{bmatrix}$$





The convergence is linear. It easy to see this by running several different number of elements and looking at the size of the error.

2.6 Part f) total potential

$$\mathbf{\Pi} = \sum_e \frac{1}{2}\mathbf{u}^\dagger \cdot \mathbf{k_e} \cdot \mathbf{u} + \mathbf{u}^\dagger \cdot \mathbf{p_e} - p(L) * u(L) = \frac{1}{2}\mathbf{U}^\dagger \cdot \mathbf{K} \cdot \mathbf{U} + \mathbf{U}^\dagger \cdot \mathbf{P_0} - \mathbf{U}^\dagger \cdot \mathbf{P}(L)$$

For the 3-element solution, $\mathbf{\Pi} = -0.3362\frac{p_0^2 L^3}{EA}$ and for the 6-element solution, $\mathbf{\Pi} = -0.3365\frac{p_0^2 L^3}{EA}$.

3 Problem 3

3.1 3-Node Element

For a 3-node element, it is easily verified that the following choice of shape functions give the desired behavior:

$$\mathbf{N}(x) = \begin{bmatrix} \frac{(x - x_b)(x - x_c)}{(x_a - x_b)(x_a - x_c)} & \frac{(x - x_a)(x - x_c)}{(x_b - x_a)(x_b - x_c)} & \frac{(x - x_a)(x - x_b)}{(x_c - x_a)(x_c - x_b)} \end{bmatrix}$$

where x_a is location of first node on element, x_b is location of middle node on element, and x_c is location of last node. Forming the element stiffness matrix as before,

$$\mathbf{k} = \frac{EA}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

and element load vector,

$$\mathbf{p} = \frac{p_0 L}{6} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

solving for the only degree of freedom at node 2,

$$\boxed{u = \frac{p_0 L^2}{8EA}}$$

3.2 2-Node Element

Recall, the element stiffness matrix and load vector are:

$$\mathbf{k} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{p} = \frac{L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

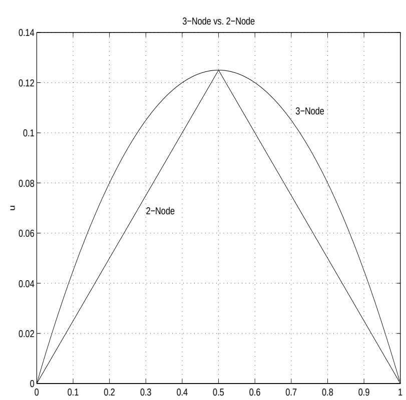
since we are asked to use two elements, the length in the above equations is half the total. Recognizing this, and assembling:

$$\mathbf{K} = \frac{EA}{L}[4] \qquad \mathbf{P} = p_0 \frac{L}{2}$$

solving,

$$\boxed{u = \frac{p_0 L^2}{8EA}}$$

The accuracy at the node is the same - they are both exact. However, anywhere else, the single 3-node element is much better, as you recall from PS #1, the exact solution to this problem is a quadratic. Since the 3-node element has quadratic functions, we expect to get the exact solution. The 2-node element only has linear functions, so it can never give exact solutions to problems that have quadratic behavior.



4 Problem 4

The BVP we want to solve is:

$$\frac{\partial}{\partial x}\left[EA\frac{\partial u}{\partial x}\right] - ku = 0 \quad EAu'(0) = P_0 \quad EAu'(L) = 0 \quad k = \frac{\alpha EA}{L^2} \tag{2}$$

Note that the force from the distributed spring depends upon the unknown displacement u.

4.1 Weak Form

Developing the weak form consists of choosing a suitable space of functions from which to select a trial function, multiplying the differential equation by an arbitrary member from the trial solution space, integrating by parts to reduce the order of derivative on the unknown, and incorporating the boundary conditions.

For this problem, since there are no displacement boundary conditions, the suitable function space is:

$$\mathfrak{V} = \left\{ v(x) \mid \quad v \in C^1 \right\}$$

Construct the weak form as follows:

Let $v \in \mathfrak{V}$, then

$$\int_0^L \frac{\partial}{\partial x}\left[EA\frac{\partial u}{\partial x}\right] v dx - \int_0^L kuvdx = 0$$

Integrate by parts,

$$-\int_0^L \left[EA\frac{\partial u}{\partial x}\right]\frac{\partial v}{\partial x}dx + EAu'v\Big|_0^L - \int_0^L kuvdx = 0$$

Applying the boundary conditions $EAu'(0) = P_0$ and $EAu'(L) = 0$

$$\boxed{EAu'(0)v(0) - \int_0^L \left[EA\frac{\partial u}{\partial x}\right]\frac{\partial v}{\partial x}dx - \int_0^L kuvdx = 0}$$

4.2 Element Stiffness for 2-Node Element

The key feature to this problem is to notice that the distributed spring depends upon the unknown displacement. Hence, assuming we use the Galerkin form, when we integrate the spring term, we will get a matrix term $\mathbf{N}^\dagger\mathbf{N}$, not a vector as we did for a body force independent of u.

Choose,

$$u(x) = \mathbf{N}\mathbf{u} \qquad v(x) = \mathbf{N}\mathbf{v}$$

For 2-node element,

$$\mathbf{N} = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix}$$

$$\mathbf{k} = \int_0^L EA\mathbf{B}^\dagger\mathbf{B} + \int_0^L k\mathbf{N}^\dagger\mathbf{N}$$

$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{kL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1+\frac{\alpha}{3} & -1+\frac{\alpha}{6} \\ -1+\frac{\alpha}{6} & 1+\frac{\alpha}{3} \end{bmatrix}$$

4.3 3-Node Element

Following exactly the same procedure, but using the 3-node element shape functions from Problem 3,

$$\mathbf{k} = \frac{EA}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{\alpha EA}{30L}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$