CE 222 HW #1 Solution

Dan Simkins

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1 Problem 1

The boundary value problem (BVP) to be solved is:

$$\frac{\partial}{\partial x}\left[EA\frac{\partial u}{\partial x}\right] + p(x) = 0 \qquad u(-\frac{L}{2}) = 0 \qquad EAu'(\frac{L}{2}) = 0 \tag{1}$$

1.1 Part a) $p(x) = p_0$

1.1.1 Exact Solution

This is a second-order linear differential equation with constant coefficients. We can solve it by direct integration. The general solution is:

$$u(x) = \frac{p_0}{EA}\left(-\frac{x^2}{2} + Cx + D\right)$$

Applying the two boundary conditions, we find the solution to be:

$$\boxed{u(x) = \frac{p_0}{EA}\left(-\frac{x^2}{2} + \frac{xL}{2} + \frac{3L^2}{8}\right)}$$

The axial force is $P(x) = EAu'$,

$$\boxed{P(x) = p_0\left(-x + \frac{L}{2}\right)}$$

1.1.2 Rayleigh Soluton

The Rayleigh method requires us to choose a form for the solution with one unknown parameter. We then determine the value of the parameter that gives the best approximation of our chosen form for the particular problem at hand. The *only* requirements on the selection of approximate solution is that it satisfy the displacement (also known as *essential*) boundary conditions and posess a derivative. For this problem, I selected $\bar{u} = c(x + \frac{L}{2})$. Note the use of the symbol $\bar{u}$, it is to distinguish the *approximate* solution from the exact solution.

$$\bar{u} = c(x + \frac{L}{2}) \qquad \bar{\epsilon} = c$$

$$\delta u = \delta c(x + \frac{L}{2}) \qquad \delta\bar{\epsilon} = \delta c$$

Plug into the virtual work expression, after integrating over the cross section,

$$W_i = \int_{-\frac{L}{2}}^{\frac{L}{2}} EAc\delta c dx = EAc\delta cL$$

$$W_e = \int_{-\frac{L}{2}}^{\frac{L}{2}} p_0 \delta c dx = \delta c L$$

Solving for c,

$$c = \frac{p_0 L}{2EA}$$

$$\boxed{\bar{u} = \frac{p_0 L}{2EA}\left(x + \frac{L}{2}\right) \qquad \bar{P} = \frac{p_0 L}{2}}$$

1.2 Part b) $p(x) = a_0 x$

1.2.1 Exact Solution

The general solution is:

$$u(x) = \frac{a_0}{EA}\left(-\frac{x^3}{6} + Cx + D\right)$$

Applying the two boundary conditions, we find the solution to be:

$$\boxed{u(x) = \frac{a_0}{EA}\left(-\frac{x^3}{6} + \frac{xL^2}{8} + \frac{L^3}{24}\right)}$$

The axial force is $P(x) = EAu'$,

$$\boxed{P(x) = a_0\left(-\frac{x^2}{2} + \frac{L^2}{8}\right)}$$

1.2.2 Rayleigh Soluton

Use the same trial function as before.

$$\bar{u} = c(x + \frac{L}{2}) \qquad \bar{\epsilon} = c$$

$$\delta u = \delta c(x + \frac{L}{2}) \qquad \delta\bar{\epsilon} = \delta c$$

Plug into the virtual work expression, after integrating over the cross section,

$$W_i = \int_{-\frac{L}{2}}^{\frac{L}{2}} EAc\delta c dx = EAc\delta c L$$

$$W_e = \int_{-\frac{L}{2}}^{\frac{L}{2}} a_0 \delta c x dx = \delta c \frac{L^2}{2}$$

Solving for c,

$$c = \frac{a_0 L^2}{24EA}$$

$$\boxed{\bar{u} = \frac{a_0 L^2}{24EA}\left(x + \frac{L}{2}\right) \qquad \bar{P} = \frac{a_0 L^2}{24EA}}$$

2 Problem 2

The difference between this problem and the first is the boundary condition. The BVP to be solved is:

$$\frac{\partial}{\partial x}\left[EA\frac{\partial u}{\partial x}\right] + p(x) = 0 \qquad u(-\frac{L}{2}) = 0 \qquad u(\frac{L}{2}) = 0 \tag{2}$$

2.1 Part a) $p(x) = p_0$

2.1.1 Exact Solution

The general solution is:

$$u(x) = \frac{p_0}{EA}\left(-\frac{x^2}{2} + Cx + D\right)$$

Applying the two boundary conditions, we find the solution to be:

$$\boxed{u(x) = \frac{p_0}{EA}\left(-\frac{x^2}{2} + \frac{L^2}{8}\right)}$$

We could have guessed that $C = 0$ due to the symmetry of the problem. The axial force is $P(x) = EAu'$,

$$\boxed{P(x) = -p_0 x}$$

2.1.2 Rayleigh Soluton

I could choose a polynomial for my trial solution, as I did in Problem 1, but I don't have to. Whatever trial function I choose, it must satisfy the essential boundary conditions. Three candidates are:

$$\bar{u} = c(x + \frac{L}{2})(x - \frac{L}{2}) \tag{3}$$

$$\bar{u} = c\, cos(\frac{x\pi}{L}) \tag{4}$$

$$\bar{u} = c\, sin(\frac{2x\pi}{L}) \tag{5}$$

$$\bar{u} = c(e^x - e^{-L/2})(e^{-x} - e^{-L/2})e^x \tag{6}$$

We will discuss these choices later. Obviously, since the first one is the exact solution, it is the best. However, one generally doesn't know the exact solution, for if one did, there wouldn't be much point in finding an approximate soln. I chose to use Eqn. (4).

$$\bar{u} = c\, cos(\frac{x\pi}{L}) \qquad \bar{\epsilon} = -c\frac{\pi}{L}sin(\frac{x\pi}{L})$$

$$\delta u = \delta c\, cos(\frac{x\pi}{L}) \qquad \delta\bar{\epsilon} = -\delta c\frac{\pi}{L}sin(\frac{x\pi}{L})$$

Plug into the virtual work expression, after integrating over the cross section,

$$W_i = \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(-EAc\frac{\pi}{L}sin(\frac{x\pi}{L})\right)\left(-\delta c\frac{\pi}{L}sin(\frac{x\pi}{L})\right)dx = EAc\delta c\frac{\pi^2}{2L}$$

$$W_e = \int_{-\frac{L}{2}}^{\frac{L}{2}} p_0 \delta c \, cos(\frac{\pi x}{L}) dx = \frac{2 p_0 L}{\pi} \delta c$$

Solving for c,

$$c = \frac{4 p_0 L^2}{\pi^3 EA}$$

$$\boxed{\bar{u} = \frac{4 p_0 L^2}{\pi^3 EA} cos(\frac{x\pi}{L}) \qquad \bar{P} = -\frac{4 p_0 L}{\pi^2} sin(\frac{x\pi}{L})}$$

2.2 Part b) $p(x) = a_0 x$

2.2.1 Exact Solution

The general solution is:

$$u(x) = \frac{p_0}{EA} \left(-\frac{x^3}{6} + Cx + D \right)$$

Applying the two boundary conditions, we find the solution to be:

$$\boxed{u(x) = \frac{p_0}{EA} \left(-\frac{x^3}{6} + \frac{xL^2}{24} \right)}$$

The axial force is $P(x) = EAu'$,

$$\boxed{P(x) = p_0 \left(-\frac{x^2}{2} + \frac{L^2}{24} \right)}$$

2.2.2 Rayleigh Soluton

As instructed, use the same trial function as in part a). The internal virtual work is not dependent on the load, so it is the same as in part a):

$$W_i = \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(-EAc\frac{\pi}{L} sin(\frac{x\pi}{L}) \right) \left(-\delta c \frac{\pi}{L} sin(\frac{x\pi}{L}) \right) dx = EAc\delta c \frac{\pi^2}{2L}$$

$$W_e = \int_{-\frac{L}{2}}^{\frac{L}{2}} p_0 \delta c \, x \, cos(\frac{\pi x}{L}) dx = 0$$

Solving for c,

$$c = 0$$

What happened? This tells us that the best possible solution for a cosine function is zero! The problem goes back to a fundamental property of our methodology. What we do is find the *best possible solution, restricted to the given trial function.* Sometimes, the best is zero. When is this? Well, consider a completely anti-symmetric function, x, on the interval $[-1, 1]$. If one were to approximate this function by a constant, what is the best possible constant to use? The only fair choice is the constant function 0, because any non-zero value will be biased for one half of the interval over the other. In particular, for this problem, the exact solution is a completely anti-symmetric function of x, yet the trial function I chose is completely symmetric. The best possible symmetric function to approximate a completely anti-symmetric function is the zero function (and vice-versa). I would do much better choosing as a trial function one that is completely anti-symmetric, e.g., the sine function. I could also choose a function that is neither symmetric nor anti-symmetric, that is the choice in Eqn. (6).

3 Problem 3

3.1 Rayleigh Solution

I chose to use the same trial function as I did for problem 2a).

$$\bar{u} = c\,cos(\frac{x\pi}{L}) \qquad \bar{\epsilon} = -c\frac{\pi}{L}sin(\frac{x\pi}{L})$$

$$\delta u = \delta c\,cos(\frac{x\pi}{L}) \qquad \delta\bar{\epsilon} = -\delta c\frac{\pi}{L}sin(\frac{x\pi}{L})$$

Plug into the virtual work expression, after integrating over the cross section,

$$W_i = \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(-EA\left(1-\frac{x}{L}\right)c\frac{\pi}{L}sin(\frac{x\pi}{L})\right)\left(-\delta c\frac{\pi}{L}sin(\frac{x\pi}{L})\right)dx = EAc\delta c\frac{\pi^2}{2L}$$

The resulting integral did not change from the previous problem because sin^2 is even, and x is odd, and the integral of an odd function over symmetric limits is zero, so the variation in the cross-sectional area did not change the resulting approximate solution. The external virtual work term does not change from that in Problem 2a), so the value of c is the same:

$$c = \frac{4p_0L^2}{\pi^3 EA}$$

$$\boxed{\bar{u} = \frac{4p_0L^2}{\pi^3 EA}cos(\frac{x\pi}{L}) \qquad \bar{P} = -\frac{4p_0L}{\pi^2}sin(\frac{x\pi}{L})}$$

3.2 Exact Solution

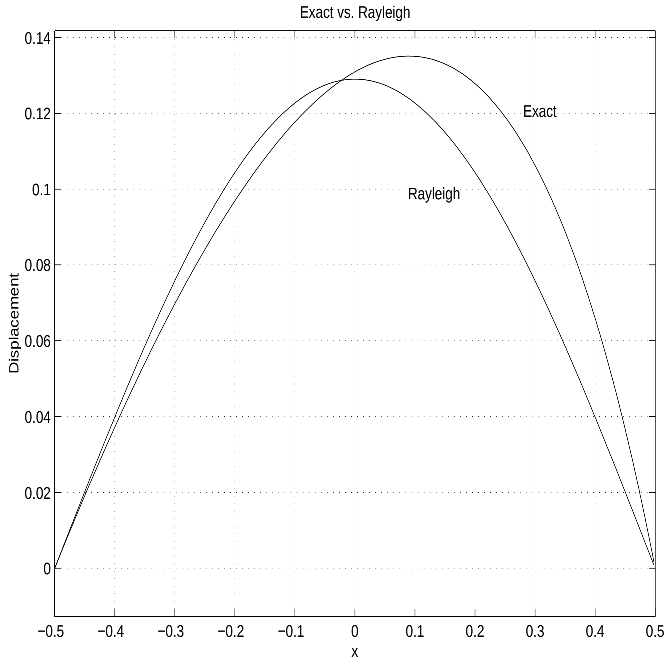
The exact solution was not required for this problem, but I computed it for comparison sake. The BVP is the same as for problem 2, part a. The difference, of course, is that the cross-sectional area changes.

$$A(x) = A\left(1-\frac{x}{L}\right)$$

After some work, the equation can be solved to yield:

$$u(x) = \frac{p_0L^2}{EA}\left\{\frac{x}{L} + \frac{\ln\left(1-\frac{x}{L}\right)}{\ln 3} + \frac{\ln\frac{4}{3}}{2\ln 3}\right\}$$

I have included in Figure 1 a plot comparing the displacement from the approximate solution to the exact solution. All constants set to 1.



4 Problem 4

This problem has no loads, but an initial strain. Since the structure is unrestrained, the initial strain can not cause stress. I like to think of this as a thermal strain. The BVP we want to solve is:

$$\frac{\partial}{\partial x}[EA\epsilon_{el}] = 0 \quad u(0) = 0 \quad EA\epsilon_{el}(L) = 0 \quad \frac{\partial u}{\partial x} = \epsilon_{tot} \quad \epsilon_{tot} = \epsilon_{el} + \epsilon_0 \tag{7}$$

Note that there are two strains, one from the initial, one due to elastic deformation. Note also, that the equilibrium equation involves the elastic strain, not the initial strain.

4.1 Exact Solution

$$\frac{\partial}{\partial x}[EA\epsilon_{el}] = 0$$

$$\epsilon_{el} = \frac{C}{EA}$$

Apply boundary condition,

$$\epsilon_{el}(L) = 0 = \frac{C}{EA} \qquad \Longrightarrow \qquad C = 0$$

$$\epsilon_{el} \equiv 0 \qquad \Longrightarrow \qquad \epsilon_{tot} = \epsilon_0$$

$$\boxed{u = \epsilon_0 \frac{x^2}{2} \qquad P = 0}$$

This is what we expect, if a simply supported bar is heated, no internal stress is generated and the total strain is just the thermal strain.

4.2 Weak Form

Developing the weak form consists of choosing a suitable space of functions from which to select a trial function, multiplying the differential equation by an arbitrary member from the trial solution space, integrating by parts to reduce the order of derivative on the unknown, and incorporating the boundary conditions.

For this problem, the suitable function space is:

$$\mathfrak{V} = \left\{ v(x) \mid v(0) = 0 \quad ; \quad v \in C^1 \right\}$$

Construct the weak form as follows:
Let $v \in \mathfrak{V}$, then

$$\int_0^L \frac{\partial}{\partial x} \left[EA\epsilon_{el} \right] v dx = 0$$

Integrate by parts,

$$-\int_0^L \left[EA\epsilon_{el} \right] \frac{\partial v}{\partial x} dx + \left[EA\epsilon_{el} \right] v \Big|_0^L = 0$$

Applying the boundary conditions $v(0) = 0$ and $EA\epsilon_{el}(L) = 0$

$$-\int_0^L \left[EA\epsilon_{el} \right] \frac{\partial v}{\partial x} dx = 0$$

We would like to put this into a form that contains only the known information and the unknown, u:

$$\boxed{-\int_0^L EA \left[\frac{\partial u}{\partial x} - \epsilon_a x \right] \frac{\partial v}{\partial x} dx = 0}$$

To solve this using Rayleigh's Method, choose a form for $\bar{u}$,

$$\bar{u} = a \sin\left(\frac{\pi x}{2L}\right) \qquad \bar{\epsilon} = a \frac{\pi}{2L} \cos\left(\frac{\pi x}{2L}\right)$$

$$\delta u = \delta a \sin\left(\frac{\pi x}{2L}\right) \qquad \delta \epsilon = \delta a \frac{\pi}{2L} \cos\left(\frac{\pi x}{2L}\right)$$

$$W_i = \int_0^L EA \left[a \frac{\pi}{2L} \cos\left(\frac{\pi x}{2L}\right) - \epsilon_a x \right] \left[\delta a \frac{\pi}{2L} \cos\left(\frac{\pi x}{2L}\right) \right] dx$$

Since there are no external loads, the external work term is 0. Performing the integrals, cancelling the δa, and solving for a,

$$a = \epsilon_a 4L^2 \frac{\pi - 2}{\pi^3} \sin\left(\frac{\pi x}{2L}\right)$$

$$\boxed{\bar{u} = \epsilon_a 4L^2 \frac{\pi - 2}{\pi^3} \sin\left(\frac{\pi x}{2L}\right)}$$