QUANTUM DYNAMICS

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Abstract

CONTENTS

1. Introduction

Quantum dynamics is quantization of classical dynamics. In quantum theory physicists despite several schemes of quantization. First by the restriction of smallest possible action in nature (quantum hypothesis, Planck, 1900) and then by the transformation of quantities to operators (Schrödinger, 1926). Transformation of classical (poisson) brackets to quantum commutators is a further step of quantization (Heisenberg, 1927). In present paper Schrödinger quantization rule is considered as a basic postulate for the quantization of Newtonian, Hamilton-Jacobi and Euler-Lagrange dynamics and for other quantum problems. Schrödinger proposed to quantize quantities to their operators by an interpretation of Schrödinger (psi) wave. Dirac later advanced this analysis by description of *bra-ket* notations. Heisenberg transformed poisson brackets to quantum commutators and putted quantum dynamics forward with *Canonical quantization*. Scheme of operator representation of dynamical properties instead of functional transformations is regarded Schrödinger quantization rule which would be used for the construction of future dynamics following above postulates.

2. Quantum description of systems

Quantities in quantum language are considered as the expectation of their quantum mechanical (operator) values with description of perturbation theory

$$\alpha = \frac{\langle\psi|\hat{\mathcal{A}}|\psi\rangle}{\langle\psi|\mathbb{I}|\psi\rangle} \tag{2.1}$$

with eigenvalues equation $\hat{\mathcal{A}}|\psi\rangle = \alpha|\psi\rangle$. For the sake of complete description of systems in *configuration space* we consider *normalization condition*

$$\langle\psi|\mathbb{I}|\psi\rangle = 1 \tag{2.2}$$

Quantities in *normalized configuration* are given as expectation of their operators as follows

$$\alpha = \langle\hat{\mathcal{A}}\rangle = \langle\psi|\hat{\mathcal{A}}|\psi\rangle \tag{2.3}$$

Differential study of this description with many order derivatives yield some quantum description of systems. We must use following differential-integral rule at all

$$\hat{f}\int K(\tau,\tau')\ d\tau = \int \hat{f}K(\tau,\tau')\ d\tau \tag{2.4}$$

§ *2.1. First order quantum description of systems*

Differentiate $\langle\hat{\mathcal{A}}\rangle$ w.r.t. time

$$\frac{\partial}{\partial t}\langle\hat{\mathcal{A}}\rangle = \langle\frac{\partial\psi}{\partial t}|\hat{\mathcal{A}}|\psi\rangle + \langle\psi|\frac{\partial\hat{\mathcal{A}}}{\partial t}|\psi\rangle + \langle\psi|\hat{\mathcal{A}}|\frac{\partial\psi}{\partial t}\rangle \tag{2.1.1}$$

Energy eigenvalue equation

$$|\frac{\partial\psi}{\partial t}\rangle = -\frac{i}{\hbar}|\hat{H}\psi\rangle \qquad ; \qquad \langle\frac{\partial\psi}{\partial t}| = \frac{i}{\hbar}\langle\hat{H}^{+}\psi| \tag{2.1.2}$$

leads

$$\frac{\partial}{\partial t}\langle\hat{\mathcal{A}}\rangle = \langle\psi|\frac{\partial\hat{\mathcal{A}}}{\partial t}|\psi\rangle + \frac{i}{\hbar}\Big[\langle\hat{H}^{+}\psi|\hat{\mathcal{A}}|\psi\rangle - \langle\psi|\hat{\mathcal{A}}\hat{H}|\psi\rangle\Big] \tag{2.1.3}$$

Hermiticity of Hamiltonian

$$\langle\hat{H}^{\dagger}\psi|\hat{\mathcal{A}}|\psi\rangle = \langle\psi|\hat{H}\hat{\mathcal{A}}|\psi\rangle \tag{2.1.4}$$

Yields

$$\frac{\partial}{\partial t}\langle\hat{\mathcal{A}}\rangle=\langle\frac{\partial\hat{\mathcal{A}}}{\partial t}\rangle+\frac{i}{\hbar}\langle[\hat{H},\hat{\mathcal{A}}]_{-}\rangle \tag{2.1.5}$$

This is first order quantum description of systems. Exact prime of $\langle\hat{\mathcal{A}}\rangle$

$$\frac{d}{dt}\langle\hat{\mathcal{A}}\rangle=\langle\frac{d\psi}{dt}|\hat{\mathcal{A}}|\psi\rangle+\langle\psi|\frac{d\hat{\mathcal{A}}}{dt}|\psi\rangle+\langle\psi|\hat{\mathcal{A}}|\frac{d\psi}{dt}\rangle \tag{2.1.6}$$

Lagrangian eigenvalue equation

$$|\frac{d\psi}{dt}\rangle=\frac{i}{\hbar}|\hat{L}\psi\rangle \qquad ; \qquad \langle\frac{d\hat{\mathcal{A}}}{dt}|=-\frac{i}{\hbar}\langle\hat{L}^{+}\psi| \tag{2.1.7}$$

leads

$$\frac{d}{dt}\langle\hat{\mathcal{A}}\rangle=\langle\psi|\frac{d\hat{\mathcal{A}}}{dt}|\psi\rangle-\frac{i}{\hbar}\Big[\langle\hat{L}^{+}\psi|\hat{\mathcal{A}}|\psi\rangle-\langle\psi|\hat{\mathcal{A}}\hat{L}|\psi\rangle\Big] \tag{2.1.8}$$

Hermiticity of Lagrangian

$$\langle\hat{L}^{\dagger}\psi|\hat{\mathcal{A}}|\psi\rangle=\langle\psi|\hat{L}\hat{\mathcal{A}}|\psi\rangle \tag{2.1.9}$$

Yield

$$\frac{d}{dt}\langle\hat{\mathcal{A}}\rangle=\langle\frac{d\hat{\mathcal{A}}}{dt}\rangle-\frac{i}{\hbar}\langle[\hat{L},\hat{\mathcal{A}}]_{-}\rangle \tag{2.1.10}$$

This is first order quantum description of systems. Differentiate $\langle\hat{\mathcal{A}}\rangle$ w.r.t. q_{α}

$$\frac{\partial}{\partial q_{\alpha}}\langle\hat{\mathcal{A}}\rangle=\langle\frac{\partial\psi}{\partial q_{\alpha}}|\hat{\mathcal{A}}|\psi\rangle+\langle\psi|\frac{\partial\hat{\mathcal{A}}}{\partial q_{\alpha}}|\psi\rangle+\langle\psi|\hat{\mathcal{A}}|\frac{\partial\psi}{\partial q_{\alpha}}\rangle \tag{2.1.11}$$

momentum eigenvalue equation

$$|\frac{\partial\psi}{\partial q_{\alpha}}\rangle=\frac{i}{\hbar}|\hat{p}_{\alpha}\psi\rangle \qquad ; \qquad \langle\frac{\partial\psi}{\partial q_{\alpha}}|=-\frac{i}{\hbar}\langle\hat{p}_{\alpha}^{+}\psi| \tag{2.1.12}$$

leads

$$\frac{\partial}{\partial q_{\alpha}}\langle\hat{\mathcal{A}}\rangle=\langle\psi|\frac{\partial\hat{\mathcal{A}}}{\partial q_{\alpha}}|\psi\rangle-\frac{i}{\hbar}\Big[\langle\hat{p}_{\alpha}^{+}\psi|\hat{\mathcal{A}}|\psi\rangle-\langle\psi|\hat{\mathcal{A}}\hat{p}_{\alpha}|\psi\rangle\Big] \tag{2.1.13}$$

Hermiticity of momentum

$$\langle \hat{p}_\alpha^\dagger \psi | \hat{\mathcal{A}} | \psi \rangle = \langle \psi | \hat{p}_\alpha \hat{\mathcal{A}} | \psi \rangle \tag{2.1.14}$$

Yields

$$\frac{\partial}{\partial q_\alpha} \langle \hat{\mathcal{A}} \rangle = \langle \frac{\partial \hat{\mathcal{A}}}{\partial q_\alpha} \rangle - \frac{i}{\hbar} \langle [\hat{p}_\alpha, \hat{\mathcal{A}}]_- \rangle \tag{2.1.15}$$

This is first order quantum description of systems. Differentiate $\langle \hat{\mathcal{A}} \rangle$ w.r.t. p_α

$$\frac{\partial}{\partial p_\alpha} \langle \hat{\mathcal{A}} \rangle = \langle \frac{\partial \psi}{\partial p_\alpha} | \hat{\mathcal{A}} | \psi \rangle + \langle \psi | \frac{\partial \hat{\mathcal{A}}}{\partial p_\alpha} | \psi \rangle + \langle \psi | \hat{\mathcal{A}} | \frac{\partial \psi}{\partial p_\alpha} \rangle \tag{2.1.16}$$

Position eigenvalue equation

$$| \frac{\partial \psi}{\partial p_\alpha} \rangle = -\frac{i}{\hbar} | \hat{q}_\alpha \psi \rangle \quad ; \quad \langle \frac{\partial \psi}{\partial p_\alpha} | = \frac{i}{\hbar} \langle \hat{q}_\alpha^+ \psi | \tag{2.1.17}$$

leads

$$\frac{\partial}{\partial p_\alpha} \langle \hat{\mathcal{A}} \rangle = \langle \psi | \frac{\partial \hat{\mathcal{A}}}{\partial p_\alpha} | \psi \rangle + \frac{i}{\hbar} \Big[\langle \hat{q}_\alpha^+ \psi | \hat{\mathcal{A}} | \psi \rangle - \langle \psi | \hat{\mathcal{A}} \hat{q}_\alpha | \psi \rangle \Big] \tag{2.1.18}$$

Hermiticity of position

$$\langle \hat{q}_\alpha^+ \psi | \hat{\mathcal{A}} | \psi \rangle = \langle \psi | \hat{q}_\alpha \hat{\mathcal{A}} | \psi \rangle \tag{2.1.19}$$

Yields

$$\frac{\partial}{\partial p_\alpha} \langle \hat{\mathcal{A}} \rangle = \langle \frac{\partial \hat{\mathcal{A}}}{\partial p_\alpha} \rangle + \frac{i}{\hbar} \langle [\hat{q}_\alpha, \hat{\mathcal{A}}]_- \rangle \tag{2.1.20}$$

This is first order quantum description of systems. Consider quantum description of systems (2.1.5) and (2.1.10), one gets

$$\left(\frac{\partial}{\partial t} + \frac{d}{dt} \right) \langle \hat{\mathcal{A}} \rangle = \langle \left(\frac{\partial}{\partial t} + \frac{d}{dt} \right) \hat{\mathcal{A}} \rangle + \frac{i}{\hbar} \langle [(\hat{H} - \hat{L}), \hat{\mathcal{A}}]_- \rangle \tag{2.1.21}$$

Consider

$$2\hat{V} = \hat{H} - \hat{L} \quad ; \quad [2\hat{V}, \hat{\mathcal{A}}]_- = 2[\hat{V}, \hat{\mathcal{A}}]_- \tag{2.1.22}$$

One obtains

$$\left(\frac{\partial}{\partial t}+\frac{d}{dt}\right)\langle\hat{\mathcal{A}}\rangle=\langle\left(\frac{\partial}{\partial t}+\frac{d}{dt}\right)\hat{\mathcal{A}}\rangle+2\frac{i}{\hbar}\langle[\hat{V},\hat{\mathcal{A}}]_{-}\rangle \tag{2.1.23}$$

This is first order quantum description of systems. Consider again (2.1.5) and (2.1.10), one gets

$$\left(\frac{\partial}{\partial t}-\frac{d}{dt}\right)\langle\hat{\mathcal{A}}\rangle=\langle\left(\frac{\partial}{\partial t}-\frac{d}{dt}\right)\hat{\mathcal{A}}\rangle+\frac{i}{\hbar}\langle[(\hat{H}+\hat{L}),\hat{\mathcal{A}}]_{-}\rangle \tag{2.1.24}$$

Consider

$$2\hat{T}=\hat{H}+\hat{L} \quad ; \qquad [2\hat{T},\hat{\mathcal{A}}]_{-}=2[\hat{T},\hat{\mathcal{A}}]_{-} \tag{2.1.25}$$

One obtains

$$\left(\frac{\partial}{\partial t}-\frac{d}{dt}\right)\langle\hat{\mathcal{A}}\rangle=\langle\left(\frac{\partial}{\partial t}-\frac{d}{dt}\right)\hat{\mathcal{A}}\rangle+2\frac{i}{\hbar}\langle[\hat{T},\hat{\mathcal{A}}]_{-}\rangle \tag{2.1.26}$$

This is first order quantum description of systems. Operate unity operator to $\langle\hat{\mathcal{A}}\rangle$

$$\mathbb{I}\langle\hat{\mathcal{A}}\rangle=\langle\mathbb{I}\psi\,|\,\hat{\mathcal{A}}\,|\,\psi\rangle+\langle\psi\,|\,\mathbb{I}\hat{\mathcal{A}}\,|\,\psi\rangle+\langle\psi\,|\,\hat{\mathcal{A}}\,|\,\mathbb{I}\psi\rangle \tag{2.1.27}$$

Action eigenvalue equation

$$|\,\mathbb{I}\psi\rangle=\frac{i}{\hbar}\,|\,\hat{S}\psi\rangle \quad ; \qquad \langle\mathbb{I}\psi\,|=-\frac{i}{\hbar}\langle\hat{S}^{+}\psi\,| \tag{2.1.28}$$

leads

$$\mathbb{I}\langle\hat{\mathcal{A}}\rangle=\langle\psi\,|\,\mathbb{I}\hat{\mathcal{A}}\,|\,\psi\rangle-\frac{i}{\hbar}\Big[\langle\hat{S}^{+}\psi\,|\,\hat{\mathcal{A}}\,|\,\psi\rangle-\langle\psi\,|\,\hat{\mathcal{A}}\hat{S}\,|\,\psi\rangle\Big] \tag{2.1.29}$$

Hermiticity of action

$$\langle\hat{S}^{+}\psi\,|\,\hat{\mathcal{A}}\,|\,\psi\rangle=\langle\psi\,|\,\hat{S}\hat{\mathcal{A}}\,|\,\psi\rangle \tag{2.1.30}$$

Yields

$$\mathbb{I}\langle\hat{\mathcal{A}}\rangle=\langle\mathbb{I}\hat{\mathcal{A}}\rangle-\frac{i}{\hbar}\langle[\hat{S},\hat{\mathcal{A}}]_{-}\rangle \tag{2.1.31}$$

This is zero order quantum description of systems. Differentiate $\langle\hat{\mathcal{A}}\rangle$ w.r.t. action

$$\frac{d}{dS}\langle\hat{\mathcal{A}}\rangle = \langle\frac{d\psi}{dS}|\hat{\mathcal{A}}|\psi\rangle + \langle\psi|\frac{d\hat{\mathcal{A}}}{dS}|\psi\rangle + \langle\psi|\hat{\mathcal{A}}|\frac{d\psi}{dS}\rangle \tag{2.1.32}$$

Action eigenvalue-eigenoperator equation

$$|\frac{d\psi}{dS}\rangle = -\frac{1}{\hbar^2}|\hat{S}\psi\rangle \qquad ; \qquad \langle\frac{d\psi}{dS}| = -\frac{1}{\hbar^2}\langle\hat{S}^+\psi| \tag{2.1.33}$$

leads

$$\frac{d}{dS}\langle\hat{\mathcal{A}}\rangle = \langle\psi|\frac{d\hat{\mathcal{A}}}{dS}|\psi\rangle - \frac{1}{\hbar^2}\Big[\langle\hat{S}^+\psi|\hat{\mathcal{A}}|\psi\rangle + \langle\psi|\hat{\mathcal{A}}\hat{S}|\psi\rangle\Big] \tag{2.1.34}$$

Hermiticity of action yields

$$\frac{d}{dS}\langle\hat{\mathcal{A}}\rangle = \langle\frac{d\hat{\mathcal{A}}}{dS}\rangle - \frac{1}{\hbar^2}\langle[\hat{S},\hat{\mathcal{A}}]_+\rangle \tag{2.1.35}$$

This is first order quantum description of systems. Let consider derivative of $\langle\hat{\mathcal{A}}\rangle$ w.r.t. action (2.1.32) with *probability eigenvalue equation*

$$|\frac{d\psi}{dS}\rangle = \frac{i}{\hbar}|\hat{w}\psi\rangle \qquad ; \qquad \langle\frac{d\psi}{dS}| = -\frac{i}{\hbar}\langle\hat{w}^+\psi| \tag{2.1.36}$$

One obtains

$$\frac{d}{dS}\langle\widehat{\mathcal{A}}\rangle = \langle\frac{d\widehat{\mathcal{A}}}{dS}\rangle - \frac{i}{\hbar}\Big[\langle\hat{w}^\dagger\psi|\widehat{\mathcal{A}}|\psi\rangle - \langle\psi|\widehat{\mathcal{A}}\widehat{w}|\psi\rangle\Big] \tag{2.1.37}$$

Since the probability is a real aspect of nature, i.e., in operator representation, it must be hermitian

$$\langle\hat{w}^\dagger\psi|\widehat{\mathcal{A}}|\psi\rangle = \langle\psi|\widehat{w}\widehat{\mathcal{A}}|\psi\rangle \tag{2.1.38}$$

which yields

$$\frac{d}{dS}\langle\widehat{\mathcal{A}}\rangle = \langle\frac{d\widehat{\mathcal{A}}}{dS}\rangle - \frac{i}{\hbar}\langle[\widehat{w},\widehat{\mathcal{A}}]_-\rangle \tag{2.1.39}$$

This is first order quantum description of systems.

§ *2.2. Second order quantum description of systems*

Differentiate first order quantum description of systems (2.1.5) w.r.t. time and consider (2.1.5) for operators $\hat{\mathcal{A}} := \partial\hat{\mathcal{A}}/\partial t$ and $\hat{\mathcal{A}} := [\hat{H},\mathcal{A}]_-$ respectively, one obtains

$$\frac{\partial^2}{\partial t^2}\langle\hat{\mathcal{A}}\rangle=\langle\frac{\partial^2\hat{\mathcal{A}}}{\partial t^2}\rangle+\frac{i}{\hbar}\Big\langle\Big[[\hat{H},\frac{\partial\hat{\mathcal{A}}}{\partial t}]_-+\frac{\partial}{\partial t}[\hat{H},\hat{\mathcal{A}}]_-+\frac{i}{\hbar}[\hat{H},[\hat{H},\hat{\mathcal{A}}]_-]_-\Big]\Big\rangle \tag{2.2.1}$$

This is second order quantum description of systems. Differentiate first order quantum description of systems (2.1.10) exactly w.r.t. and consider (2.1.10) for operators $\hat{\mathcal{A}}:=d\hat{\mathcal{A}}/dt$ and $\hat{\mathcal{A}}:=[\hat{L},\hat{\mathcal{A}}]_-$ respectively, one obtains

$$\frac{d^2}{dt^2}\langle\hat{\mathcal{A}}\rangle=\langle\frac{d^2\hat{\mathcal{A}}}{dt^2}\rangle-\frac{i}{\hbar}\Big\langle\Big[[\hat{L},\frac{d\hat{\mathcal{A}}}{dt}]_-+\frac{d}{dt}[\hat{L},\hat{\mathcal{A}}]_--\frac{i}{\hbar}[\hat{L},[\hat{L},\hat{\mathcal{A}}]_-]_-\Big]\Big\rangle \tag{2.2.2}$$

This is second order quantum description of systems. Differentiate first order quantum description of systems (2.1.15) w.r.t. q_α and consider (2.1.15) for operators $\hat{\mathcal{A}}:=\partial\hat{\mathcal{A}}/\partial q_\alpha$ and $\hat{\mathcal{A}}:=[\hat{p}_\alpha,\hat{\mathcal{A}}]_-$ respectively, one obtains

$$\frac{\partial^2}{\partial q_\alpha^2}\langle\hat{\mathcal{A}}\rangle=\langle\frac{\partial^2\hat{\mathcal{A}}}{\partial t^2}\rangle-\frac{i}{\hbar}\Big\langle\Big[[\hat{p}_\alpha,\frac{\partial\hat{\mathcal{A}}}{\partial q_\alpha}]_-+\frac{\partial}{\partial q_\alpha}[\hat{p}_\alpha,\hat{\mathcal{A}}]_--\frac{i}{\hbar}[\hat{p}_\alpha,[\hat{p}_\alpha,\hat{\mathcal{A}}]_-]_-\Big]\Big\rangle \tag{2.2.3}$$

This is second order quantum description of systems. Differentiate first order quantum description of systems (2.1.20) w.r.t. p_α and consider (2.1.20) for operators $\hat{\mathcal{A}}:=\partial\hat{\mathcal{A}}/\partial p_\alpha$ and $\hat{\mathcal{A}}:=[\hat{q}_\alpha,\hat{\mathcal{A}}]_-$ respectively, one obtains

$$\frac{\partial^2}{\partial p_\alpha^2}\langle\hat{\mathcal{A}}\rangle=\langle\frac{\partial^2\hat{\mathcal{A}}}{\partial p_\alpha^2}\rangle+\frac{i}{\hbar}\Big\langle\Big[[\hat{q}_\alpha,\frac{\partial\hat{\mathcal{A}}}{\partial p_\alpha}]_-+\frac{\partial}{\partial p_\alpha}[\hat{q}_\alpha,\hat{\mathcal{A}}]_-+\frac{i}{\hbar}[\hat{q}_\alpha,[\hat{q}_\alpha,\hat{\mathcal{A}}]_-]_-\Big]\Big\rangle \tag{2.2.4}$$

This is second order quantum description of systems. Differentiate first order quantum description of systems (2.1.35) exactly w.r.t. action and consider (2.1.35) for operators $\hat{\mathcal{A}}:=d\hat{\mathcal{A}}/dS$ and $\hat{\mathcal{A}}:=[\hat{S},\hat{\mathcal{A}}]_+$ respectively, one obtains

$$\frac{d^2}{dS^2}\langle\hat{\mathcal{A}}\rangle=\langle\frac{d^2\hat{\mathcal{A}}}{dS^2}\rangle-\frac{1}{\hbar^2}\Big\langle\Big[[\hat{S},\frac{d\hat{\mathcal{A}}}{dS}]_++\frac{d}{dS}[\hat{S},\hat{\mathcal{A}}]_+-\frac{1}{\hbar^2}[\hat{S},[\hat{S},\hat{\mathcal{A}}]_+]_+\Big]\Big\rangle \tag{2.2.5}$$

This is second order quantum description of systems. Differentiate first order quantum description of systems (2.1.39) exactly w.r.t. action and consider (2.1.39) for operators $\hat{\mathcal{A}}:=d\hat{\mathcal{A}}/dS$ and $\hat{\mathcal{A}}:=[\hat{w},\hat{\mathcal{A}}]_-$ respectively, one obtains

$$\frac{d^2}{dS^2}\langle\widehat{\mathcal{A}}\rangle=\langle\frac{d^2\widehat{\mathcal{A}}}{dS^2}\rangle-\frac{i}{\hbar}\Big\langle\Big[[\widehat{w},\frac{d\widehat{\mathcal{A}}}{dS}]_-+\frac{d}{dS}[\widehat{w},\widehat{\mathcal{A}}]_--\frac{i}{\hbar}[\widehat{w},[\widehat{w},\widehat{\mathcal{A}}]_-]_-\Big]\Big\rangle \tag{2.1.6}$$

This is second order quantum description of systems.

§ *2.3. Third order quantum description of systems*

Differentiate second order quantum description of systems (2.2.1) w.r.t. time and consider first order quantum description of systems (2.1.5) for operators $\hat{\mathcal{A}} := \partial^2 \hat{\mathcal{A}} / \partial t^2$, $\hat{\mathcal{A}} := [\hat{H}, \partial \hat{\mathcal{A}} / \partial t]_-$, $\hat{\mathcal{A}} := \frac{\partial}{\partial t}[\hat{H}, \hat{\mathcal{A}}]_-$ and $\hat{\mathcal{A}} := [\hat{H}, [\hat{H}, \hat{\mathcal{A}}]_-]_-$ respectively, one obtains

$$
\begin{aligned}
\frac{\partial^3}{\partial t^3}\langle \hat{\mathcal{A}} \rangle = \langle \frac{\partial^3 \hat{\mathcal{A}}}{\partial t^3} \rangle \\
&+ \frac{i}{\hbar} \Big\langle \Big[[\hat{H}, \frac{\partial^2 \hat{\mathcal{A}}}{\partial t^2}]_- + \frac{\partial}{\partial t}[\hat{H}, \frac{\partial \hat{\mathcal{A}}}{\partial t}]_- + \frac{\partial^2}{\partial t^2}[\hat{H}, \hat{\mathcal{A}}]_- \\
&+ \frac{i}{\hbar} \Big[[\hat{H}, [\hat{H}, \frac{\partial \hat{\mathcal{A}}}{\partial t}]_-]_- + [\hat{H}, \frac{\partial}{\partial t}[\hat{H}, \hat{\mathcal{A}}]_- + \frac{\partial}{\partial t}[\hat{H}, [\hat{H}, \hat{\mathcal{A}}]_-]_- \\
&+ \frac{i}{\hbar}[\hat{H}, [\hat{H}, [\hat{H}, \hat{\mathcal{A}}]_-]_-]_- \Big] \Big] \Big\rangle
\end{aligned}
\qquad (2.3.1)
$$

This is third order quantum description of systems. Differentiate second order quantum description of systems (2.2.2) exactly w.r.t. time and consider first order quantum description of systems (2.1.10) for operators $\hat{\mathcal{A}} := d^2 \hat{\mathcal{A}} / dt^2$, $\hat{\mathcal{A}} := [\hat{L}, d\hat{\mathcal{A}} / dt]_-$, $\hat{\mathcal{A}} := \frac{d}{dt}[\hat{L}, \hat{\mathcal{A}}]_-$ and $\hat{\mathcal{A}} := [\hat{L}, [\hat{L}, \hat{\mathcal{A}}]_-]_-$ respectively, one obtains

$$
\begin{aligned}
\frac{d^3}{dt^3}\langle \hat{\mathcal{A}} \rangle = \langle \frac{d^3 \hat{\mathcal{A}}}{dt^3} \rangle \\
&- \frac{i}{\hbar} \Big\langle \Big[[\hat{L}, \frac{d^2 \hat{\mathcal{A}}}{dt^2}]_- + \frac{d}{dt}[\hat{L}, \frac{d \hat{\mathcal{A}}}{dt}]_- + \frac{d^2}{dt^2}[\hat{L}, \hat{\mathcal{A}}]_- \\
&- \frac{i}{\hbar} \Big[[\hat{L}, [\hat{L}, \frac{d \hat{\mathcal{A}}}{dt}]_-]_- + [\hat{L}, \frac{d}{dt}[\hat{L}, \hat{\mathcal{A}}]_-]_- + \frac{d}{dt}[\hat{L}, [\hat{L}, \hat{\mathcal{A}}]_-]_- \\
&- \frac{i}{\hbar}[\hat{L}, [\hat{L}, [\hat{L}, \hat{\mathcal{A}}]_-]_-]_- \Big] \Big] \Big\rangle
\end{aligned}
\qquad (2.3.2)
$$

This is third order quantum description of systems. Differentiate second order quantum description of systems (2.2.3) w.r.t. q_α and consider first order quantum description of systems (2.1.15) for operators $\hat{\mathcal{A}} := \partial^2 \hat{\mathcal{A}} / \partial q_\alpha^2$, $\hat{\mathcal{A}} := [\hat{p}_\alpha, \frac{\partial \hat{\mathcal{A}}}{\partial q_\alpha}]_-$, $\hat{\mathcal{A}} := \frac{\partial}{\partial q_\alpha}[\hat{p}_\alpha, \hat{\mathcal{A}}]_-$ and $\hat{\mathcal{A}} := [\hat{p}_\alpha, [\hat{p}_\alpha, \hat{\mathcal{A}}]_-]_-$ respectively, one obtains

$$
\begin{aligned}
\frac{\partial^3}{\partial q_\alpha^3}\langle\hat{\mathcal{A}}\rangle = \langle\frac{\partial^3\hat{\mathcal{A}}}{\partial q_\alpha^3}\rangle \\
&-\frac{i}{\hbar}\Bigg\langle\Bigg[[\hat{p}_\alpha,\frac{\partial^2\hat{\mathcal{A}}}{\partial q_\alpha^2}]_- + \frac{\partial}{\partial q_\alpha}[\hat{p}_\alpha,\frac{\partial\hat{\mathcal{A}}}{\partial q_\alpha}]_- + \frac{\partial^2}{\partial q_\alpha^2}[\hat{p}_\alpha,\hat{\mathcal{A}}]_- \\
&-\frac{i}{\hbar}\Bigg[[\hat{p}_\alpha,[\hat{p}_\alpha,\frac{\partial\hat{\mathcal{A}}}{\partial q_\alpha}]_-]_- + [\hat{p}_\alpha,\frac{\partial}{\partial q_\alpha}[\hat{p}_\alpha,\hat{\mathcal{A}}]_-]_- + \frac{\partial}{\partial q_\alpha}[\hat{p}_\alpha,[\hat{p}_\alpha,\hat{\mathcal{A}}]_-]_- \\
&-\frac{i}{\hbar}[\hat{p}_\alpha,[\hat{p}_\alpha,[\hat{p}_\alpha,\hat{\mathcal{A}}]_-]_-]_-\Bigg]\Bigg]\Bigg\rangle
\end{aligned}
\qquad (2.3.3)
$$

This is third order quantum description of systems. Differentiate second order quantum description of systems (2.2.4) w.r.t. p_α and consider first order quantum description of systems (2.1.20) for operators $\hat{\mathcal{A}} := \partial^2\hat{\mathcal{A}}/\partial p_\alpha^2$, $\hat{\mathcal{A}} := [\hat{q}_\alpha,\frac{\partial\hat{\mathcal{A}}}{\partial p_\alpha}]_-$ $\hat{\mathcal{A}} := \frac{\partial}{\partial p_\alpha}[\hat{q}_\alpha,\hat{\mathcal{A}}]_-$ and $\hat{\mathcal{A}} := [\hat{q}_\alpha,[\hat{q}_\alpha,\hat{\mathcal{A}}]_-]_-$ respectively, one obtains

$$
\begin{aligned}
\frac{\partial^3}{\partial p_\alpha^3}\langle\hat{\mathcal{A}}\rangle = \langle\frac{\partial^3\hat{\mathcal{A}}}{\partial p_\alpha^3}\rangle \\
&+\frac{i}{\hbar}\Bigg\langle\Bigg[[\hat{q}_\alpha,\frac{\partial^2\hat{\mathcal{A}}}{\partial p_\alpha^2}]_- + \frac{\partial}{\partial p_\alpha}[\hat{q}_\alpha,\frac{\partial\hat{\mathcal{A}}}{\partial p_\alpha}]_- + \frac{\partial^2}{\partial p_\alpha^2}[\hat{q}_\alpha,\hat{\mathcal{A}}]_- \\
&+\frac{i}{\hbar}\Bigg[[\hat{q}_\alpha,[\hat{q}_\alpha,\frac{\partial\hat{\mathcal{A}}}{\partial p_\alpha}]_-]_- + [\hat{q}_\alpha,\frac{\partial}{\partial p_\alpha}[\hat{q}_\alpha,\hat{\mathcal{A}}]_- + \frac{\partial}{\partial p_\alpha}[\hat{q}_\alpha,[\hat{q}_\alpha,\hat{\mathcal{A}}]_-]_- \\
&+\frac{i}{\hbar}[\hat{q}_\alpha,[\hat{q}_\alpha,[\hat{q}_\alpha,\hat{\mathcal{A}}]_-]_-]_-\Bigg]\Bigg]\Bigg\rangle
\end{aligned}
\qquad (2.3.4)
$$

This is third order quantum description of systems. Differentiate second order quantum description of systems (2.2.5) exactly w.r.t. action and consider first order quantum description of systems (2.1.35) for operators $\hat{\mathcal{A}} := d^2\hat{\mathcal{A}}/dS^2$, $\hat{\mathcal{A}} := [\hat{S},\frac{d\hat{\mathcal{A}}}{dS}]_+$, $\hat{\mathcal{A}} := \frac{d}{dS}[\hat{S},\hat{\mathcal{A}}]_+$ and $\hat{\mathcal{A}} := [\hat{S},[\hat{S},\hat{\mathcal{A}}]_+]_+$ respectively, one obtains

$$
\begin{aligned}
\frac{d^3}{dS^3}\langle\hat{\mathcal{A}}\rangle = \langle\frac{d^3\hat{\mathcal{A}}}{dS^3}\rangle \\
&-\frac{1}{\hbar^2}\Bigg\langle\Bigg[[\hat{S},\frac{d^2\hat{\mathcal{A}}}{dS^2}]_+ + \frac{d}{dS}[\hat{S},\frac{d\hat{\mathcal{A}}}{dS}]_+ + \frac{d^2}{dS^2}[\hat{S},\hat{\mathcal{A}}]_+ \\
&-\frac{1}{\hbar^2}\Bigg[[\hat{S},[\hat{S},\frac{d\hat{\mathcal{A}}}{dS}]_+]_+ + [\hat{S},\frac{d}{dS}[\hat{S},\hat{\mathcal{A}}]_+]_+ + \frac{d}{dS}[\hat{S},[\hat{S},\hat{\mathcal{A}}]_+]_+ \\
&-\frac{1}{\hbar^2}[\hat{S},[\hat{S},[\hat{S},\hat{\mathcal{A}}]_+]_+]_+\Bigg]\Bigg]\Bigg\rangle
\end{aligned}
\qquad (2.3.5)
$$

This is third order quantum description of systems. Differentiate second order quantum description of systems (2.2.6) exactly w.r.t. action and consider first order quantum description of systems (2.1.39) for operators $\hat{\mathcal{A}} := d^2\hat{\mathcal{A}}/dS^2$, $\widehat{\mathcal{A}} := [\hat{w}, \frac{d\widehat{\mathcal{A}}}{dS}]_-$, $\widehat{\mathcal{A}} := \frac{d}{dS}[\hat{w}, \widehat{\mathcal{A}}]_-$ and $\widehat{\mathcal{A}} := [\hat{w}, [\hat{w}, \widehat{\mathcal{A}}]_-]_-$ respectively, one obtains

$$
\begin{aligned}
\frac{d^3}{dS^3}\langle\widehat{\mathcal{A}}\rangle = \langle\frac{d^3\widehat{\mathcal{A}}}{dS^3}\rangle \\
&-\frac{i}{\hbar}\Big\langle\Big[[\widehat{w}, \frac{d^2\widehat{\mathcal{A}}}{dS^2}]_- + \frac{d}{dS}[\widehat{w}, \frac{d\widehat{\mathcal{A}}}{dS}]_- + \frac{d^2}{dS^2}[\widehat{w}, \widehat{\mathcal{A}}]_- \\
&-\frac{i}{\hbar}\Big[[\widehat{w}, [\widehat{w}, \frac{d\widehat{\mathcal{A}}}{dS}]_-]_- + [\widehat{w}, \frac{d}{dS}[\widehat{w}, \widehat{\mathcal{A}}]_-]_- + \frac{d}{dS}[\widehat{w}, [\widehat{w}, \widehat{\mathcal{A}}]_-]_- \\
&-\frac{i}{\hbar}[\widehat{w}, [\widehat{w}, [\widehat{w}, \widehat{\mathcal{A}}]_-]_-]_-\Big]\Big]\Big\rangle
\end{aligned}
\tag{2.3.6}
$$

This is third order quantum description of systems.

3. Quantum Newtonian dynamics

Newton's second law

$$
\mathcal{F} = \frac{dp_\alpha}{dt} = -\frac{\partial V}{\partial q_\alpha} \tag{3.1}
$$

is given in quantum language

$$
\langle\hat{\mathcal{F}}\rangle = \frac{d}{dt}\langle\hat{p}_\alpha\rangle = -\frac{\partial}{\partial q_\alpha}\langle\hat{V}\rangle \tag{3.2}
$$

quantum description of systems (2.1.10) and (2.1.15) for operators $\hat{\mathcal{A}} := \hat{p}_\alpha$ and $\hat{\mathcal{A}} := \hat{V}$

$$
\begin{aligned}
\frac{d}{dt}\langle\hat{p}_\alpha\rangle &= \langle\frac{d\hat{p}_\alpha}{dt}\rangle - \frac{i}{\hbar}\langle[\hat{L}, \hat{p}_\alpha]_-\rangle \\
\frac{\partial}{\partial q_\alpha}\langle\hat{V}\rangle &= \langle\frac{\partial\hat{V}}{\partial q_\alpha}\rangle - \frac{i}{\hbar}\langle[\hat{p}_\alpha, \hat{V}]_-\rangle
\end{aligned}
\tag{3.3}
$$

leads

$$\hat{\mathcal{F}} = \frac{d\hat{p}_\alpha}{dt} - \frac{i}{\hbar}[\hat{L}, \hat{p}_\alpha]_-$$
$$\hat{\mathcal{F}} = -\frac{\partial \hat{V}}{\partial q_\alpha} + \frac{i}{\hbar}[\hat{p}_\alpha, \hat{V}]_- \tag{3.4}$$

This is quantum dynamical Newton's second law. Its comparison yields

$$\frac{d\hat{p}_\alpha}{dt} = -\frac{\partial \hat{V}}{\partial q_\alpha} + \frac{i}{\hbar}[\hat{p}_\alpha, (\hat{V} - \hat{L})]_- \tag{3.5}$$

Following $\hat{L} = -i\hbar d/dt$ and $\hat{p}_\alpha = -i\hbar\partial/\partial q_\alpha$ (3.4) reduces to

$$\hat{\mathcal{F}} = \frac{i}{\hbar}\hat{p}_\alpha \hat{L} = -\frac{i}{\hbar}\hat{V}\hat{p}_\alpha = -i\hbar\frac{\partial}{\partial q_\alpha}\frac{d}{dt} \tag{3.6}$$

which follow *Ehrenfest* theorem

$$\mathcal{F} = \int \psi^* \frac{\partial \hat{L}}{\partial q_\alpha}\psi \, d\tau$$
$$\mathcal{F} = \frac{1}{2}\int \psi^* \left(\frac{\partial \hat{p}_\alpha}{\partial t} + \frac{d\hat{p}_\alpha}{dt}\right)\psi \, d\tau \equiv \int \psi^* \frac{d\hat{p}_\alpha}{dt}\psi \, d\tau \tag{3.7}$$

or

$$\mathcal{F} = \frac{i}{\hbar}\int \psi^* \hat{p}_\alpha \hat{L}\psi \, d\tau$$
$$\mathcal{F} = -\frac{i}{\hbar}\int \psi^* \hat{V}\hat{p}_\alpha\psi \, d\tau \tag{3.8}$$

(3.5) reduces to

$$\hat{V}\hat{p}_\alpha + \hat{p}_\alpha\hat{L} = 0 \tag{3.9}$$

and follows *Ehrenfest* theorem

$$\int \psi^* \frac{\partial \hat{L}}{\partial q_\alpha}\psi \, d\tau = \frac{1}{2}\int \psi^* \left(\frac{\partial}{\partial t} + \frac{d}{dt}\right)\hat{p}_\alpha\psi \, d\tau \equiv \int \psi^* \frac{d\hat{p}_\alpha}{dt}\psi \, d\tau \tag{3.10}$$

Newton's second law

$$\mathcal{F} = m\frac{d^2 q_\alpha}{dt^2} \tag{3.11}$$

is given in quantum language

$$\langle \hat{\mathcal{F}} \rangle = m \frac{d^2}{dt^2} \langle \hat{q}_\alpha \rangle \qquad (3.12)$$

here we have assumed arbitrary (non-quantizable) mass. Second order quantum description of systems (2.2.2) for operator $\hat{\mathcal{A}} := \hat{q}_\alpha$ leads

$$\langle \hat{\mathcal{F}} \rangle = m \langle \frac{d^2 \hat{q}_\alpha}{dt^2} \rangle - \frac{i}{\hbar} m \left\langle \Big[[\hat{L}, \frac{d\hat{q}_\alpha}{dt}]_- + \frac{d}{dt}[\hat{L}, \hat{q}_\alpha]_- - \frac{i}{\hbar}[\hat{L},[\hat{L},\hat{q}_\alpha]_-]_- \Big] \right\rangle \qquad (3.13)$$

One obtains

$$\hat{\mathcal{F}} = m \frac{d^2 \hat{q}_\alpha}{dt^2} - \frac{i}{\hbar} m \Big[[\hat{L}, \frac{d\hat{q}_\alpha}{dt}]_- + \frac{d}{dt}[\hat{L}, \hat{q}_\alpha]_- - \frac{i}{\hbar}[\hat{L},[\hat{L},\hat{q}_\alpha]_-]_- \Big] \qquad (3.14)$$

This is quantum dynamical Newton’s second law. It reduces to

$$\hat{\mathcal{F}} = -\frac{m}{\hbar^2} \hat{q}_\alpha \hat{L}^2 \qquad (3.15)$$

which follows *Ehrenfest* theorem

$$\mathcal{F} = -\frac{m}{\hbar^2} \int \psi^* \hat{q}_\alpha \hat{L}^2 \psi \, d\tau \qquad (3.16)$$

Newton's equation of motion

$$m \frac{d^2 q_\alpha}{dt^2} + \frac{\partial V}{\partial q_\alpha} = 0 \qquad (3.17)$$

is given in quantum language

$$m \frac{d^2}{dt^2} \langle \hat{q}_\alpha \rangle + \frac{\partial}{\partial q_\alpha} \langle \hat{V} \rangle = 0 \qquad (3.18)$$

quantum description of systems (2.2.2) and (2.1.15) for operators $\hat{\mathcal{A}} := \hat{q}_\alpha$ and $\hat{\mathcal{A}} := \hat{V}$ leads

$$\begin{aligned} & m \frac{d^2 \hat{q}_\alpha}{dt^2} + \frac{\partial \hat{V}}{\partial q_\alpha} \\ & - \frac{i}{\hbar} m \Big[[\hat{L}, \frac{d\hat{q}_\alpha}{dt}]_- + \frac{d}{dt}[\hat{L}, \hat{q}_\alpha]_- + \frac{1}{m}[\hat{p}_\alpha, \hat{V}]_- - \frac{i}{\hbar}[\hat{L},[\hat{L},\hat{q}_\alpha]_-]_- \Big] = 0 \end{aligned} \qquad (3.19)$$

This is quantum dynamical Newton's equation of motion. Following $\hat{L} = -i\hbar d/dt$ and $\hat{p}_\alpha = -i\hbar\partial/\partial q_\alpha$, it reduces to

$$m\frac{i}{\hbar}\hat{q}_\alpha\hat{L}^2 + \hat{V}\hat{p}_\alpha = 0 \tag{3.20}$$

which follows *Ehrenfest* theorem

$$m\int \psi^* \hat{q}_\alpha \frac{d}{dt}\hat{L}\psi\, d\tau + \int \psi^* \hat{V}\hat{p}_\alpha\psi\, d\tau = 0 \tag{3.21}$$

Consider (3.5) and (3.19), one obtains

$$\begin{aligned} &\frac{d\hat{p}_\alpha}{dt} - m\frac{d^2\hat{q}_\alpha}{dt^2} \\ &+m\frac{i}{\hbar}\Big[[\hat{L},\frac{d\hat{q}_\alpha}{dt}]_- + \frac{d}{dt}[\hat{L},\hat{q}_\alpha]_- + \frac{1}{m}[\hat{p}_\alpha,\hat{L}]_- - \frac{i}{\hbar}[\hat{L},[\hat{L},\hat{q}_\alpha]_-]_-\Big] = 0 \end{aligned} \tag{3.22}$$

This is another quantum dynamical Newtonian equation of motion. Following $\hat{L} = -i\hbar d/dt$ and $\hat{p}_\alpha = -i\hbar\partial/\partial q_\alpha$, it reduces to

$$m\frac{i}{\hbar}\hat{q}_\alpha\hat{L}^2 - \hat{p}_\alpha\hat{L} = 0 \tag{3.23}$$

which follows *Ehrenfest* theorem

$$m\int \psi^* \hat{q}_\alpha \frac{d\hat{L}}{dt}\psi\, d\tau = \int \psi^* \hat{p}_\alpha\hat{L}\psi\, d\tau \tag{3.24}$$

4. Quantum Hamilton-Jacobi principle

Hamilton equations

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \qquad ; \qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \tag{4.1}$$

are given in quantum language

$$\langle \dot{\hat{q}}_\alpha \rangle = \frac{\partial}{\partial p_\alpha}\langle \hat{H} \rangle \qquad ; \qquad \langle \dot{\hat{p}}_\alpha \rangle = -\frac{\partial}{\partial q_\alpha}\langle \hat{H} \rangle \tag{4.2}$$

quantum description of systems (2.1.20) and (2.1.15) for operators $\hat{A} := \hat{H}$

$$\frac{\partial}{\partial p_\alpha}\langle \hat{H}\rangle = \langle \frac{\partial \hat{H}}{\partial p_\alpha}\rangle + \frac{i}{\hbar}\langle [\hat{q}_\alpha, \hat{H}]_-\rangle$$
$$\frac{\partial}{\partial q_\alpha}\langle \hat{H}\rangle = \langle \frac{\partial \hat{H}}{\partial q_\alpha}\rangle - \frac{i}{\hbar}\langle [\hat{p}_\alpha, \hat{H}]_-\rangle \tag{4.3}$$

leads

$$\dot{\hat{q}}_\alpha = \frac{\partial \hat{H}}{\partial p_\alpha} + \frac{i}{\hbar}[\hat{q}_\alpha, \hat{H}]_-$$
$$\dot{\hat{p}}_\alpha = -\frac{\partial \hat{H}}{\partial q_\alpha} + \frac{i}{\hbar}[\hat{p}_\alpha, \hat{H}]_- \tag{4.4}$$

These are quantum dynamical Hamilton equations. These reduces to

$$\dot{\hat{q}}_\alpha = -\frac{i}{\hbar}\hat{H}\hat{q}_\alpha \quad ; \quad \dot{\hat{p}}_\alpha = -\frac{i}{\hbar}\hat{H}\hat{p}_\alpha \tag{4.5}$$

which follow *Ehrenfest* theorem

$$\dot{q}_\alpha = \int \psi^* \frac{\partial \hat{q}_\alpha}{\partial t}\psi \, d\tau$$
$$\dot{p}_\alpha = \int \psi^* \frac{\partial \hat{p}_\alpha}{\partial t}\psi \, d\tau \tag{4.6}$$

or

$$\dot{q}_\alpha = -\frac{i}{\hbar}\int \psi^* \hat{H}\hat{q}_\alpha \psi \, d\tau$$
$$\dot{p}_\alpha = -\frac{i}{\hbar}\int \psi^* \hat{H}\hat{p}_\alpha \psi \, d\tau \tag{4.7}$$

Canonical variables

$$p_\alpha = \frac{\partial S}{\partial q_\alpha} \quad ; \quad q_\alpha = -\frac{\partial S}{\partial p_\alpha} \tag{4.8}$$

are given in quantum language

$$\langle \hat{p}_\alpha \rangle = \frac{\partial}{\partial q_\alpha}\langle \hat{S}\rangle \quad ; \quad \langle \hat{q}_\alpha \rangle = -\frac{\partial}{\partial p_\alpha}\langle \hat{S}\rangle \tag{4.9}$$

quantum description of systems (2.1.15) and (2.1.20) for operators $\hat{\mathcal{A}} := \hat{S}$

$$\begin{aligned} \frac{\partial}{\partial q_\alpha}\langle \hat{S}\rangle &= \langle \frac{\partial \hat{S}}{\partial q_\alpha}\rangle - \frac{i}{\hbar}\langle [\hat{p}_\alpha, \hat{S}]_-\rangle \\ \frac{\partial}{\partial p_\alpha}\langle \hat{S}\rangle &= \langle \frac{\partial \hat{S}}{\partial p_\alpha}\rangle + \frac{i}{\hbar}\langle [\hat{q}_\alpha, \hat{S}]_-\rangle \end{aligned} \tag{4.10}$$

leads

$$\begin{aligned} \hat{p}_\alpha &= \frac{\partial \hat{S}}{\partial q_\alpha} - \frac{i}{\hbar}[\hat{p}_\alpha, \hat{S}]_- \\ \hat{q}_\alpha &= -\frac{\partial \hat{S}}{\partial p_\alpha} - \frac{i}{\hbar}[\hat{q}_\alpha, \hat{S}]_- \end{aligned} \tag{4.11}$$

These reduces to

$$\hat{p}_\alpha = \frac{i}{\hbar}\hat{S}\hat{p}_\alpha \quad ; \quad \hat{q}_\alpha = \frac{i}{\hbar}\hat{S}\hat{q}_\alpha \tag{4.12}$$

which follow *Ehrenfest* theorem

$$\begin{aligned} p_\alpha &= \frac{i}{\hbar}\int \psi^* \hat{S}\hat{p}_\alpha \psi \; d\tau \\ q_\alpha &= \frac{i}{\hbar}\int \psi^* \hat{S}\hat{q}_\alpha \psi \; d\tau \end{aligned} \tag{4.13}$$

Force and momentum

$$\mathcal{F} = \frac{\partial L}{\partial q_\alpha} \quad ; \quad p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \tag{4.14}$$

are given in quantum language

$$\langle \hat{\mathcal{F}}\rangle = \frac{\partial}{\partial q_\alpha}\langle \hat{L}\rangle \quad ; \quad \langle \hat{p}_\alpha\rangle = \frac{\partial}{\partial \dot{q}_\alpha}\langle \hat{L}\rangle \tag{4.15}$$

Consider $p_\alpha = m\dot{q}_\alpha$ and quantum description of systems (2.1.15) and (2.1.20) for operator $\hat{\mathcal{A}} := \hat{L}$

$$\frac{\partial}{\partial q_{\alpha}}\langle \hat{L}\rangle = \langle \frac{\partial \hat{L}}{\partial q_{\alpha}}\rangle - \frac{i}{\hbar}\langle [\hat{p}_{\alpha}, \hat{L}]_{-}\rangle$$
$$\frac{\partial}{\partial p_{\alpha}}\langle \hat{L}\rangle = \langle \frac{\partial \hat{L}}{\partial p_{\alpha}}\rangle + \frac{i}{\hbar}\langle [\hat{q}_{\alpha}, \hat{L}]_{-}\rangle \tag{4.16}$$

leads

$$\hat{\mathcal{F}} = \frac{\partial \hat{L}}{\partial q_{\alpha}} - \frac{i}{\hbar}[\hat{p}_{\alpha}, \hat{L}]_{-}$$
$$\hat{p}_{\alpha} = \frac{\partial \hat{L}}{\partial \dot{q}_{\alpha}} + m\frac{i}{\hbar}[\hat{q}_{\alpha}, \hat{L}]_{-} \tag{4.17}$$

These reduces to

$$\hat{\mathcal{F}} = \frac{i}{\hbar}\hat{L}\hat{p}_{\alpha} \quad ; \quad \hat{p}_{\alpha} = -m\frac{i}{\hbar}\hat{L}\hat{q}_{\alpha} \tag{4.18}$$

which follow *Ehrenfest* theorem

$$\mathcal{F} = \frac{i}{\hbar}\int \psi^{*}\hat{L}\hat{p}_{\alpha}\psi \, d\tau = \int \psi^{*}\frac{d\hat{p}_{\alpha}}{dt}\psi \, d\tau$$
$$p_{\alpha} = -m\frac{i}{\hbar}\int \psi^{*}\hat{L}\hat{q}_{\alpha}\psi \, d\tau = -m\int \psi^{*}\frac{d\hat{q}_{\alpha}}{dt}\psi \, d\tau \tag{4.19}$$

Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H = 0 \tag{4.20}$$

is given in quantum language

$$\frac{\partial}{\partial t}\langle \hat{S}\rangle + \langle \hat{H}\rangle = 0 \tag{4.21}$$

quantum description of systems (2.1.5) for operator $\widehat{\mathcal{A}} := \hat{S}$

$$\frac{\partial}{\partial t}\langle \hat{S}\rangle = \langle \frac{\partial \hat{S}}{\partial t}\rangle + \frac{i}{\hbar}\langle [\hat{H}, \hat{S}]_{-}\rangle \tag{4.22}$$

leads

$$\frac{\partial \hat{S}}{\partial t} + \hat{H} + \frac{i}{\hbar}[\hat{H}, \hat{S}]_{-} = 0 \tag{4.23}$$

This is quantum dynamical Hamilton-Jacobi equation. It reduces to

$$\hat{H} = \frac{i}{\hbar}\hat{S}\hat{H} \tag{4.24}$$

and follows *Ehrenfest* theorem

$$H = \frac{i}{\hbar}\int \psi^* \hat{S}\hat{H}\psi\, d\tau \tag{4.25}$$

5. Quantum Euler-Lagrange principle

Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = Q^{!}_\alpha \tag{5.1}$$

is given in quantum language

$$\frac{d}{dt}\left(\frac{\partial}{\dot{q}_\alpha}\langle \hat{L}\rangle\right) - \frac{\partial}{\partial q_\alpha}\langle \hat{L}\rangle = \langle \hat{Q}^{!}_\alpha\rangle \tag{5.2}$$

here $\hat{Q}^{!}_\alpha$ is quantum dynamical non-potential Constraint. Newton's equation $p_\alpha = m\dot{q}_\alpha$ and quantum description of systems (2.1.20) and (2.1.15) for operators $\hat{\mathcal{A}} := \hat{L}$ leads

$$m\frac{d}{dt}\Big[\langle \frac{\partial \hat{L}}{\partial p_\alpha}\rangle + \frac{i}{\hbar}\langle [\hat{q}_\alpha, \hat{L}]_-\rangle\Big] - \langle \frac{\partial \hat{L}}{\partial q_\alpha}\rangle + \frac{i}{\hbar}\langle [\hat{p}_\alpha, \hat{L}]_-\rangle = \langle \hat{Q}^{!}_\alpha\rangle \tag{5.3}$$

Furthermore, quantum description of systems (2.1.10) for operators $\hat{\mathcal{A}} := \partial \hat{L}/\partial p_\alpha$ and $\hat{\mathcal{A}} := [\hat{q}_\alpha, \hat{L}]_-$ yields

$$\begin{aligned} &m\Big[\Big\langle \frac{d}{dt}\frac{\partial \hat{L}}{\partial p_\alpha} + \frac{i}{\hbar}\frac{d}{dt}[\hat{q}_\alpha, \hat{L}]_-\Big\rangle - \frac{i}{\hbar}\Big\langle [\hat{L}, \frac{\partial \hat{L}}{\partial p_\alpha}]_- + \frac{i}{\hbar}[L, [\hat{q}_\alpha, \hat{L}]_-]_-\Big\rangle\Big] \\ &- \langle \frac{\partial \hat{L}}{\partial q_\alpha}\rangle + \frac{i}{\hbar}\langle [\hat{p}_\alpha, \hat{L}]_-\rangle = \langle \hat{Q}^{!}_\alpha\rangle \end{aligned} \tag{5.4}$$

Again with $p_\alpha = m\dot{q}_\alpha$, one obtains

$$\frac{d}{dt}\left(\frac{\partial\hat{L}}{\partial\dot{\hat{q}}_\alpha}\right)-\frac{\partial\hat{L}}{\partial\hat{q}_\alpha}$$

$$+m\frac{i}{\hbar}\Big[\frac{d}{dt}[\hat{q}_\alpha,\hat{L}]_- -[\hat{L},\frac{\partial\hat{L}}{\partial p_\alpha}]_- -\frac{i}{\hbar}[\hat{L},[\hat{q}_\alpha,\hat{L}]_-]_- +\frac{1}{m}[\hat{p}_\alpha,\hat{L}]_-\Big]=\hat{\mathcal{Q}}^!_\alpha \tag{5.5}$$

This is quantum dynamical Euler-Lagrange equation. Let us discuss dissipation. Euler-Lagrange-Rayleigh equation for dissipation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}+\frac{\partial\mathcal{D}}{\partial\dot{q}_\alpha}=0 \tag{5.6}$$

is given in quantum language

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{q}_\alpha}\langle\hat{L}\rangle\right)-\frac{\partial}{\partial q_\alpha}\langle\hat{L}\rangle+\frac{\partial}{\partial\dot{q}_\alpha}\langle\hat{\mathcal{D}}\rangle=0 \tag{5.7}$$

here $\mathcal{D}=\frac{1}{2}\sum k_\alpha\dot{q}_\alpha^2$ is Rayleigh dissipation function and $\mathcal{F}_\alpha^{(d)}=-\partial\mathcal{D}/\partial\dot{q}_\alpha$ is dissipative force (rather non-potential) replaced with $\mathcal{Q}^!_\alpha$. Following (5.5) with $p_\alpha=m\dot{q}_\alpha$ and quantum description of systems (2.1.20) for operator $\hat{\mathcal{A}}:=\hat{\mathcal{D}}$, one obtains

$$m\Big[\Big\langle\frac{d}{dt}\frac{\partial\hat{L}}{\partial p_\alpha}+\frac{i}{\hbar}[\hat{q}_\alpha,\hat{L}]_-\Big\rangle-\frac{i}{\hbar}\Big\langle[\hat{L},\frac{\partial\hat{L}}{\partial p_\alpha}]_- +\frac{i}{\hbar}[\hat{L},[\hat{q}_\alpha,\hat{L}]_-]_-\Big\rangle\Big]$$

$$-\langle\frac{\partial\hat{L}}{\partial q_\alpha}\rangle+\frac{i}{\hbar}\langle[\hat{p}_\alpha,\hat{L}]_-\rangle+m\langle\frac{\partial\hat{\mathcal{D}}}{\partial p_\alpha}\rangle+m\frac{i}{\hbar}\langle[\hat{q}_\alpha,\hat{\mathcal{D}}]_-\rangle=0 \tag{5.8}$$

which yields

$$\frac{d}{dt}\left(\frac{\partial\hat{L}}{\partial\dot{q}_\alpha}\right)-\frac{\partial\hat{L}}{\partial q_\alpha}+\frac{\partial\hat{\mathcal{D}}}{\partial\dot{q}_\alpha}$$

$$m\frac{i}{\hbar}\Big[\frac{d}{dt}[\hat{q}_\alpha,\hat{L}]_- -[\hat{L},\frac{\partial\hat{L}}{\partial p_\alpha}]_- -\frac{i}{\hbar}[\hat{L},[\hat{q}_\alpha,\hat{L}]_-]_- +[\hat{q}_\alpha,\hat{\mathcal{D}}]_- +\frac{1}{m}[\hat{p}_\alpha,\hat{L}]_-\Big]=0 \tag{5.9}$$

This is quantum dynamical Euler-Lagrange-Rayleigh equation. Dissipative force is given in quantum language

$$\langle\widehat{\mathcal{F}}_\alpha^{(d)}\rangle=-\frac{\partial}{\partial\dot{q}_\alpha}\langle\widehat{\mathcal{D}}\rangle \tag{5.10}$$

Following $p_\alpha=m\dot{q}_\alpha$ and quantum description of systems (2.1.20) for operator $\hat{\mathcal{A}}:=\hat{\mathcal{D}}$, one obtains

$$\hat{\mathcal{F}}_\alpha^{(d)} = -\frac{\partial \hat{\mathcal{D}}}{\partial \dot{q}_\alpha} - \frac{i}{\hbar} m[\hat{q}_\alpha, \hat{\mathcal{D}}]_- \tag{5.11}$$

This is quantum dynamical dissipative force with quantum dynamical Rayleigh dissipative function

$$\hat{\mathcal{D}} = \frac{1}{2}\sum k_\alpha \hat{\dot{q}}_\alpha = -\frac{i\hbar}{2m}\sum k_\alpha \frac{\partial}{\partial q_\alpha} \tag{5.12}$$

work performed by dissipation (W) is related

$$\frac{d\mathcal{W}}{dt} + 2\mathcal{D} = 0 \tag{5.13}$$

and given in quantum language

$$\frac{d}{dt}\langle \hat{\mathcal{W}} \rangle + 2\langle \hat{\mathcal{D}} \rangle = 0 \tag{5.14}$$

quantum description of systems (2.1.10) yields

$$\frac{d\hat{\mathcal{W}}}{dt} + 2\hat{\mathcal{D}} - \frac{i}{\hbar}[\hat{L}, \hat{\mathcal{W}}]_- = 0 \tag{5.15}$$

This is quantum dynamical Rayleigh dissipative work equation. It reduces to

$$\frac{i}{\hbar}\hat{\mathcal{W}}\hat{L} + 2\hat{\mathcal{D}} = 0 \tag{5.16}$$

which follows *Ehrenfest* theorem

$$\mathcal{D} = -\frac{i}{2\hbar}\int \psi^* \hat{\mathcal{W}}\hat{L}\psi \, d\tau \tag{5.17}$$

(5.11) reduces to

$$\hat{\mathcal{F}}_\alpha^{(d)} = m\frac{i}{\hbar}\hat{\mathcal{D}}\hat{q}_\alpha \tag{5.18}$$

which follows *Ehrenfest* theorem

$$\mathcal{F}_\alpha^{(d)} = m\frac{i}{\hbar}\int \psi^* \hat{\mathcal{D}}\hat{q}_\alpha \psi \, d\tau = \frac{1}{2}k_\alpha \int \psi^* \frac{\partial \hat{q}_\alpha}{\partial q_\alpha}\psi \, d\tau \tag{5.19}$$