The *canonical* wave transformation

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Abstract

The work is a theoretical treatment of canonical ψ-wave and action. Fundamental equations of Quantum mechanics have been considered for canonical wave and action. Canonical field and canonical field operator has been considered. Quantities are obtained in the terms of poisson bracket of action. Canonical eigenvalue equations have been formulated with the consideration of poisson bracket eigenvalue equation.

Keyword(s): Canonical transformation; exact and partial prime; classical and quantum mechanical poisson bracket; second quantization principle; expectation and perturbation.

1. Introduction

The present work is a theoretical study of canonical ψ-wave and action. Quantum mechanical operators do not depend upon the choice of variable of wave function. Therefore, fundamental equations of quantum mechanics remain unchanged in canonical treatment. These are simply replaced with canonical wave function. The comparing of these equations with canonical wave transformation provides us some further equations in canonical field theory. Comparing of canonical results with second quantization principle provides second quantized equations. Interpretation of poisson bracket operator and canonical action leads us to formulate canonical eigenvalue equations.

2. Canonical ψ-wave

Let consider the canonical ψ-wave

$$\psi := \psi(q_\alpha, p_\alpha, t) := \exp\left(\frac{i}{\hbar} S(q_\alpha, p_\alpha, t)\right) \tag{2.1}$$

with canonical action $S(q_\alpha, p_\alpha, t)$. The exact prime of $\psi(q_\alpha, p_\alpha, t)$ is given

$$\frac{d}{dt}\psi(q_\alpha, p_\alpha, t) = \frac{\partial}{\partial t}\psi(q_\alpha, p_\alpha, t) + \{H, \psi\}_{p_\alpha, q_\alpha} \tag{2.2}$$

Consider Saurav equation for canonical $\psi(q_\alpha, p_\alpha, t)$ [1]

$$\frac{d}{dt}\psi(q_\alpha, p_\alpha, t) = \frac{\partial}{\partial t}\psi(q_\alpha, p_\alpha, t) - \frac{i\hbar}{m}\nabla_\alpha^2 \psi(q_\alpha, p_\alpha, t) \tag{2.3}$$

One obtains

$$\{H,\psi\}_{p_\alpha,q_\alpha} + \frac{i\hbar}{m}\nabla_\alpha^2\psi(q_\alpha, p_\alpha, t) = 0 \tag{2.4}$$

which is Saurav equation for canonical $\psi(q_\alpha, p_\alpha, t)$. Consider (2.2) for operator values of $\psi(q_\alpha, p_\alpha, t)$, Quantum mechanical poisson bracket [4] and expectation of equation:

$$\left\langle \frac{d}{dt}\hat{\psi}(q_\alpha, p_\alpha, t)\right\rangle = \left\langle \frac{\partial}{\partial t}\hat{\psi}(q_\alpha, p_\alpha, t)\right\rangle + \left\langle \{\hat{H},\hat{\psi}\}_{\hat{p}_\alpha,\hat{q}_\alpha}\right\rangle \tag{2.5}$$

Similarly, consider (2.4) for quantum mechanical poisson bracket and expectation of equation:

$$\left\langle \{\hat{H},\hat{\psi}\}_{\hat{p}_\alpha,\hat{q}_\alpha}\right\rangle + \frac{i\hbar}{m}\left\langle \hat{\nabla}_\alpha^2\hat{\psi}(q_\alpha, p_\alpha, t)\right\rangle = 0 \tag{2.6}$$

here $\hat{\nabla}_\alpha^2$ is quantum mechanical Laplacian operator [1]

$$\hat{\nabla}_\alpha^2 = \frac{i}{\hbar}\left\{[\nabla_\alpha^2 S] + \frac{i}{\hbar}[\nabla_\alpha S]^2\right\} \tag{2.7}$$

Second quantization principle for $\psi(q_\alpha, p_\alpha, t)$ is given [4]

$$\left\langle \{\hat{H},\hat{\psi}\}_{\hat{p}_\alpha,\hat{q}_\alpha}\right\rangle = 2\frac{i}{\hbar}\left\{\left\langle [\hat{T},\hat{\psi}]_-\right\rangle + \hat{T}\left\langle \hat{\psi}\right\rangle\right\} \tag{2.8}$$

One obtains

$$\frac{\hbar^2}{2m}\left\langle \hat{\nabla}_\alpha^2\hat{\psi}(q_\alpha, p_\alpha, t)\right\rangle + \left\{\left\langle [\hat{T},\hat{\psi}]_-\right\rangle + \hat{T}\left\langle \hat{\psi}\right\rangle\right\} = 0 \tag{2.9}$$

which is second quantized Saurav equation for $\psi(q_\alpha, p_\alpha, t)$. Consider the poisson bracket of $\psi(q_\alpha, p_\alpha, t)$

$$\Lambda\psi = \{H,\psi\}_{p_\alpha,q_\alpha} = \left\{\frac{\partial H}{\partial p_\alpha}\frac{\partial \psi}{\partial q_\alpha} - \frac{\partial H}{\partial q_\alpha}\frac{\partial \psi}{\partial p_\alpha}\right\} \tag{2.10}$$

with classical poisson bracket operator [4]

$$\Lambda = \{H,\quad\}_{p_\alpha,q_\alpha} \tag{2.11}$$

Consider it for my equation (2.4), one obtains

$$\left(\Lambda + \frac{i\hbar}{m}\nabla_\alpha^2\right)\psi(q_\alpha, p_\alpha, t) = 0 \tag{2.12}$$

Consider poisson bracket eigenoperator equation [4]

$$\hat{\Lambda}\hat{\psi} = \Lambda\psi \tag{2.13}$$

with quantum mechanical poisson bracket operator

$$\hat{\Lambda} = \left\{\hat{H}, \quad \right\}_{\hat{p}_\alpha, \hat{q}_\alpha} \tag{2.14}$$

One obtains

$$\hat{\Lambda}\hat{\psi} = \left\{\hat{H}, \hat{\psi}\right\}_{\hat{p}_\alpha, \hat{q}_\alpha} = \left\{\frac{\partial \hat{H}}{\partial \hat{p}_\alpha}\frac{\partial \hat{\psi}}{\partial \hat{q}_\alpha} - \frac{\partial \hat{H}}{\partial \hat{q}_\alpha}\frac{\partial \hat{\psi}}{\partial \hat{p}_\alpha}\right\} \tag{2.15}$$

Consider it for (2.6) canceling for expectation, one obtains

$$\left(\hat{\Lambda} + \frac{i\hbar}{m}\hat{\nabla}_\alpha^2\right)\hat{\psi}(q_\alpha, p_\alpha, t) = 0 \tag{2.16}$$

(2.12) and (2.16) are my equations for canonical field theory with canonical field $\psi(q_\alpha, p_\alpha, t)$ and canonical field operator $\hat{\psi}(q_\alpha, p_\alpha, t)$.

3. Canonical action

Let consider canonical action

$$S := S(q_\alpha, p_\alpha, t) \tag{3.1}$$

The exact prime of $S(q_\alpha, p_\alpha, t)$ is given

$$\frac{d}{dt}S(q_\alpha, p_\alpha, t) = \frac{\partial}{\partial t}S(q_\alpha, p_\alpha, t) + \left\{H, S\right\}_{p_\alpha, q_\alpha} \tag{3.2}$$

Consider my equation [1]

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{[\nabla_\alpha S]^2}{m} - \frac{i\hbar}{m}[\nabla_\alpha^2 S] \tag{3.3}$$

One obtains

$$\frac{[\nabla_\alpha S]^2}{m} - \frac{i\hbar}{m}[\nabla_\alpha^2 S] - \left\{H, S\right\}_{p_\alpha, q_\alpha} = 0 \tag{3.4}$$

which is my equation for canonical action. Consider (3.2) for operator values of action, Quantum mechanical poisson bracket and expectation of equation:

$$\left\langle \frac{d}{dt}\hat{S}(q_\alpha, p_\alpha, t)\right\rangle = \left\langle \frac{\partial}{\partial t}\hat{S}(q_\alpha, p_\alpha, t)\right\rangle + \left\langle \{\hat{H}, \hat{S}\}_{\hat{p}_\alpha, \hat{q}_\alpha}\right\rangle \tag{3.5}$$

Similarly, consider (3.4) for expectation of equation, Quantum mechanical poisson bracket and operator values of action, gradient and Laplacian:

$$\left\langle \{\hat{H}, \hat{S}\}_{\hat{p}_\alpha, \hat{q}_\alpha}\right\rangle = \left\langle \frac{[\hat{\nabla}_\alpha \hat{S}]^2}{m}\right\rangle - \frac{i\hbar}{m}\left\langle [\hat{\nabla}_\alpha^2 \hat{S}]\right\rangle \tag{3.6}$$

Second quantization principle for $S(q_\alpha, p_\alpha, t)$ is given [4]

$$\left\langle \{\hat{H}, \hat{S}\}_{\hat{p}_\alpha, \hat{q}_\alpha}\right\rangle = 2\frac{i}{\hbar}\left\{\left\langle [\hat{T}, \hat{S}]_-\right\rangle + \hat{T}\left\langle \hat{S}\right\rangle\right\} \tag{3.7}$$

One obtains

$$\left\langle \frac{[\hat{\nabla}_\alpha \hat{S}]^2}{2m}\right\rangle - \frac{i}{\hbar}\left\{\left\langle [\hat{T}, \hat{S}]_-\right\rangle + \hat{T}\left\langle \hat{S}\right\rangle\right\} - \frac{i\hbar}{2m}\left\langle [\hat{\nabla}_\alpha^2 \hat{S}]\right\rangle = 0 \tag{3.8}$$

Consider the classical form of second quantization principle [4] and assume the expectation as the eigenvalue of its operator, one obtains

$$\frac{[\nabla_\alpha S]^2}{2m} - \frac{i}{\hbar}\left\{[\hat{T}, \hat{S}]_- + \hat{T}\Omega\right\} - \frac{i\hbar}{2m}[\nabla_\alpha^2 S] = 0 \tag{3.9}$$

with action eigenvalue Ω [1].

4. Canonical action-Classical quantities

Lagrangian in the terms of $S(q_\alpha, p_\alpha, t)$ is given

$$\frac{d}{dt}S(q_\alpha, p_\alpha, t) = \frac{\partial}{\partial t}S(q_\alpha, p_\alpha, t) + L(q_\alpha, \dot{q}_\alpha, t) - \frac{\partial}{\partial t}S(q_\alpha, p_\alpha, t) \tag{3.10}$$

Consider (3.2), one obtains

$$\frac{\partial}{\partial t}S(q_\alpha, p_\alpha, t) + \{H, S\}_{p_\alpha, q_\alpha} - L(q_\alpha, \dot{q}_\alpha, t) = 0 \tag{4.2}$$

which is my equation for Lagrangian. Hamilton-Jacobi equation for $S(q_\alpha, p_\alpha, t)$ is given

$$\frac{\partial}{\partial t}S(q_\alpha, p_\alpha, t) + H(q_\alpha, p_\alpha, t) = 0 \tag{4.3}$$

Consider (3.2), one obtains

$$\frac{d}{dt}S(q_\alpha, p_\alpha, t) + H(q_\alpha, p_\alpha, t) - \{H, S\}_{p_\alpha, q_\alpha} = 0 \tag{4.4}$$

which is my equation for Hamiltonian. Consider kinetic energy in terms of $S(q_\alpha, p_\alpha, t)$ [1]

$$\frac{\partial}{\partial t}S(q_\alpha, p_\alpha, t) + 2T(\dot{q}_\alpha, t) - \frac{d}{dt}S(q_\alpha, p_\alpha, t) = 0 \tag{4.5}$$

Consider (3.2), one obtains

$$\{H, S\}_{p_\alpha, q_\alpha} - 2T(\dot{q}_\alpha, t) = 0 \tag{4.6}$$

which is my equation for kinetic energy. Consider potential energy in terms of $S(q_\alpha, p_\alpha, t)$ [1]

$$\frac{\partial}{\partial t}S(q_\alpha, p_\alpha, t) + 2V(q_\alpha, t) + \frac{d}{dt}S(q_\alpha, p_\alpha, t) = 0 \tag{4.7}$$

Consider (3.2), one obtains

$$2\frac{\partial}{\partial t}S(q_\alpha, p_\alpha, t) + 2V(q_\alpha, t) + \{H, S\}_{p_\alpha, q_\alpha} = 0 \tag{4.8}$$

which is my equation for potential energy. The exact prime of $S(q_\alpha, p_\alpha, t)$ with differential rule is given

$$\frac{d}{dt}S(q_\alpha, p_\alpha, t) = \frac{\partial}{\partial t}S(q_\alpha, p_\alpha, t) + \frac{\partial}{\partial q_\alpha}S(q_\alpha, p_\alpha, t)\,\dot{q}_\alpha + \frac{\partial}{\partial p_\alpha}S(q_\alpha, p_\alpha, t)\,\dot{p}_\alpha \tag{4.9}$$

Consider from classical theoretical physics

$$\frac{\partial}{\partial q_\alpha}S(q_\alpha, p_\alpha, t) - p_\alpha = 0 \tag{4.10}$$

$$\frac{\partial}{\partial p_\alpha}S(q_\alpha, p_\alpha, t) + q_\alpha = 0 \tag{4.11}$$

Consider (3.2), one obtains

$$\{H, S\}_{p_\alpha, q_\alpha} = p_\alpha \dot{q}_\alpha - q_\alpha \dot{p}_\alpha \tag{4.12}$$

with a formal differential rule, one obtains

$$p_\alpha^2 \frac{d}{dt}\left\{\frac{q_\alpha}{p_\alpha}\right\} - \{H, S\}_{p_\alpha, q_\alpha} = 0 \tag{4.13}$$

5. Canonical eigenvalue formalism

Since poisson bracket is a first-order differential operation. One obtains poisson bracket eigenvalue equation;

$$\Lambda\psi(q_\alpha, p_\alpha, t) = \frac{i}{\hbar}[\Lambda S(q_\alpha, p_\alpha, t)]\psi(q_\alpha, p_\alpha, t) \tag{5.1}$$

$$\{H, S\}_{p_\alpha, q_\alpha}\psi(q_\alpha, p_\alpha, t) = -i\hbar\{H, \psi\}_{p_\alpha, q_\alpha} \tag{5.2}$$

Consider (4.6), one obtains

$$T(\dot{q}_\alpha, t)\psi(q_\alpha, p_\alpha, t) + \frac{i\hbar}{2}\{H, \psi\}_{p_\alpha, q_\alpha} = 0 \tag{5.3}$$

Similarly, consider (4.2), one obtains [1]

$$L(q_\alpha, \dot{q}_\alpha, t)\psi(q_\alpha, p_\alpha, t) + i\hbar\frac{\partial}{\partial t}\psi(q_\alpha, p_\alpha, t) + i\hbar\{H, \psi\}_{p_\alpha, q_\alpha} = 0 \tag{5.4}$$

Consider (4.4), one obtains

$$H(q_\alpha, p_\alpha, t)\psi(q_\alpha, p_\alpha, t) + i\hbar\{H, \psi\}_{p_\alpha, q_\alpha} - i\hbar\frac{d}{dt}\psi(q_\alpha, p_\alpha, t) = 0 \tag{5.5}$$

Consider (4.8), one obtains [1]

$$V(q_\alpha, t)\psi(q_\alpha, p_\alpha, t) - i\hbar\frac{\partial}{\partial t}\psi(q_\alpha, p_\alpha, t) - \frac{i\hbar}{2}\{H, \psi\}_{p_\alpha, q_\alpha} = 0 \tag{5.6}$$

Consider (4.13), one obtains

$$p_\alpha^2 \frac{d}{dt}\left(\frac{q_\alpha}{p_\alpha}\right)\psi(q_\alpha, p_\alpha, t) + i\hbar\{H, \psi\}_{p_\alpha, q_\alpha} = 0 \tag{5.7}$$

We obtain following canonical eigenvalue equations:

$$\hat{L}(q_\alpha, \dot{q}_\alpha, t)\psi(q_\alpha, p_\alpha, t) = \lambda(q_\alpha, \dot{q}_\alpha, t)\psi(q_\alpha, p_\alpha, t) \tag{5.8}$$

$$\hat{H}(q_\alpha, p_\alpha, t)\psi(q_\alpha, p_\alpha, t) = \varepsilon(q_\alpha, p_\alpha, t)\psi(q_\alpha, p_\alpha, t) \tag{5.9}$$

$$\hat{T}(\dot{q}_\alpha, t)\psi(q_\alpha, p_\alpha, t) = \tau(\dot{q}_\alpha, t)\psi(q_\alpha, p_\alpha, t) \tag{5.10}$$

$$\hat{V}(q_\alpha, t)\psi(q_\alpha, p_\alpha, t) = \nu(q_\alpha, t)\psi(q_\alpha, p_\alpha, t) \tag{5.11}$$

with operators;

$$\hat{L}(q_\alpha, \dot{q}_\alpha, t) = -i\hbar\partial_t - i\hbar\Lambda = -i\hbar\frac{\partial}{\partial t} - i\hbar\{H, \quad \}_{p_\alpha, q_\alpha} \tag{5.12}$$

$$\hat{H}(q_\alpha, p_\alpha, t) = i\hbar d_t - i\hbar\Lambda = i\hbar\frac{d}{dt} - i\hbar\{H, \quad \}_{p_\alpha, q_\alpha} \tag{5.13}$$

$$\hat{T}(\dot{q}_\alpha, t) = -\frac{i\hbar}{2}\Lambda = -\frac{i\hbar}{2}\{H, \quad \}_{p_\alpha, q_\alpha} \tag{5.14}$$

$$\hat{V}(q_\alpha, t) = i\hbar\partial_t + \frac{i\hbar}{2}\Lambda = i\hbar\frac{\partial}{\partial t} + \frac{i\hbar}{2}\{H, \quad \}_{p_\alpha, q_\alpha} \tag{5.15}$$

where $\alpha = (\lambda, \varepsilon, \tau, \nu ....)$ are eigenvalues of their corresponding operators and $\psi(q_\alpha, p_\alpha, t)$ is canonical eigenfunction.

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