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A Look at the Burr and Related Distributions

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Summary

The Burr distribution (Burr type XII), which yields a wide range of values of skewness and kurtosis, can be used to fit almost any given set of unimodal data. The Burr distribution has appeared in the literature under different names. The relationship between the Burr distribution and the various other distributions, namely, the Lomax, the Compound Weibull, the Weibull-Exponential, the logistic, the log logistic, the Weibull and the Kappa family of distributions is summarized. Also it is shown that the 'reciprocal Burr' distribution (Burr type III) covers a wider region in the ($\sqrt{\beta_1}$, β_2) plane and includes all the region covered by the Burr type XII distribution.

1 Introduction

Burr (1942) has suggested a number of forms of cumulative distribution functions which might be useful for fitting data. The principal aim in choosing one of these forms of distributions is to facilitate the mathematical analysis to which it is subjected, while attaining a reasonable approximation. Burr (1942, 1968, 1973) and others (see Burr and Cislak (1968), Hatke (1949), Rodriguez (1977)) devoted special attention to one of these forms, denoted by type XII, whose distribution function $F(x)$ is given as

$$F(x) = 1 - (1 + x^c)^{-k}, \quad x > 0, \quad c > 0, \quad k > 0. \tag{1}$$

Both c and k are shape parameters. Location and scale parameters can easily be introduced to make (1) a four-parameter distribution. A given set of data may be fitted by the Burr distribution by matching their mean, variances, skewness ($\sqrt{\beta_1}$) and Kurtosis (β_2). We adopt the convention that $\sqrt{\beta_1} = \mu_3/\sigma^3$ and $\beta_2 = \mu_4/\sigma^4$, where μ_i denotes the ith moment about the mean μ_1. A scale parameter can be introduced in many different ways and we rewrite the cdf given in (1) by introducing a 'scale parameter α' in two different ways, for later reference.

$$F(x) = 1 - (1 + x^c/\alpha)^{-k} \tag{2}$$

$$F(x) = 1 - (1 + (x/\alpha)^c)^{-k}. \tag{3}$$

Various authors used alternate forms of the Burr distribution in different areas applications, most of which are special cases of the Burr distribution (1), (2), or (3). The objective of this note is (i) to develop some diagrams in the ($\sqrt{\beta_1}$, β_2) plane corresponding to (1) and some related distributions and (ii) to summarize the relationship between (1) and other related distributions.

2 The Burr type XII distribution

The probability density function (pdf) corresponding to (1) is

$$f(x) = kcx^{c-1}(1 + x^c)^{-(k+1)}, \quad x \geqslant 0, \quad k > 0, \quad c > 0. \tag{4}$$

The density given in (4) is unimodal at $x = ((c-1)/(kc+1)^{1/c})$ if $c > 1$ and L-shaped if $c \leqslant 1$. The rth moment about the origin of X can easily be shown to be

$$E(X^r) = \mu_r' = k\Gamma(k - r/c)\,\Gamma(r/c + 1)/\Gamma(k + 1), \quad ck > r. \tag{5}$$

It is required that $ck > 4$ for the fourth moment, and thus β_2, to exist. Burr (1973) constructed tables to obtain the values of k and c for given $\sqrt{\beta_1}$ and β_2 (α_3 and α_4 in his notation). The Burr distribution covers a wide region in the ($\sqrt{\beta_1}$, β_2) plane as shown in Figure 1. (Also see Rodriguez (1977).)

This region includes the loci of ($\sqrt{\beta_1}$, β_2) points corresponding to Pearson types IV, VI and bell-shaped curves of type I, the gamma distribution (the section of the Pearson type III line joining the normal and exponential points) and the Weibull distribution. It also includes the points corresponding to the normal, logistic and extreme value distributions. Figure 1 also shows the loci of ($\sqrt{\beta_1}$, β_2) points for constant k and varying c. For a discussion on the lower and upper bounds of this region of coverage, see later sections of this paper and also refer to Rodriguez (1977).

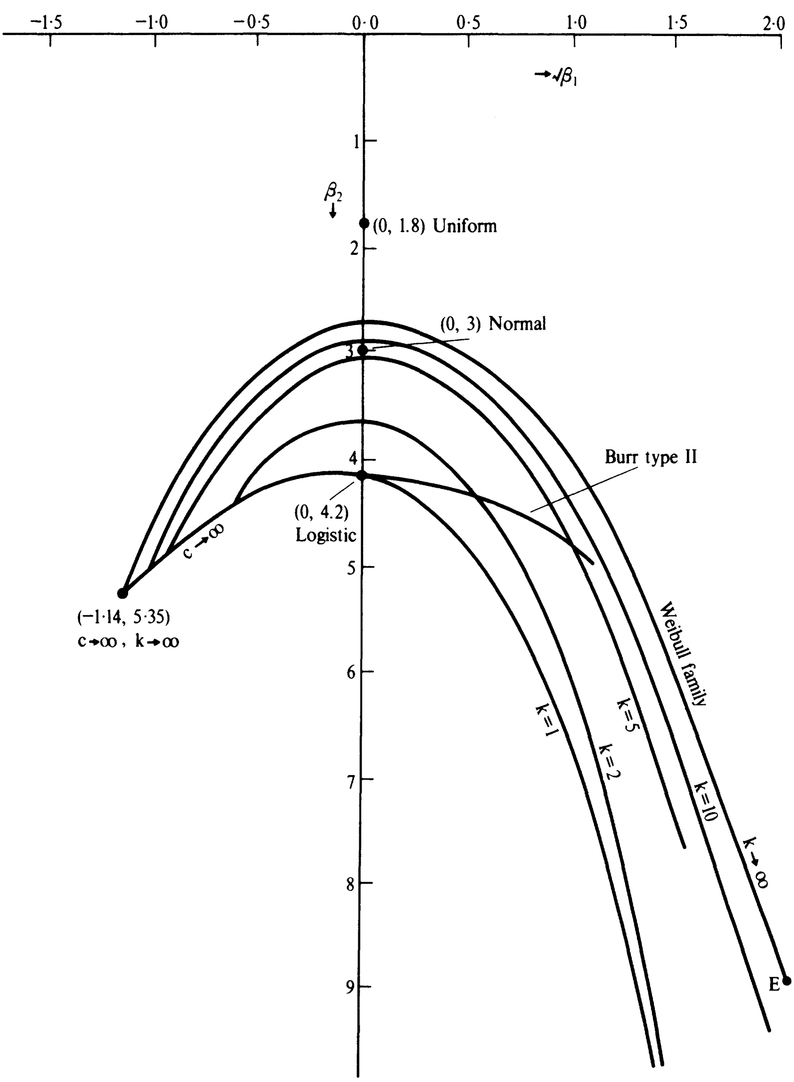

Figure 1. *The Burr type XII Distribution in ($\sqrt{\beta_1}$, β_2) Plane.*

3 Related distributions and applications

In this section, we summarize the relationship between (1), (2) and (3) and related distributions.

3.1 *Lomax distribution*

Lomax (1954) used the following functional form for cdf to fit some business failure data:

$$F(x) = 1 - (a/(x+a))^{-k} = 1 - (1 + x/a)^{-k}, \quad x \geqslant 0. \tag{6}$$

He derived this distribution based on the failure rate function $Z(x) = k/(x+a)$ where $Z(x) = f(x)/(1-F(x))$. It can be easily seen that (6) is a special case of Burr distribution (2) or (3) with scale parameter $\alpha = a$ and the shape parameter $c = 1$.

Dubey (1966a) derives (4) as a special case of a compound gamma or gamma-gamma distribution (Dubey, 1970) and calls it exponential-gamma distribution. If the conditional random variate X has the exponential distribution with pdf

$$f(x \mid \beta) = \beta\, e^{-\beta x}, \quad x \geqslant 0, \quad \beta > 0 \tag{7}$$

and if the parameter β has the gamma distribution with pdf

$$g(\beta) = a^k\, \beta^{k-1}\, e^{-a\beta}, \quad \beta \geqslant 0, \quad a > 0, \quad k > 0, \tag{8}$$

then, the pdf of X can be obtained as

$$f(x) = \int_0^\infty f(x \mid \beta)\, g(\beta)\, d\beta \tag{9}$$

$$= (k/a)/(1 + x/a)^{k+1}. \tag{10}$$

The cdf corresponding to (10) is same as (6).

3.2 *Compound-Weibull distribution* (Dubey, 1968)

If the conditional random variate X has the Weibull distribution whose pdf is given by

$$f(x \mid \beta) = \alpha\beta\, x^{\alpha-1}\, e^{-\beta x^\alpha}, \quad x \geqslant 0, \quad \alpha, \quad \beta > 0 \tag{11}$$

and if the parameter β has the gamma distribution with pdf

$$g(\beta) = \delta^\nu\, \beta^{\nu-1}\, e^{-\delta\beta}, \quad \beta > 0, \quad \nu, \quad \delta > 0, \text{ add} \tag{12}$$

then, the pdf of X can be shown to be

$$f(x) = \int_0^\infty f(x \mid \beta)\, g(\beta)\, d\beta \tag{13}$$

$$= \alpha\nu\delta^\nu\, x^{\alpha-1}/(x^\alpha + \delta)^{\nu+1}, \quad x \geqslant 0, \quad \alpha, \quad \nu, \quad \delta > 0. \tag{14}$$

The cdf corresponding to (14) is given by

$$F(x) = 1 = (1 + x^\alpha/\delta)^{-\nu}. \tag{15}$$

Hence (15) is the same as the Burr distribution (2) with scale parameter δ ($c = \alpha$ and $k = \nu$). Dubey considers (15) to be the generalized Burr distribution and refers to it as a compound Weibull or Weibull-Gamma. Also note that by substituting $\alpha = 1$, (15) is reduced to the Lomax distribution (6). Dubey (1968) also discusses the fact that by means of suitable transformations the pdf (14) can be reduced to some special cases of the beta and F distributions.

3.3 *The Weibull-Exponential distribution*

By setting $\nu = 1$ in (15), we obtain

$$F(x) = (x^{\alpha}/\delta)/(1 + x^{\alpha}/\delta). \tag{16}$$

Dubey (1966b) shows that (16) can be fitted better to the business failure data given by Lomax (1954). The loci of $(\sqrt{\beta_1}, \beta_2)$ points corresponding to (16) is given in Figure 1. This loci corresponds to the curve $k = 1$ and forms part of the upperbound for the region of coverage. the other part of the upperbound for the region of coverage is provided by the loci $c \to \infty$ and k varying.

3.4 *The Weibull distribution*

In Figure 1, it is shown that the lower bound of $(\sqrt{\beta_1}, \beta_2)$ region for (1) coincides with the Weibull distribution. As shown by Rodriguez (1977), the identification of the lower bound with the Weibull family can be demonstrated as follows:

$$\begin{aligned} \Pr(x < (1/k)^{1/c} y) &= 1 - (1 + y^c/k)^{-k} \\ &= 1 - \exp(-k \log(1 + y^c/k) \\ &= 1 - \exp[-k(y^c/k - \tfrac{1}{2}(y^c/k)^2 + \ldots] \\ &= 1 - e^{-y^c} \quad \text{as } k \to \infty. \end{aligned} \tag{17}$$

Thus the limiting distribution of (1) as $k \to \infty$ is the Weibull distribution (17).

3.5 *The logistic distribution*

The limiting Burr curve $c \to +\infty$ forms vart of the upper bound for the Burr region, passing through the logistic point ($\sqrt{\beta_1} = 0$, $\beta_2 = 4.2$, which corresponds to $k = 1$ and $c \to \infty$) and approaching Weibull curve asymptotically as $k \to 0$.

3.6 *The log logistic distribution*

Consider the logistic random variable Y with cdf given by

$$F(y) = 1 + \exp(-(y - \log \alpha)/\beta)^{-1} \tag{18}$$

where β is a scale parameter and $\log \alpha$ is chosen as the location parameter for convenience. The distribution of $X = e^Y$ can be shown to be

$$f(x) = (\beta \alpha^{\beta} x^{\beta-1})/(\alpha^{\beta} + x^{\beta})^2, \quad x \geqslant 0. \tag{19}$$

The cdf corresponding to (19) can be written as

$$\begin{aligned} F(x) &= x^{\beta}/(\alpha^{\beta} + x^{\beta}) \\ &= (x/\alpha)^{\beta}/(1 + (x/\alpha)^{\beta}). \end{aligned} \tag{20}$$

It can be noted that (20) and (16) are similar (with different scale parameters) and (20) is a special case of (3) with $k = 1$ and $c = \beta$. Cheng (1977) uses (20) as an 'enveloping' distribution to obtain an algorithm for sampling from the gamma distribution.

3.7 *Burr type III distributions*

Let X be the random variable with cdf given by (1) and consider the transformation $Y = 1/X$. This yields the distribution function

$$G(y) = (1 + y^{-c})^{-k} \\ = (y^c/(1 + y^c))^k \tag{21}$$

which is one of the many forms of distribution functions (Burr type III) given by Burr. The pdf and the rth moment corresponding (21) can be written as

$$g(y) = kcy^{-c-1}(1 + y^{-c})^{-k-1} \tag{22}$$

and

$$E(Y^r) = k\Gamma(k + r/c)\,\Gamma(1 - r/c)/\Gamma(k + 1). \tag{23}$$

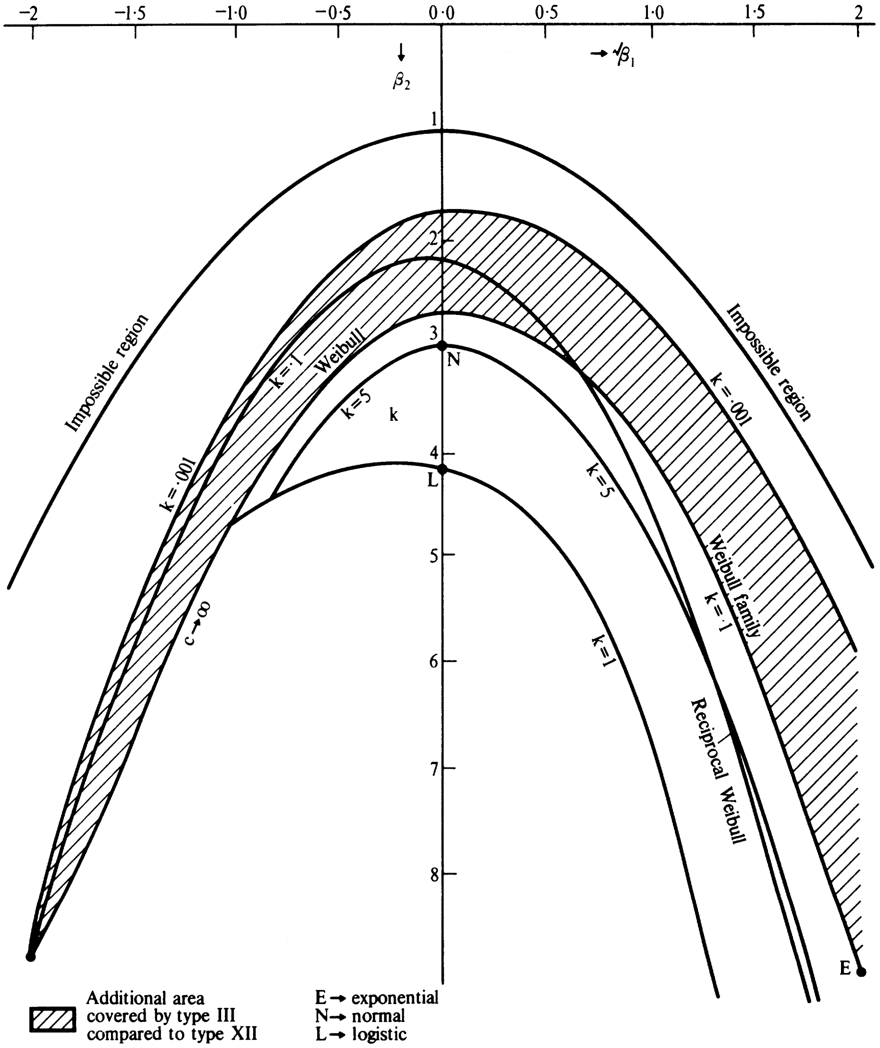

Figure 2. *The Burr type III Distribution in ($\sqrt{\beta_1}, \beta_2$) Plane.*

By using (22), the region of coverage is greatly extended toward low β_2's for given $\sqrt{\beta_1}$. This extension of the region covers virtually all of the bell-shaped and J-shaped regions of the beta region as shown in Figure 2. It is interesting to note that although the Burr type III distribution (22) covers all of the region in $(\sqrt{\beta_1}, \beta_2)$ plane as covered by the Burr type XII distribution and more, much attention has not been paid to this distribution. Figure 2 shows the loci of $(\sqrt{\beta_1}, \beta_2)$ points for constant k and varying c. As in the case of type XII, the upper bound for the region of coverage of type III distribution is provided by the loci of $(\sqrt{\beta_1}, \beta_2)$ points for $k = 1$ and c varying and $c \to \infty$ and k varying. The lower bound for the region of coverage is provided by $k \to 0$ and c varying. Also note that as $k \to \infty$, analogues to (17), the limiting form of the distribution function (21) is given by

$$F(x) = e^{-x^{-c}} \tag{24}$$

We can consider (24) as the distribution function corresponding to a reciprocal Weibull variate *i.e.* if X has the distribution function given by (17), then $Y = 1/X$ has the distribution function (24). The loci of $(\sqrt{\beta_1}, \beta_2)$ points corresponding to (24) is also shown in Figure 2.

3.8 *Burr type II distribution*

Using the transformation $X = c \log y$ in (21), we obtain

$$F(x) = (1 + e^{-x})^{-k} \tag{25}$$

which is another form (type II) of the distribution functions given by Burr. Figure 2 shows the loci of $(\sqrt{\beta_1}, \beta_2)$ points corresponding to (25).

3.9 *Two parameter Kappa family of distributions*

Mielke (1973) introduced a two parameter (shape and scale) family of distributions for describing and analyzing precipitation data. The cdf of the two parameter Kappa distribution, as given by Mielke (1973), is

$$F(x) = (x/\beta)^{\alpha}/[\alpha + (x/\beta)^{\alpha}]^{1/\alpha}, \quad x \geqslant 0, \quad \alpha > 0, \quad \beta > 0. \tag{26}$$

β is a scale parameter and $c = \alpha$ and $k = 1/\alpha$ in (21) would result (26). Mielke (1973) obtained method of moments and maximum likelihood estimators for α and β in (26).

3.10 *Three parameter Kappa distributions*

A reparameterized version of the distribution (26) termed the 'three parameter Kappa family of distributions' was given by Mielke and Johnson (1973) the cdf given by,

$$F(x) = (x/\beta)^{\alpha\theta}/[\alpha + (x/\beta)^{\alpha\theta}], \quad x \geqslant 0, \quad \alpha, \beta, \theta > 0. \tag{27}$$

Later, Mielke and Johnson (1974) gave another form of (27)

$$F(x) = (x/\beta)^{\theta}/[1 + (x/\beta)^{\theta}]^{\alpha}, \quad x \geqslant 0, \quad \beta, \theta, > 0. \tag{28}$$

They derived (28) as a special case of the generalized beta distributions of the second kind and called it the beta $-k$ distribution. Both the distributions (27) and (28) were used in hydrology and meteorology as alternate models to the gamma and lognormal distributions for fitting stream flow and precipitation data. It can be seen that (28) is the same as (21) with scale parameter β, $c = \theta$ and $k = \alpha$

3.11 We observe an *interesting relationship* between two special cases of the Burr type XII and type II distributions. Let X be a random variable with cdf given by

$$F(x) = 1 - (1 + x^c)^{-1} \tag{29}$$

Note that (29) is merely a special case of (1) with $k = 1$. Consider the distribution of $Y = 1/X$ and the cdf of Y can be written as

$$F(y) = (1 + y^{-c})^{-1} \tag{30}$$

(Note that (30) is a special case of (21) with $k = 1$.) With very little algebraic manipulation it can be shown that both (29) and (30) are the same and can be rewritten as

$$F(x) = x^c/(1 + x^c) \tag{31}$$

Also note that (31) is the same as the log logistic distribution (20) with scale parameter $\alpha = 1$ and $\beta = c$. In other words, if X is a random variable with cdf given by (29), the distribution of $Y = 1/X$ is the same as that of X.

4 Applications

The distribution function and the inverse of the distribution function for the Burr type III and type XII distributions exist in simple closed form. This fact plays an important role in the selection of a particular family of distributions as a stochastic model in simulation studies. Since the inverses of the Burr distributions exist in simple closed form, random samples from these distributions can be obtained by the so called 'direct method' or 'the inverse transformation method'. Tadikamalla and Ramberg (1975) and Tadikamalla (1977) used the four-parameter Burr distribution as an approximation to the gamma distribution.

Both the Pearson and the Johnson systems cover a wider region than the Burr type III distribution in the $(\sqrt{\beta_i}, \beta_2)$ plane. However they both use more than one functional form for densities. No simple direct methods are available for obtaining random samples from the Pearson distributions. The estimation of parameters for both the Pearson and the Johnson systems of distributions is limited to the method of moments. Parameters for the Johnson system of distributions can also be estimated numerically by the sample percentiles. A brief discussion of the parameter estimation and random variate generation for these two systems of distributions can be found in Tadikamalla (1975), among many others.

The relationship between different types of the Burr distribution and various other distributions is summarized. It is noted that the 'reciprocal Burr' distribution (Burr type III) covers a wider region in the $(\sqrt{\beta_1}, \beta_2)$ plane than the commonly used Burr type XII distribution. This region of coverage included all the region covered by the commonly used distributions such as the gamma family, the Weibull family, the lognormal family, the normal distribution, the logistic distribution, the bell-shaped and J-shaped beta distributions. The Burr type III and type XII distributions can be used to fit almost any unimodal data and are comparable to the Pearson and the Johnson systems of distributions. The applicability of the Burr distributions in simulation modeling is enhanced by the fact that the inverses of their distribution functions exist in simple closed forms.

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Résumé

La distribution de Burr (type XII de Burr) dont les coefficients d'asymétrie et d'applatissement recouvrent un large champ de valeurs, peur servir à ajuster presque n'importe quel ensemble unimodal de données. La distribution de Burr est apparue dans la littérature sous des noms divers. La relation existant entre la distribution de Burr et les diverses autres distributions ainsi désignées: Lomax, Weibull-composée, Weibull-exponentielle, Logistique, Log-logistique, Weibull et Kappa, est résumée ici. On montre aussi que la distribution dite réciproque de Burr (type III de Burr) recouvre un champ plus vaste que la distribution de Burr dans le plan des coefficients de Pearson $(\sqrt{\beta_1}, \beta_2)$, champ renfermant en totalité celui de la distribution de type XII de Burr.