The Matter-Wave Background of the Titius-Bode Rule

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In today's physics, there is an increased interest in the supposed properties of the future Quantum Gravity in the fields of theoretical foundation and as well as experimental verification. The ultimate goal of this research is the creation of a definitive, universally accepted theory of Quantum Gravity. In the present work, the two hundred and fifty years of history Titius-Bode rule is investigated, under the assumption that this empirical fact is a key evidence of the quantized feature of the already long known classical gravity.
Keywords: Titius-Bode rule, Bohr-Sommerfeld quantization, de Broglie matter wave, Wave-Gravitational Theory.

1. Introduction

Currently an increased interest is seen in the properties of Quantum Gravity in order to its foundation in both theoretical and experimental area. The currently favored Standard Model, which unifies the basic physical interactions into a single theoretical framework, does not contain the gravitational interaction. Root of this problem is the fact that the most advanced theory of gravity, Einstein's general theory of relativity neither in vision, nor in the mathematical formulation can be linked to the philosophy of modern quantum physics.

In recent decades, most attempts related to unify the gravity and the other fundamental physical interactions into a common theoretical description are associated with the various intention to increase the dimension of the well-known four-dimensional (relativistic) space-time. These so called *string theories* and *membrane theories* use very complicated mathematical tools; in addition, the experimental support can seem hopeless, since its assumed effects appear in unobservable small space-time domains.

By simpler theoretical considerations, the quantum of the gravitational field, if it exists at all, is a spin-2, massless particle would be called ‘graviton’, by analogy with the photon. The direct detection of the graviton has so far been no available due to its estimated extremely low energy. It seems that in the other fields successfully used methods of *Quantum Mechanics (QM)* and *quantum field theories* did not lead to breakthrough results. In the passage of time the returning failures of these attempts incite us to approach the problem with completely different physical considerations.

About two hundred and fifty years ago for the known planets the so-called. *Titius-Bode rule* (T-B rule) has been found which describes the approximate distances of the planets from the Sun with exponentially quantized function [1, 2, 3]. The *semi-major axes* of the planets *in astronomical unit* are approximately:

$$a_n \cong 0.4 + 0.3 \cdot 2^n; \; (n = -\infty, 0, 1, 2, \ldots). \tag{1.1}$$

In case of the innermost planet (*Mercury*) the exponent n is minus infinity (in this case the second term is zero), but for the other planets the second term in the formula includes non-negative exponents of integers. Especially in the case of the *Earth* $n = 1$, when the formula gives a *unit value* for the Sun-Earth distance, according to the definition of the *astronomical distance unit*. For details, we find a lot of information on the Internet.

The aim of the present work is a new, alternative physical interpretation of T-B rule, which would be a starting point of a final *Quantum Gravity Theory* in the future. Over the past centuries, and over in recent decades a number of attempts have been made to decipher of the physical background of T-B rule. Unfortunately disturbing evidence exists, that for the moons of the planets in the Solar System the T-B rule only partially or not at all satisfied in some case, but the exponential distance distribution is seemed to be their common property [4].

Physicists, astronomers cannot see any new physical law in the T-B rule, since the Solar System billions of years developed trough chaotic, dissipative processes, series of random mass collisions, which played a crucial role in the generation of the Solar System. However, some physicists accept the existence of some kind of regularity trends which are interpreted as 'path resonances' [5]. In their view this is the direct consequence of the long gravitational couplings between the planets, what resulted for the orbital radii simple rational fractions as: 1:2, 2:3, 2:5, etc.

Some physicists and the author of present work think a deeper physical law behind the T-B rule. In the Internet and especially in recognized astronomical journals as well there are many references, scientific articles related to the theoretical modeling of this mysterious behavior of the planet's orbits. In this work we are trying to understand the T-B rule by a supposed quantum property of the gravity. It should be noted, that we are not the first to combine the gravity with the macroscopic manifestations of quantum mechanics, see [6,...,10].

Table 1. gives the results of T-B rule calculations by (1.1) including the real and calculated planetary distances; the relative errors of the calculated values are shown as percentages. The standard deviation of the calculated distances is very high; it is about 33%.

Table 1. Demonstration of the Titius-Bode rule

Planet	Real distance	Calculated distance	Relative error
Mercury	0.39	0.4	2.56%
Venus	0.72	0.7	2.78%
Earth	1	1	0%
Mars	1.52	1.6	5.26%
Ceres	2.77	2.8	1.08%
Jupiter	5.2	5.2	0%
Saturn	9.54	10	4.82%
Uranus	19.2	19.6	2.08%
Neptune	30.06	38.8	29.08%
Pluto	39.44	77.2	95.74%

2. The Exponential Approximation

Many authors in the related literature conclude that the distribution of the distances of the planets around the Sun, by the mathematical essence is a typical exponential distribution [11, 12, 13]. The concerned physicists try to explain the very physical cause by different theories, but really reassuring and generally accepted theory does not exist to this day. The planetary system was formed during billions of years, and we all can agree that in this period random processes played the decisive role. However, this long period also allowed, due to until now unknown property of gravity, to form the exponential distribution of the distances for the majority of planets and planet's moons with of course, limited accuracy. In this light there is an obvious thing to fit an exponential function to the known planetary distances, what has been realized also in the recent past in many places. However, we cannot speak about a final canonized result in this respect.

In present work we have carried out the fitting procedure for the planetary distances of the Solar System, assuming the exponential distribution. Using the real distance data from **Table 1**., the obtained best result is:

$$a_n = a_0\alpha^n; \ (n = 1, 2, 3, \ldots 10), \tag{2.1}$$

where $n = 1$ belongs to the distance of the most inner planet Mercury (i.e. semi-major axis of the ellipse). The further planetary distances belong to the powers $n = 2, 3, 4, ...$, etc. The result of the math fitting:

$$a_0 = 0.2108...; \ \alpha = 1.7078...; \ \sigma = 0.130... = 13\%. \tag{2.2}$$

It cannot be said that the obtained standard deviation σ is too large or too small, but if we insist that the exponential distribution cannot be a coincidence; the 13 percent of the standard deviation confirms our belief.

Of course, an additional examination has been carried out, when we omitted from the calculation some 'irregular' planets. Regrettably, this way led not to a significant improvement of the (2.1) exponential rule. Now we exemplify it with two calculations. In first example we omitted from the calculation the Uranus and Neptune:

$$a_0 = 0.2211...; \ \alpha = 1.6799...; \ \sigma = 0.099... = 9.9\%. \tag{2.3}$$

In the second example, we omitted from the calculation five planets, namely Venus, Mars, Saturn, Uranus, Neptune planets. The matching result is:

$$a_0 = 0.2207...; \ \alpha = 1.6778...; \ \sigma = 0.052... = 5.2\%. \tag{2.4}$$

The obtained results indicate that the a_0 and α constants did not change significantly by omitting of 'irregular' planets, however, the accuracy of the fitted exponential model improved noticeably. These facts ultimately contribute to our belief that in the background of T-B rule an unknown, surely important real physical law hides.

3. Refining the Titius-Bode rule

The fitted exponential functions of the planetary distances described in the previous chapter contain only one '*quantum number*'. In this relation, however, an important question arises whether there can be such exponential functions with two or more quantum numbers which are capable of calculating planetary distances more accurately than the simple exponential formula (2.1). The mathematical relevance of the question is whether we are able to discover such functions. The other side of this question is far more important, namely whether there can be found such a multi-quantum-variable function which, in one way or another, can be connected to a real or perceivably real physical explanation. In the Internet we have found this kind of functions for descriptions of planetary distances completed more or less with the analysis of the physical background; for example [14,...,17].

In the last short period, regarding to the mathematical point of view, we found surprisingly good mathematical functions for the high precision description of planetary distances. Each of these has two quantum numbers, which we named them by principal quantum number (*n*) and orbital quantum number (*j*) referring to the analogy of the well-known quantum numbers of the hydrogen atom. The studied to date, considering successful for the quantized distance functions are the next:

1./ The first example of the distance function contains two fitting parameters:

$$a_n \cong \mathrm{a}_0\left(\alpha^n + \alpha^j\right);\ \left(n = 1,2,...N; j = 0,1,...N-1\right). \tag{3.1}$$

In the fitting procedure we have taken into account all the distance data from the **Table 1**. The obtained fitting parameters and the standard deviation of the model are:

$$a_0 = 0.143913...;\ \alpha = 1.746846...;\ N = 10;\ \sigma = 0.0271... = 2.71\%. \tag{3.2}$$

To understand the above statement correctly, each planet's distance assigned with two quantum numbers what are specific to the given planet. Despite of this surprisingly simple formula, depending on only two fitting parameters, gives very good values for the real planetary distances. The only problem with this formula that in the case *j = 0*, it does not return to the quantized exponential function (2.1), what we have studied in the previous chapter.

2./ The second example of the distance function contains three fitting parameters (**Table 2**.):

$$a_n \cong \mathrm{a}_0\alpha^n\beta^{-j};\ \left(n = 1,2,...N; j = 0,1,...N-1\right). \tag{3.3}$$

It can be seen that in the case *j = 0*, this formula returns the tested exponential function (2.1) in the previous section. The fitted result is also accurate for the planetary distances; the relative standard deviation is around 2.3 per cent:

$$a_0 = 0.220153...;\ \alpha = 1.813371...;\ \beta = 1.157176...;\ N = 10;\ \sigma = 0.0228... = 2.28\%. \tag{3.4}$$

Table 2. Generalization of T-B rule
with the double quantum-numbered calculation model of (3.3).

Planet	n	j	Real distance	Calculated distance	Relative error
Mercury	1	0	0.39	0.3992	2.36%
Venus	2	0	0.72	0.7239	0.55%
Earth	3	2	1	0.9804	-1.96%
Mars	4	3	1.52	1.5363	1.07%
Ceres	5	3	2.77	2.7859	0.57%
Jupiter	6	3	5.2	5.0518	-2.85%
Saturn	7	3	9.54	9.1608	-3.97%
Uranus	8	2	19.2	19.2230	0.12%
Neptune	9	3	30.06	30.1236	0.21%
Pluto	10	5	39.44	40.7939	3.43%

In advance it is important to mention, that this latter introduced distance function exactly corresponds to our newly established quantized model of gravity based on the old quantum theory of *Bohr-Sommerfeld*.

Remark: The fitting procedure of the above shown distance functions has been realized by method '*Monte-Carlo*'. For the calculation of the standard deviation of the fitted planet distances has been used the usual method:

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{n=1}^{N}\left(\frac{a_n - a_n'}{a_n}\right)^2}. \tag{3.5}$$

In this formula the real planet distances are represented by a_n, the calculated planet distances are represented by a_n', and finally N is the number of the planets have been involved into the calculation.

4. The 'Wave Nature' of the Matter

In the previous chapter the planet's distances have been described by a double quantum-numbered formula what remind us of the quantum mechanical model of the hydrogen atom. The simple fact that the planets occupy approximately exponentially quantized orbits around the Sun, itself does not imply any genetic link to the Quantum Mechanics (QM). However, we are going to show that the exponential distribution of the planetary orbits intrinsically connected to the previous version of the QM; namely to the *old quantum theory*. This new recognition is indeed a possible scientific direction to a really solid foundation of the long-sought quantum theory of gravity. In the following we show that our unusual way leads to the surely real physical interpretation of the Titius-Bode law.

The starting point of the old quantum theory is known as *Bohr-Sommerfeld (B-S) quantization theory* [19,…, 21]. In the B-S theory the quantization of a closed physical system can be realized by the following rule:

$$S = \oint p_i dq_i = n_i h;\ (i = 1,2,3,...;\ n_i = 1,2,3,...), \tag{4.1}$$

where p_i are the momentum components of the particles, q_i are the coordinates of the particles, i is the *freedom degree* of the system, and h is the *Planck's constant*. The quantum numbers n_i are positive integers and the integral is taken over one period of the motion at constant energy (as described by the Hamiltonian). The integral S is an area in the *phase space*, which quantity called '*action*' and quantized in units of Planck's constant. For this reason, Planck's constant was often called '*elementary quantum of action*'.

The initial successes of the B-S theory fed high hopes to understanding of quantum phenomena, for example using this theory, successfully has been derived the known quantized energy levels of harmonic oscillator, and perhaps the most important result was the clarifying of the Bohr's atomic model. A little later a relativistic formulation has been applied for this model by *A. Sommerfeld,* also on the principle of the B-S quantization. The '*relativistic atom-model*' led to the interpretation of the *fine structure* of the *hydrogen spectrum*, which remained appropriate also today. At the same time, despite all effort, the B-S quantization was not suitable for the description of spectra of two- or many electron atoms, for the general solution have had to wait until the middle of the 1920's, the birth of QM.

The first real outbreak came in 1925 in this field, when epoch-making article of *W. Heisenberg* appeared in which he gave a really operable mathematical background for the description of quantum phenomena. Heisenberg assigned infinite-dimensional matrices to the physical quantities (coordinates, momentums) hence the name of his theory is matrix mechanics. *E. Schrödinger* in 1926 found another version of QM, what has been named wave mechanics and it was equivalent to the Heisenberg's matrix mechanics as Schrödinger himself showed from the beginning. The basic idea of wave mechanics came from a French physicist, namely from *Louis de Broglie*, who already in 1924 created the theory of electron waves, at that time without any remarkable scientific echo [22]. Today, the de Broglie's concept of matter waves is fully accepted by the physicist's community.

De Broglie's idea of matter waves was based on the theory of relativity. *M. Planck* in 1900 showed that the most experimentally known laws of the thermal (black body) radiation can be interpreted only by quantized energy radiation:

$$E = h\nu \equiv \hbar\omega. \tag{4.2}$$

Here h is the Planck's constant, ν is the frequency of the thermal (electromagnetic) radiation, $\hbar = h/2\pi$ and $\omega = 2\pi\nu$. In the relativity the energy and the three components of momentum form a four-vector (c is the speed of light):

$$p_\mu = \{E/c, p_x, p_y, p_z\}. \tag{4.3}$$

Planck's law of thermal radiation orders frequency ω to the energy E according to (4.2), in this relation whether what physical quantities can be associated with the momentum components? In the relativity the electromagnetic wave assigned to the *wave number four-vector* (shortly wave four-vector), which its first component is just equal to the frequency of the electromagnetic wave divided by the light speed:

$$k_\mu = \{k_0 = \omega / c,\ k_x = 2\pi / \lambda_x,\ k_y = 2\pi / \lambda_y,\ k_z = 2\pi / \lambda_z\}. \quad (4.4)$$

The quantities λ-s are the spatial components of the wavelength. The relativistic generalization of Planck's law by the above statements can be only the next (it was recognized by de Broglie):

$$p_\mu = \hbar k_\mu. \quad (4.5)$$

The rest mass of the electromagnetic waves (the mass of photons) is zero; the four-momentum squared satisfies the following equation:

$$p_\mu p^\mu = \hbar^2 k_\mu k^\mu = 0. \quad (4.6)$$

De Broglie supposed that this equation must also be met for the rest massive particles, especially electrons as well. According to the relativity, the above equation will change into the next form:

$$p_\mu p^\mu = \hbar^2 k_\mu k^\mu = m^2 c^2. \quad (4.7)$$

In this case this equation associates the mass with some kind of wave, which is called matter wave today. De Broglie's important outcome can be found in the majority of textbooks in a simplified form (this is the *de Broglie wavelength*):

$$\lambda = h / p = h / mv, \quad (4.8)$$

where p is the momentum, m is the mass and v is the velocity of the particle. A simple calculation can easily show that the wavelengths of the macroscopic bodies are unobservable short. However, in case of the electron having very small mass, its matter wave can be detect with interference experiment [23].

5. The 'Wave-Gravity' Hypothesis

After the overwhelming success of the initial results of the obscured preliminary quantum theory; the Bohr-Sommerfeld (B-S) quantization rule remained only a curiosity of the physics history. Understandably, de Broglie's theory of matter waves was not taken into account later in the obsolete B-S quantization method. However, this old quantization method is able to give a new, very interesting physical outcome. Continuing the use of relativistic notation, the B-S quantization of the matter waves can be written into a simple form:

$$S = \oint p_\mu dx^\mu = \hbar \oint k_\mu dx^\mu = \sum_{\mu=0}^{3} n_{(\mu)} = nh;\ (n = 1, 2, 3, ...). \quad (5.1)$$

By this condition, the 'action integral' S is associated to the matter wave can be only a whole-number multiple of the Planck's constant h. In the usual procedure of the B-S quantization the momentum components must express in function of the coordinates, the only question remains what the matter-wave vector dependence on the space-time coordinates. It is important to note that both sides of (5.1) equation have *action dimension* (energy x time), so the *loop integral* can only be a dimensionless quantity. Clear that the simplest choice of the matter-wave vector satisfying the (5.1) condition is the following:

$$k_\mu \Rightarrow 2\pi \{1 / x_0,\ 1 / x_1,\ 1 / x_2,\ 1 / x_3\}, \quad (5.2)$$

where the space-time four-vector usually has the form:

$$x_\mu = \{x_0,\ x_1,\ x_2,\ x_3\} = \{ct,\ x,\ y,\ z\}. \quad (5.3)$$

Using the above definitions, the B-S quantization condition can be written:

$$S = \hbar \oint k_\mu dx^\mu = h \oint \frac{dx_\mu}{x_\mu} = h \sum_{\mu=0}^{3} n_{(\mu)} = nh;\ (n = 1, 2, 3, ...). \quad (5.4)$$

Surprisingly, the Planck's constant with its microscopic property 'vanishes' from the relativistic B-S quantization, so its dominant role what regularly occurs in the atomic, molecular, nuclear, particle, etc. physics, but here becomes irrelevant. In this situation we shall use the B-S quantization rule for macroscopic physical system in the following format:

$$S = D_{(\mu)} \oint \frac{dx_\mu}{x_\mu} = C \sum_{\mu=0}^{3} n_{(\mu)} = Cn > 0; \ (n = 1,2,3,...). \tag{5.5}$$

where $D_{(\mu)}$ are appropriate dimensioned factors and C is an '*action*' dimensioned, unknown yet constant having only positive value. Applying the usual space-time metric, this condition can be written:

$$S = S_T + S_R = D_{(0)} \oint \frac{dx_0}{x_0} - D_{(a)} \sum_{a=1}^{3} \oint \frac{dx_a}{x_a} = Cn > 0; \ (n = 1,2,3,...). \tag{5.6}$$

This quantum condition is equivalent with the next two conditions:

$$S_T = D_{(0)} \oint \frac{dx_0}{x_0} = Cn; \ \left(n = n_{(0)} = 1,2,3,..\right); \tag{5.6a}$$

$$S_R = -D_{(a)} \sum_{a=1}^{3} \oint \frac{dx_a}{x_a} = -C \sum_{a=1}^{3} n_{(a)} \equiv -Cj; \ \left(j = 0,1,2,..,n-1\right). \tag{5.6b}$$

The requirement for the j quantum variable in (5.6b) provides the action S in (5.6) will be positive in all circumstances. Firstly we investigate the time-component (5.6a). The loop integral in this case means that the movement is periodical with finite periods:

$$S_T = D_T \oint \frac{dx_0}{x_0} \equiv D_T \int_{T_0}^{T} \frac{dt}{t} = Cn; \ \left(n = 1,2,3,...; \ D_T \equiv D_{(0)}\right). \tag{5.7}$$

It is useful to replace the time variable for distance variable with the help of a simple integral transformation. Supposing that the velocity of the matter wave is equal to a constant v, we can introduce new variables:

$$R_0 = vT_0; \ R_A = vT; \ dt = dr / v; \ 1/t = v/r, \tag{5.8}$$

what leads to the next quantum condition being equivalent to (5.7):

$$S_T = D_T \int_{T_0}^{T} \frac{dt}{t} = D_T \int_{R_0}^{R_A} \frac{dr}{r} = Cn; \ \left(n = 1,2,3,...\right). \tag{5.9}$$

On the other hand, knowing that the T-B rule is related to the gravitationally central forced Solar System, we can suppose the spatial quantum condition (5.6b) depends on only the radial distance from the gravitational centre:

$$S_R = D_{(a)} \sum_{a=1}^{3} \oint \frac{dx_a}{x_a} = D_R \int_{R_0}^{R_B} \frac{dr}{r} = Cj; \ \left(j = 0,1,2,...,n-1\right). \tag{5.10}$$

Based on the above the total macroscopic action integral will be:

$$S = D_T \int_{R_0}^{R_A} \frac{dr}{r} - D_R \int_{R_0}^{R_B} \frac{dr}{r} = C(n-j) > 0; \ \left(n = 1,2,3,..; \ j = 0,1,2,...,n-1\right). \tag{5.11}$$

Introducing the constants:

$$C_T = C / D_T; \ C_R = C / D_R, \tag{5.12}$$

the (5.11) quantum condition gets a simple form:

$$\int_{R_0}^{R_A} \frac{dr}{r} - \int_{R_0}^{R_B} \frac{dr}{r} = C_T n - C_R j > 0; \ (n = 1,2,3,..; \ j = 0,1,2,...,n-1). \tag{5.13}$$

The evaluation of the integrals leads to the next result:

$$\ln \frac{R_A}{R_B} = C_T n - C_R j > 0; \ (n = 1,2,3,..; \ j = 0,1,2,...,n-1), \tag{5.14}$$

what is equivalent with the next exponential form:

$$\begin{gathered} R_A = R_B \exp(C_T n - C_R j) = R\alpha^n \beta^{-j}; \\ \alpha^n = \exp C_T n; \ \beta^{-j} = \exp(-C_R j); \ (n = 1,2,3,..; \ j = 0,1,2,...,n-1). \end{gathered} \tag{5.15}$$

This final result in case *j = 0* is the same that we have got empirically for the planet's distances in (2.1):

$$a_n = a_0 \alpha^n; \ (a_n \equiv R_A; \ a_0 = R_B; \ n = 1,2,3,...). \tag{5.16}$$

The (5.15) entire formula is the same with the double quantum-numbered planet's distance function what we have presented in the third chapter with the (3.3) formula:

$$a_n = a_0 \alpha^n \beta^{-j}; \ (a_n \equiv R_A; \ a_0 = R_B; \ n = 1,2,3,..; \ j = 0,1,2,...,n-1). \tag{5.17}$$

By this simple way we have certainly given the real physical background of the exponential distance distribution of the planets of our Sun System. This result obtained from the old Bohr-Sommerfeld quantum theory. According to the B-S theory, the calculation of the planet's orbits remained on the level of classic mechanics applying gravitational theory of Newton. Nevertheless, our model seems to be more than the classical physics, what may be a first step for the future foundation of the long hoped Quantum Gravity theory. Having regard to the fact that the essence of our introduced model is closely linked to the de Broglie matter wave theory, we have named our new model by '*Wave-Gravitational Theory*' (WGT). This term emphasizes that our model is far from the original Quantum Mechanics what appearance is known today. In this level there is no sense to introduce the basic concepts of the QM; however some initiative exists to do this, for example in [15].

6. Summary

In this study we have given a possible physical interpretation of the Titius-Bode rule what can be a starting point of a final quantum gravity theory in the future. The basic of the presented theory is the relativistic extension of the old Bohr-Sommerfeld quantization. We have involved the well-known matter wave theory of de Broglie for the imposition of the B-S quantum condition, namely joining the momentum four-vector with the matter wave four-vector. From this simple quantum condition, not surprisingly, the Planck's constant has been vanished. This fact opened up the possibility of the extension of this quantization method for the macroscopic physical systems, especially for the Solar System. By this way we have obtained the allowed orbits of planets what have been experienced very earlier and remained to date unsolvable mystic named as Titius-Bode rule. We have titled this new theoretical construction shortly by 'Wave-Gravitational Theory' (WGT), what basically remained on the soil of classical mechanics. The undeniable simplicity and accuracy of this wave-gravity model is surprisingly.

In recent years, in the newly explored 'exosolar systems 'contain 'exoplanets' which have also exponential orbit's distributions by the observations [24,…,29]. All the results of this study and the recently observed orbital-distributions of the exoplanets further strengthen our belief in the true physical origin of the T-B rule.

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