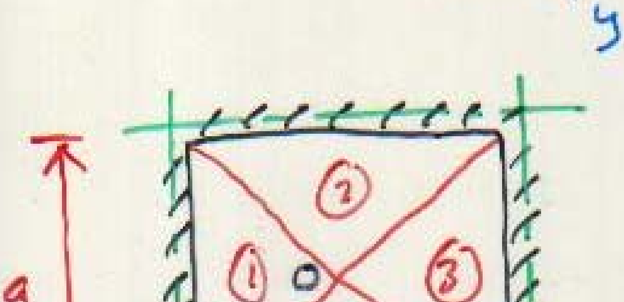
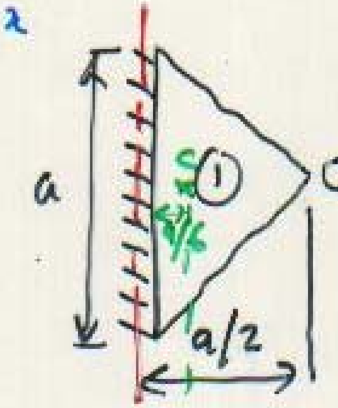
YIELD LINE THEORY (EXAMPLES)

Suhaimi Abu Bakar, PhD

Example 1

A s.s. square slab, a×a, loaded by udl of q/area. Estimate the load carrying capacity of the slab, if it is reinforced with an isotropic reinforcement system.





$M_x = ma$

$\phi_x = \frac{2\delta}{a}$

$M_y = 0$

$\phi_y = 0$

Assuming 'O' displaced by δ

$$\tan\phi_x = \frac{\delta}{a/2} = \frac{2\delta}{a}$$

$$\Rightarrow \phi_x = \frac{2\delta}{a}.$$

$$K_D = \sum M_x \phi_x + \sum M_y \phi_y$$

$$= \left[(ma)\frac{2\delta}{a} + 0\right] \times 4 = 8m\delta$$

$$K_L = \sum \iint (q\,dx\,dy)\cdot w$$

$$= \left(q(a)\left(\frac{a}{2}\right)\left(\frac{1}{2}\right)\times\frac{\delta}{3}\right)4$$

$$= \frac{qa^2\delta}{3}.$$

Set up work equation:

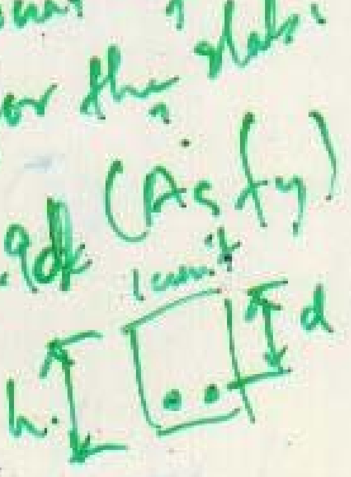
$$K_D = K_L$$

$$8m\delta = \frac{qa^2\delta}{3}$$

$$\Rightarrow m = \frac{qa^2}{24} = 0.0417\,qa^2 \quad \#$$

The amount of steel for the slab:

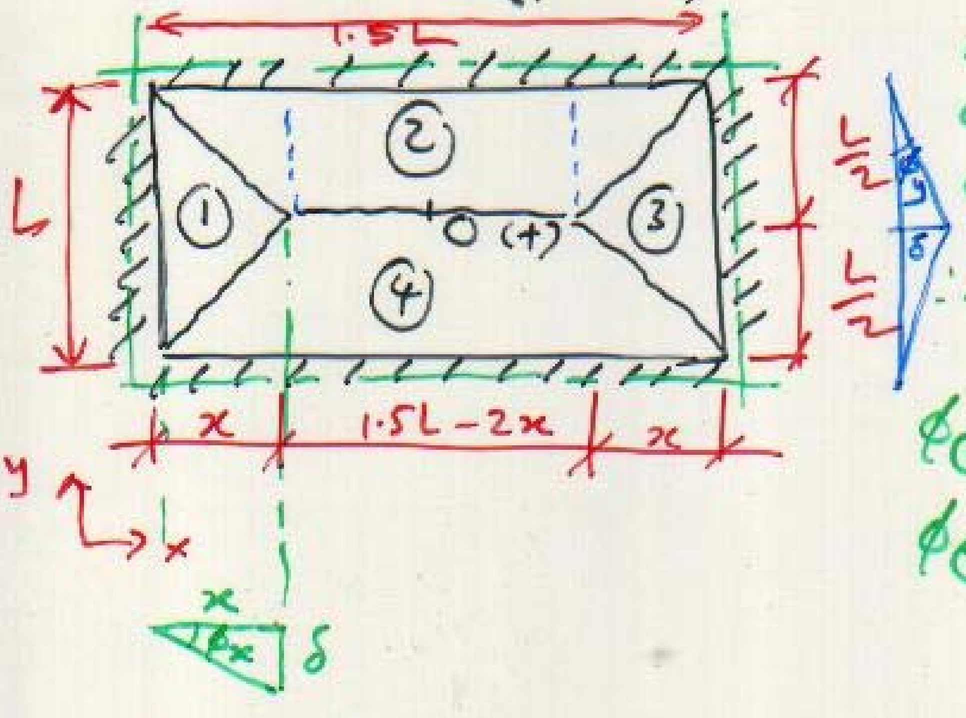
$$M = 0.9d\,(A_s f_y)$$



Example 2

Rectangular shape, $L \times 1.5L$ Simply supported along all edges, Isotropic reinforcement.

load = q (area)



Assuming displacement at $O = \delta$

$\therefore$ Rotation of the parts

$$\phi_{①} = \phi_x = \frac{\delta}{x}$$

$$\phi_{②} = \phi_y = \frac{\delta}{L/2} = \frac{2\delta}{L}$$

$$K_D = \sum m\phi$$

$$= 2\left[(mL)\frac{\delta}{x}\right]_{①\&③} + 2\left[(m)(1.5L)\left(\frac{2\delta}{L}\right)\right]_{②\&④}$$

$$= \frac{2mL\delta}{x} + 6m\delta$$

$$= 2\delta mL\left(\frac{1}{x} + \frac{3}{L}\right) \text{ ——— } ⓐ$$

$$K_L = \sum \iint q\,dx\,dy \cdot w$$

$$= 2\left[q\left(\frac{Lx}{2}\right)\left(\frac{\delta}{3}\right)\right]_{①} + 2\left[q\left(\frac{3L}{2} - 2x\right)\frac{L}{2}\left(\frac{\delta}{2}\right) + \right.$$

$$2\frac{g}{\delta}\left(\frac{xL}{4}\right)\left(\frac{\delta}{3}\right)\Big]$$

$$= \frac{g}{\delta}\left(\frac{3L^2}{4} - \frac{Lx}{3}\right)\delta \quad \text{———} \; (b)$$

$K_p = K_L$

$$2\delta mL\left(\frac{1}{x} + \frac{3}{L}\right) = \frac{g}{\delta}\left(\frac{3L^2}{4} - \frac{Lx}{3}\right)\delta$$

$$2mL\left(\frac{1}{x} + \frac{3}{L}\right) = \frac{g}{\delta}\left(\frac{3L^2}{4} - \frac{Lx}{3}\right)$$

$$m = \frac{\frac{g}{\delta}\left(\frac{3L^2}{4} - \frac{Lx}{3}\right)}{2L\left(\frac{1}{x} + \frac{3}{L}\right)}$$

$$= \frac{\frac{1}{12}\frac{g}{\delta}\left(9L^2 - 4Lx\right)}{\frac{2L}{xL}\left(L + 3x\right)}$$

$$= \frac{gx}{24\delta}\left(\frac{9L^2 - 4Lx}{L + 3x}\right)$$

$$= \frac{g}{24\delta}\frac{\left(9L^2x - 4Lx^2\right)}{\left(L + 3x\right)} \quad \text{—} \; (c)$$

$m = f(x)$

– Look for best x to optimise m. $\frac{dm}{dx} = 0, x = ?$

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~~$m = \frac{q}{24}L$~~ $m = \frac{q}{24}\left(\frac{9L^2x - 4Lx^2}{L+3x}\right)$

$$\frac{dm}{dx} = \frac{q}{24}\left[\frac{(L+3x)(9L^2-8Lx)-3(9L^2x-4Lx^2)}{(L+3x)^2}\right]$$

$= 0$

$\frac{q}{24} \neq 0$

$$\frac{(L+3x)(9L^2-8Lx)-3(9L^2x-4Lx^2)}{(L+3x)^2}$$

$= 0$

$\vdots$

$\Rightarrow \quad 12Lx^2 + 8L^2x - 9L^3 = 0$

$x = -1.26L, \ \underline{0.595L}$

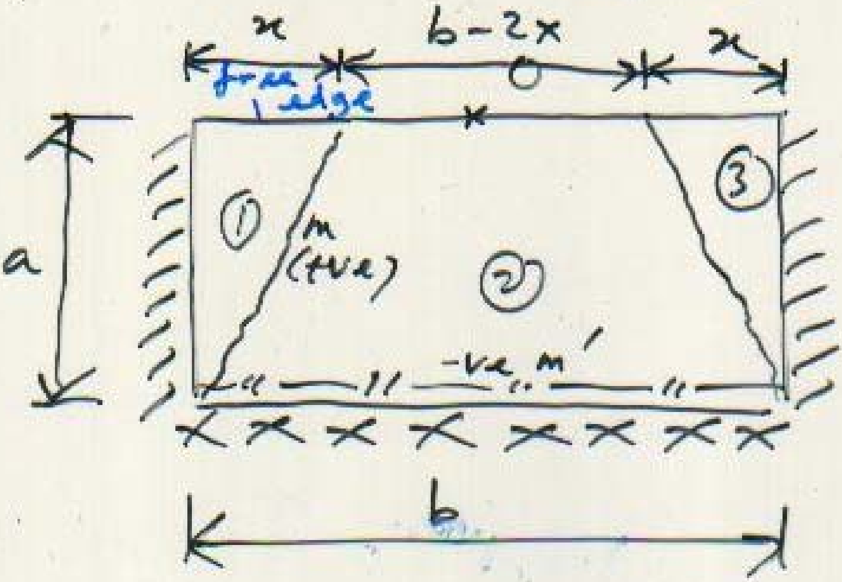
$$m = \frac{q}{24}\left(\frac{9L^2(0.595L) - 4L(0.595L)^2}{L + 3(0.595L)}\right)$$

$$= \frac{q}{24}\left(\frac{5.355L^3 - 1.4161L^3}{2.785L}\right)$$

$= 0.05893qL^2$ #

Example 3 Load = udl

$-ve = +ve \quad m' = m$



$$m = \frac{qa^2}{6}\left[\frac{3bx - 2x^2}{2x^2 + bx + 2a^2}\right]$$

$\frac{dm}{dx} = 0$

$m = ?$

Solution

$$\phi_1 = \frac{\delta}{x}.$$

$$\phi_2 = \frac{\delta}{a}.$$

$$K_D = 2\left[(ma)\left(\frac{\delta}{x}\right)\right]_{1+3} + \left[(m'b+2mx)\left(\frac{\delta}{a}\right)\right]_{3}$$

$$= \frac{m}{ax}\left(2x^2+bx+2a^2\right) \quad \text{— (a)}$$

$$K_L = 2\left(q\times\tfrac{1}{2}\times a\times x\times\frac{\delta}{3}\right) + 2\left(q\times\tfrac{1}{2}\times x\times a\times\frac{\delta}{3}\right)$$

$$+\, q\times(b-2x)(a)\times\frac{\delta}{2}$$

$$= \frac{qax\delta}{3} + \frac{qax\delta}{3} + \frac{qa\delta}{2}(b-2x)$$

$$= \frac{qa}{6}(3b-2x) \quad \text{— (b)}$$

$$\text{Hence } m = \frac{qa^2}{6}\left[\frac{3bx-2x^2}{2x^2+bx+2a^2}\right] \quad \text{— (c)}$$

$$\frac{dm}{dx} = \frac{ga^2}{6}\left\{\frac{(2x^2+bx+2a^2)(3b-4x)-(3bx-2x^2)(4x+b)}{(2x^2+bx+2a^2)^2}\right\}$$

$$= 0$$

$$m = \frac{u}{v} \rightarrow \frac{dm}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

$$\frac{ga^2}{6}\left\{\frac{6x^2b - 8x^3 + 3b^2x - 4bx^2 + 6a^2b - 8a^2x}{(2x^2+bx+2a^2)^2} - \frac{12bx^2 + 3b^2x - 8x^3 - 2x^2b}{(2x^2+bx+2a^2)^2}\right\}$$

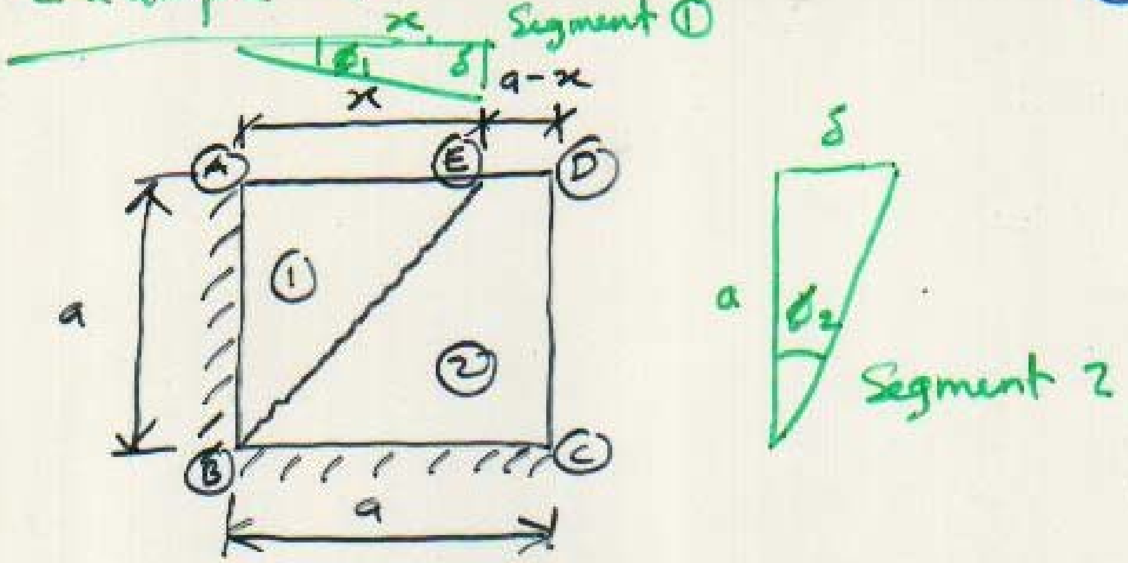
$$= \frac{ga^2}{6}\left\{\frac{(6b-4b-12b+2b)x^2 + (3b^2 - 8a^2 - 3b^2)x + 6a^2b}{(2x^2+bx+2a^2)^2}\right\}$$

$$= 0$$

$$\Rightarrow -8bx^2 - 8a^2x + 6a^2b = 0$$

$$x = \frac{-(-8a^2) \pm \sqrt{(-8a^2)^2 - 4(-8b)(6a^2b)}}{2(-8b)}$$

Example 4



Assuming displacement at Ⓔ = δ

$$\phi_1 = \frac{\delta}{x}, \quad \phi_2 = \frac{\delta}{a}$$

The work equation:

$$ma\left(\frac{\delta}{x}\right) + mx\left(\frac{\delta}{a}\right) = \frac{1}{6} q a x^2\left(\frac{\delta}{x}\right) + \frac{1}{6} q x a^2\left(\frac{\delta}{a}\right) + \frac{1}{2} q(a-x)a^2\left(\frac{\delta}{a}\right)$$

$$q = \frac{6m}{a^2}\left(\frac{a^2+x^2}{3ax-x^2}\right) \Rightarrow q = \frac{5.55m}{a^2}\left(\text{for } \frac{x}{a} = 0.72\right)$$

$$\frac{ma}{x} + \frac{mx}{a} = \frac{qax}{6} + \frac{qxa}{6} + \frac{q(a-x)a}{2}$$

$$m\left(\frac{a}{x} + \frac{x}{a}\right) = q\left(\frac{ax}{6} + \frac{ax}{6} + \frac{a(a-x)}{2}\right)$$

$$\frac{m}{xa}\left(a^2+x^2\right)=q\left(\frac{ax}{3}+\frac{a^2}{2}-\frac{ax}{2}\right)$$

$$q=\frac{\frac{m}{xa}\left(a^2+x^2\right)}{\left(\frac{a^2}{2}-\frac{ax}{6}\right)}=\frac{\frac{m}{xa}\left(a^2+x^2\right)}{\frac{a}{6}\left(3a-x\right)}$$

$$=\frac{6m}{a^2x}\left(\frac{a^2+x^2}{3a-x}\right)$$

$$=\frac{6m}{a^2}\left(\frac{a^2+x^2}{3ax-x^2}\right)$$

$$m=\frac{qa^2}{6}\left(\frac{3ax-x^2}{a^2+x^2}\right)$$

$$\frac{dm}{dx}=\frac{qa^2}{6}\left[\frac{(a^2+x^2)(3a-2x)-(2x)(3ax-x^2)}{(a^2+x^2)^2}\right]$$

$$=\frac{qa^2}{6}\left(\frac{3a^3-2a^2x+3ax^2-2x^3-6ax^2+2x^3}{(a^2+x^2)^2}\right)$$

$$=\frac{qa^2}{6}\left(\frac{-3ax^2-2a^2x+3a^3}{(a^2+x^2)^2}\right)=0$$

$$-3ax^2 - 2a^2x + 3a^3 = 0$$

$$x = \frac{-(-2a^2) \pm \sqrt{(-2a^2)^2 - 4(-3a)(3a^3)}}{2(-3a)}$$

$$= \frac{2a^2 \pm \sqrt{4a^4 + 36a^4}}{-6a}$$

$$= \frac{2a^2 \pm \sqrt{40a^4}}{-6a} = \frac{2a^2 \pm 6.32a^2}{-6a}$$

$$x = -1.39a,\ 0.72a$$

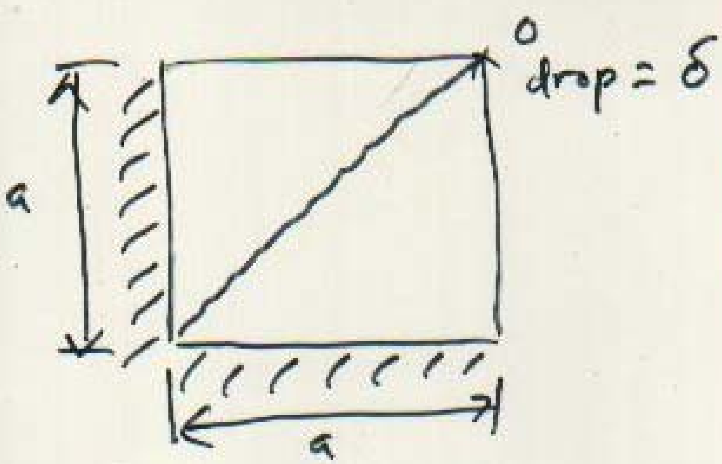
$$x = 0.72a$$

$$\frac{x}{a} = 0.72$$

$$m = \frac{ga^2}{6}\left(\frac{3a(0.72a) - 0.52a^2}{a^2 + 0.52a^2}\right)$$

$$= \frac{ga^2}{6}\left(\frac{1.64a^2}{1.52a^2}\right) = \frac{ga^2}{6}(1.08)$$

$$m = 0.18ga^2.$$



$$2(ma)\frac{\delta}{a} = 2g\left(\tfrac{1}{2}\right)(a)(a)\frac{\delta}{3}$$

$$2m\delta = \frac{ga^2\delta}{3}$$

$$m = \frac{ga^2}{6}, \quad g = \frac{6m}{a^2}$$

or $m = 0.17\,ga^2$

From example 4 :

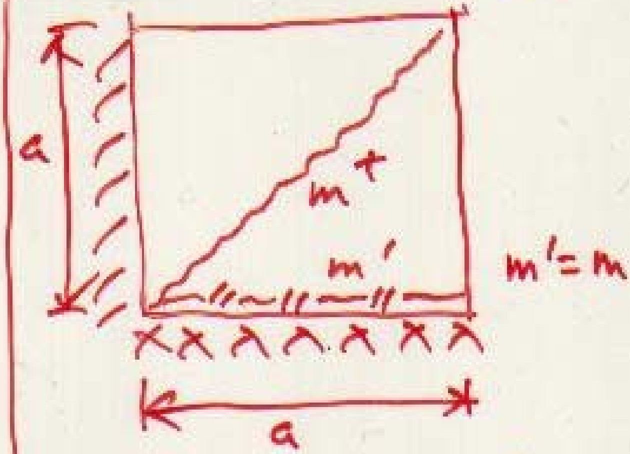
$$g = \frac{6m}{a^2}\left(\frac{a^2+x^2}{3ax-x^2}\right)$$

$x = a$

$$g = \frac{6m}{a^2}\left(\frac{a^2+a^2}{3a^2-a^2}\right)$$

$$= \frac{6m}{a^2}\left(\frac{2a^2}{2a^2}\right)$$

$$= \frac{6m}{a^2}$$



$$ma\left(\frac{\delta}{a}\right) + ma\left(\frac{\delta}{a}\right) + m'a\left(\frac{\delta}{a}\right)$$

$$\parallel$$

$$2g\left(\tfrac{1}{2}\right)(a)(a)\frac{\delta}{3}$$

$$\Rightarrow 3m\delta = \frac{ga^2\delta}{3}$$

$$m = \frac{ga^2}{9}$$

$$= 0.11\,ga^2$$

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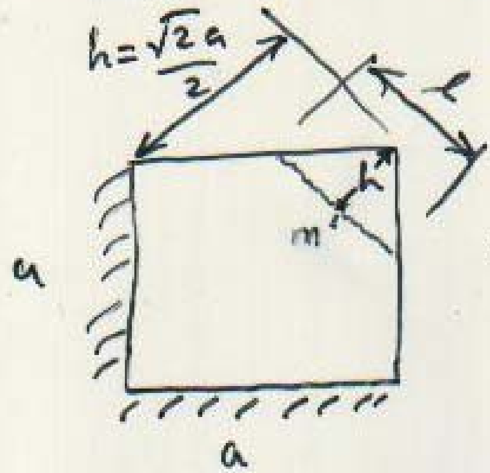
Biggest possible:



$$\frac{x}{a} = 0.72$$

$$q = \frac{5.55m}{a^2} \quad — (1)$$

Work terms,

External $W_L = q\left(\frac{1}{2} l h\right) \cdot \frac{\delta}{3} = \frac{1}{6} q l h \delta$

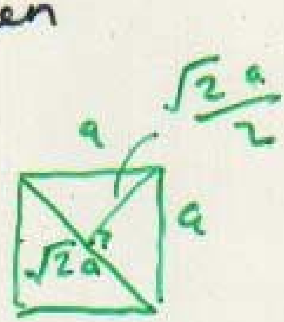
Internal $W_D = m' l \left(\frac{\delta}{h}\right)$

Work equation: $\frac{1}{6} q l h \delta = m' l \frac{\delta}{h}$

Hence $q = \frac{6 m' l \delta}{h l h \delta} = \frac{6m'}{h^2}$

For least value of q then

$$h = \frac{\sqrt{2}}{2} a$$



then $q = \frac{6m'}{\left(\frac{\sqrt{2}}{2} a\right)^2}$

$$= \frac{6m'}{\left(\frac{2a^2}{4}\right)} = \frac{12m'}{a^2} \quad — (2)$$

U.B. solution give the same value at load carrying capacity if $12m' = 5.55m$ or

$$\frac{m'}{m} = 0.46 \quad \#$$

Hence possible solution can be written as

$$q = \begin{cases} \dfrac{5.55m}{a^2} & \text{for } m' \geq 0.46m \\ \dfrac{12m'}{a^2} & \text{for } m' < 0.46m \end{cases}$$