POTENTIAL ENERGY APPLIED IN
FINITE ELEMENT METHOD

by
Suhaimi Abu Bakar, PhD
INTRODUCTION

Finite difference method is suitable to used if the structure to be analysed has the specific governing differential equation.

For structures which are so complex that is difficult or impossible to determine the governing differential equation, a powerful method for analysing such complex structures is the finite element method.
The way finite element analysis obtains the temperatures, stresses, flows, or other desired unknown parameters in the finite element model are by minimizing an energy functional. An energy functional consists of all the energies associated with the particular finite element model. Based on the law of conservation of energy, the finite element energy functional must equal zero.

The finite element method obtains the correct solution for any finite element model by minimizing the energy functional. The minimum of the functional is found by setting the derivative of the functional with respect to the unknown grid point potential for zero. Thus, the basic equation for finite element analysis is:
\[ \frac{\partial F}{\partial p} = 0 \]

where \( F \) is the energy functional and \( p \) is the unknown grid point potential (In mechanics, the potential is displacement) to be calculated. This is based on the principle of virtual work, which states that if a particle is under equilibrium, under a set of a system of forces, then for any displacement, the virtual work is zero. Each finite element will have its own unique energy functional.
PRINCIPLE OF MINIMUM POTENTIAL ENERGY

For conservative systems, of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable.

Satisfy the single-valued nature of displacements (compatibility) and the boundary conditions.
SPRING EXAMPLE

spring stiffness, $k$

$\delta$ is an equilibrium displacement, if displacement $< \delta$, the spring still moving (inequilibrium) $\Rightarrow$ has kinetic energy

We get equation of equilibrium $\Rightarrow$ satisfy equilibrium condition
STRAIN ENERGY

When loads are applied to a body, they will deform the materials. Provided no energy is lost in the form of heat, the external work done by the loads will be converted into internal work called strain energy. This energy, which is always positive, is stored in the body and is caused by the action of either normal or shear stress.
Strain energy is an amount of energy stored in the material due to work done.

The force $\sigma_x dydz$ does work on an extension $\varepsilon_x dx$.

Work done during deformation, $dW =$ Area of triangle OAB

$$dW = \frac{1}{2} \sigma_x \varepsilon_x dx dydz$$

or

$$dW = \frac{1}{2} \sigma_x \varepsilon_x dV$$

Strain energy is an amount of energy stored in the material due to work done.
In general, if the body is subjected only to a uniaxial *normal stress* \( \sigma \), acting in a specified direction, the strain energy in the body is then

\[
W = \int_V \frac{\sigma \varepsilon}{2} dV
\]

Also, if the material behaves in a linear-elastic manner, Hooke’s law applies, \( \sigma = E \varepsilon \), and therefore:

\[
W = \int_V \frac{\sigma^2}{2E} dV
\]
The force $\tau dx dy$ does work on a shear slip $\gamma dz$

Work done during deformation, $dW =$
Area of triangle OAB

$$dW = \frac{1}{2} (\tau dx dy) \gamma dz$$

or

$$dW = \frac{1}{2} \tau \gamma dV$$

Strain energy is an amount of energy stored in the material due to work done.
In general, if the body is subjected only to a shear stress $\tau$, the strain energy due to shear stress in the body is then

$$W = \int_V \frac{\tau \gamma}{2} dV$$

Also, if the material behaves in a linear-elastic manner, Hooke’s law applies, $\tau = G\gamma$, therefore:

$$W = \int_V \frac{\tau^2}{2G} dV$$
STRAIN ENERGY DUE TO MULTIPLE STRESSES

\[ dW = \frac{1}{2} \left( \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{xz} \gamma_{xz} \right) \]

or

\[ dW = \frac{1}{2} \int_{\Omega} \sigma^T \varepsilon \, dV \]

where

\[
\begin{align*}
\sigma &= \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} \\
\varepsilon &= \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}
\end{align*}
\]
Strain energy stored due to bending of a very small length of beam element, $dx$.

\[
dU = \frac{1}{2} \int_V \sigma_x \varepsilon_x dV
\]

\[
= \left( \int_A \frac{1}{2} \left( \frac{My}{I} \right) \left( \frac{\sigma_x}{E} \right) dA \right) dx
\]

\[
= \int_A \frac{1}{2} \left( \frac{My}{I} \right) \left( \frac{My}{EI} \right) dAdx
\]

\[
= \frac{1}{2} \frac{M^2}{EI^2} \int_A y^2 dAdx
\]

\[
= \frac{1}{2} \frac{M^2}{EI^2} I dx
\]

\[
= \frac{1}{2} \frac{M^2}{EI} dx
\]
STRAIN ENERGY IN BEAM DUE TO TRANSVERSE SHEAR

Define the form factor for shear as $f_s = \frac{A}{I^2} \int_A \frac{Q^2}{b^2} dA$

Substituting into the above equation, we get

$$W = \int_0^L \frac{f_s V^2}{2GA} dx$$
EXTERNAL WORK – BAR ELEMENT

Consider the work done by an axial force applied to the end of the bar. As the magnitude of force, \( F \) is \textit{gradually} increased from zero to some limiting value \( F=P \), the bar displaced from \( x=0 \) to \( x=\Delta \).

\[
\text{External Work, } W = \int_0^\Delta F \, dx
\]

\[
= \int_0^\Delta (kx) \, dx
\]

\[
= k \int_0^\Delta x \, dx
\]

\[
= k \left( \frac{\Delta^2}{2} \right)
\]

\[
= \frac{1}{2} (k \Delta) \Delta
\]

\[
= \frac{1}{2} P \Delta
\]
EXTERNAL WORK DUE TO BODY FORCES

\[ b_x, b_y, b_z = \text{Body forces per unit volume with respect to x, y and z axes, respectively} \]

\[ d = \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} \quad b = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} \]

Work due to body forces = \[ \int_{\Omega} d^T b dV \]
EXTERNAL WORK DUE TO SURFACE TRACTIONS

$q_x, q_y, q_z =$ Surface tractions with respect to x, y and z axes, respectively

\[ q = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} \]

Work due to surface tractions = 
\[ \int_\Gamma d^T q dS \]
CONSERVATION OF ENERGY

Most energy methods used in mechanics are based on a balance of energy, often referred to as the conservation of energy. The energy developed by heat effects will be neglected. As a result, if a loading is applied slowly to a body, so that kinetic energy can also be neglected, then physically the external loads tend to deform the body so that the loads do external work $W_e$ as they are displaced. This external work caused by the loads is transformed into internal work or strain energy $W_i$, which is stored in the body. Furthermore, when the loads are removed, the strain energy restores the body back to its original undeformed position, provided the material’s elastic limit is not exceeded. The conservation of energy for the body can therefore be stated mathematically as

$$W_e = W_i$$
CONSERVATION OF ENERGY - EXAMPLE

Beam

\[ \frac{1}{2} P \Delta = \int_0^L \frac{M^2}{2EI} \, dx \]

We assumed that the load is gradually increased from 0 to P

Truss

\[ \frac{1}{2} P \Delta = \sum \frac{N^2 L}{2AE} \]
DIFFERENT CONCEPT IN ENERGY METHOD

Gradually increase the load up to \( W \) (let say 0 kN to \( W \) kN)

Spring stiffness, \( k \)

Initial position

External load in spring

Internal load in spring

External Work = Internal Work

\[
\frac{1}{2} W x = \frac{1}{2} k x^2
\]

gives

\[
x = \frac{W}{k}
\]

Equilibrium Principle

\( W = k x \)

gives

\[
x = \frac{W}{k}
\]

Potential Energy Concept

\[
\Pi = \frac{1}{2} k x^2 - W x
\]

\[
\frac{d\Pi}{dx} = k x - W = 0
\]

gives

\[
x = \frac{W}{k}
\]
THE FOUR-NODE QUADRILATERAL (2D)

\[ u_1 = N_1 u_1^1 + N_2 u_1^2 + N_3 u_1^3 + N_4 u_1^4 \]

\[ u_2 = N_1 u_2^1 + N_2 u_2^2 + N_3 u_2^3 + N_4 u_2^4 \]

\[ \mathbf{d} = \mathbf{N} \mathbf{u} \]

where

\[ \mathbf{d} = \begin{bmatrix} u_1 \end{bmatrix} \]

\[ \mathbf{u} = \begin{bmatrix} u_1^1 \\ u_1^2 \\ u_1^3 \\ u_1^4 \\ u_2^1 \\ u_2^2 \\ u_2^3 \\ u_2^4 \end{bmatrix} \]

\[ \mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \]
\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]

\( \varepsilon = Ld \) where \( L = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix} \)

\( \varepsilon = LNu \)

\( \varepsilon = Bu \) where \( B = LN \)
STRAIN-STRESS RELATIONSHIP

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{zx}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{bmatrix}
\]
STRESS-STRAIN RELATIONSHIP

For plane stress:

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{E} & -\frac{\nu}{E} & 0 \\
-\frac{\nu}{E} & \frac{1}{E} & 0 \\
0 & 0 & \frac{1}{G}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \frac{E}{1-\nu^2}
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

\[
\sigma = D \varepsilon
\]
POTENTIAL ENERGY

\[ \Pi = \frac{1}{2} \int_{\Omega} \sigma^T \epsilon dV - \int_{\Omega} d^T b dV - \int_{\Gamma} d^T q dS \]

\[ = \frac{1}{2} \int_{\Omega} (D \epsilon)^T \epsilon dV - \int_{\Omega} d^T b dV - \int_{\Gamma} d^T q dS \]

\[ = \frac{1}{2} \int_{\Omega} (DBu)^T Bu dV - \int_{\Omega} d^T b dV - \int_{\Gamma} d^T q dS \]

\[ = \frac{1}{2} \int_{\Omega} (DBu)^T Bu dV - \int_{\Omega} (Nu)^T b dV - \int_{\Gamma} (Nu)^T q dS \]

\[ = \frac{1}{2} \int_{\Omega} u^T (DB)^T Bu dV - \int_{\Omega} u^T N^T b dV - \int_{\Gamma} u^T N^T q dS \quad \text{Note: } (PQ)^T = Q^T P^T \]

\[ = \frac{1}{2} \int_{\Omega} u^T (DB)^T (B^T)^T u dV - \int_{\Omega} u^T N^T b dV - \int_{\Gamma} u^T N^T q dS \]

\[ = \frac{1}{2} \int_{\Omega} u^T (B^T DB)^T u dV - \int_{\Omega} u^T N^T b dV - \int_{\Gamma} u^T N^T q dS \]
\[ \Pi = \sum_e \Pi_e \]

Thus,

\[ \Pi_e = \frac{1}{2} \int_{\Omega_e} u^T \left( B^T D B \right)^T u dV - \int_{\Omega_e} u^T N^T b dV - \int_{\Gamma} u^T N^T q dS \]

\[ = \frac{1}{2} u^T \int_{\Omega_e} \left( B^T D B \right)^T dV u - u^T \int_{\Omega_e} N^T b dV - u^T \int_{\Gamma} N^T q dS \]

\[ = \frac{1}{2} u^T \int_{\Omega_e} \left( B^T D B \right)^T dV u - u^T \left( \int_{\Omega_e} N^T b dV + \int_{\Gamma} N^T q dS \right) \]

\[ = \frac{1}{2} u^T k u - u^T f \]

\[ k=\text{element stiffness} \]
from \( \Pi_e = \frac{1}{2} u^T ku - u^T f : \)

\[
\Pi_e = \frac{1}{2} \left( u_1^1 k_{11} u_1^1 + u_1^1 k_{12} u_2^1 + \cdots + u_1^1 k_{1n} u_2^4 +
\right.
\]

\[
\left. u_2^1 k_{21} u_1^1 + u_2^1 k_{22} u_2^1 + \cdots + u_2^1 k_{2n} u_2^4 +
\right.
\]

\[
\left. \cdots +
\right.
\]

\[
\left. u_2^4 k_{n1} u_1^1 + u_2^4 k_{n2} u_2^1 + \cdots + u_2^4 k_{nn} u_2^4 \right) - \left( u_1^1 f_1 + u_2^1 f_2 + \cdots + u_2^4 f_n \right)
\]

Note: \( n = 2 \times 4 = 8 \)

Taking the derivative \( \frac{\partial \Pi_e}{\partial u_1^1} = 0 : \)

\[
\frac{\partial \Pi_e}{\partial u_1^1} = \left( k_{11} u_1^1 + \frac{1}{2} k_{12} u_2^1 + \cdots + \frac{1}{2} k_{1n} u_2^4 + \frac{1}{2} u_1^1 k_{21} + \cdots + \frac{1}{2} u_2^4 k_{n1} \right) - (f_1) = 0
\]

or

\[
\left( k_{11} u_1^1 + k_{12} u_2^1 + \cdots + k_{1n} u_2^4 \right) - (f_1) = 0 \quad (A1)
\]
\[ \Pi_e = \frac{1}{2} \left( u_1 k_{11} u_1 + u_1 k_{12} u_2 + \cdots + u_1 k_{1n} u_n + u_2 k_{21} u_1 + u_2 k_{22} u_2 + \cdots + u_2 k_{2n} u_n + \cdots + u_n k_{nn} u_n \right) - \left( u_1 f_1 + u_2 f_2 + \cdots + u_n f_n \right) \]

Taking the derivative \( \frac{\partial \Pi_e}{\partial u_1} = 0 \):

\[
\frac{\partial \Pi_e}{\partial u_2} = \left( \frac{1}{2} u_1 k_{12} + \frac{1}{2} k_{21} u_1 + k_{22} u_2 + \cdots + \frac{1}{2} k_{2n} u_n + \cdots + \frac{1}{2} u_n k_{nn} \right) - (f_2) = 0
\]

or \( (k_{21} u_1 + k_{22} u_2 + \cdots + k_{2n} u_n) - (f_2) = 0 \) \( (A2) \)

Similarly, taking the derivative \( \frac{\partial \Pi_e}{\partial u_4} = 0 \):

\[
(k_{n1} u_1 + k_{n2} u_2 + \cdots + k_{nn} u_n) - (f_n) = 0
\]

\( (A'n') \)
Collecting equations A1, A2, …, A’n’ one gets the element equilibrium equation:

\[ ku - f = 0 \]

by minimizing the potential energy, \( \frac{\partial \Pi_e}{\partial u} = 0 \)

where

\[ k = \int_{\Omega_e} \left( B^T DB \right)^T dV \]

\[ f = \int_{\Omega_e} N^T b dV + \int_{\Gamma} N^T q dS \]

Load vector

stiffness matrix

\[ k = \int_{\Omega_e} \left( DB \right)^T \left( B^T \right)^T dV \]

\[ = \int_{\Omega_e} \left( B \right)^T \left( D \right)^T \left( B^T \right)^T dV \]

\[ = \int_{\Omega_e} \left( B \right)^T \left( D \right)^T B dV \]

\[ = \int_{\Omega_e} B^T DB dV \]

\[ D^T = D \]
PROCEDURE OF FINITE ELEMENT ANALYSIS

• Assemble the element equilibrium equation \((k_u = f)\) for all finite elements in the structure and form global equilibrium equation, \(KU=F\)

• Consider boundary condition and modify the \(K\) matrix, \(U\) and \(F\) vectors. Form the following equation: \(AX=B\) where \(X\) = unknown vectors, \(B\) = known vectors.

• Solve the equation \(AX=B\)

• Calculate other parameters – displacements, strains, stresses etc