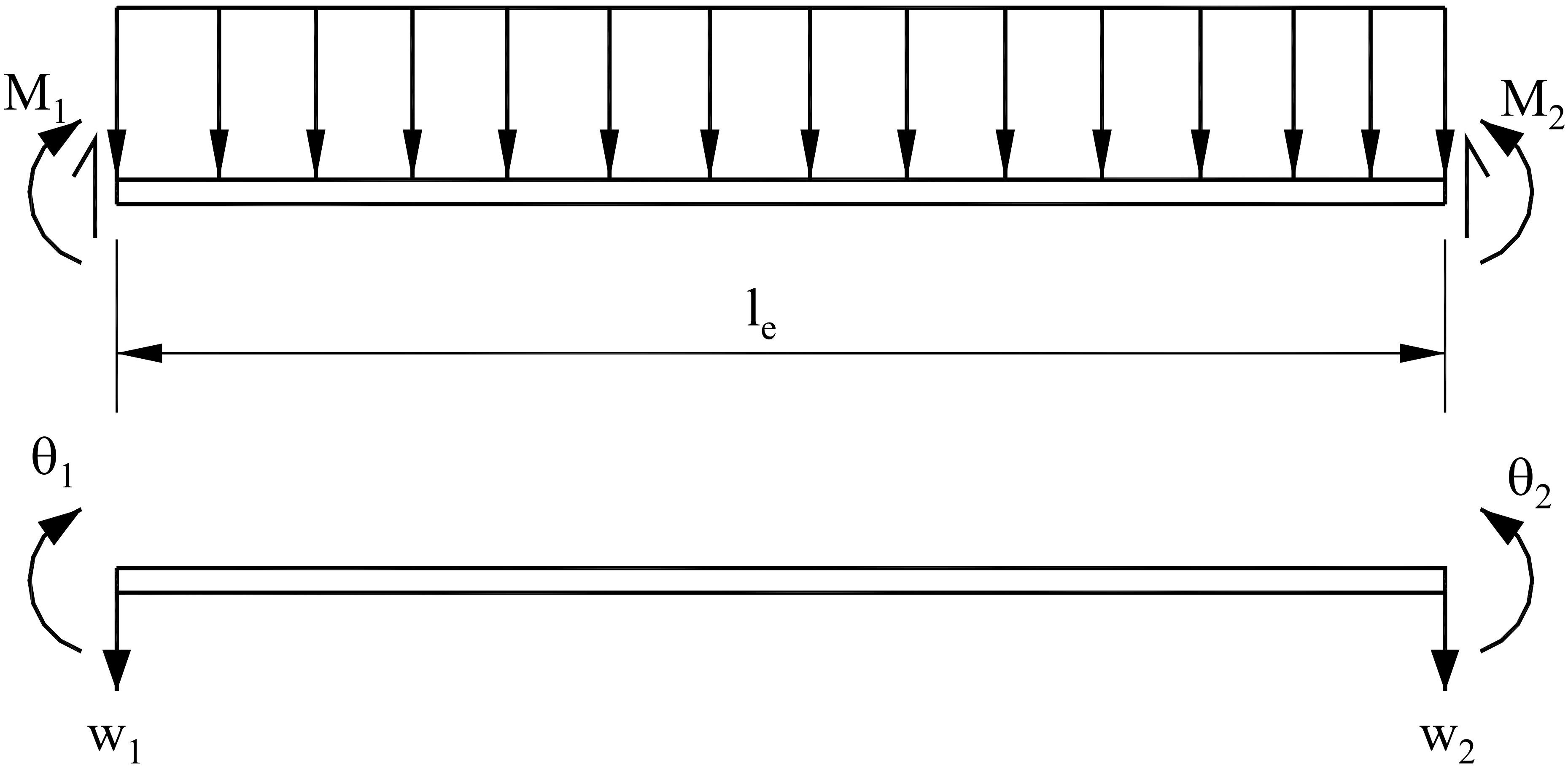
EXERCISE

By using Potential Energy Approach, prove that the stiffness matrix for the beam element (loaded by uniform distributed load) is:

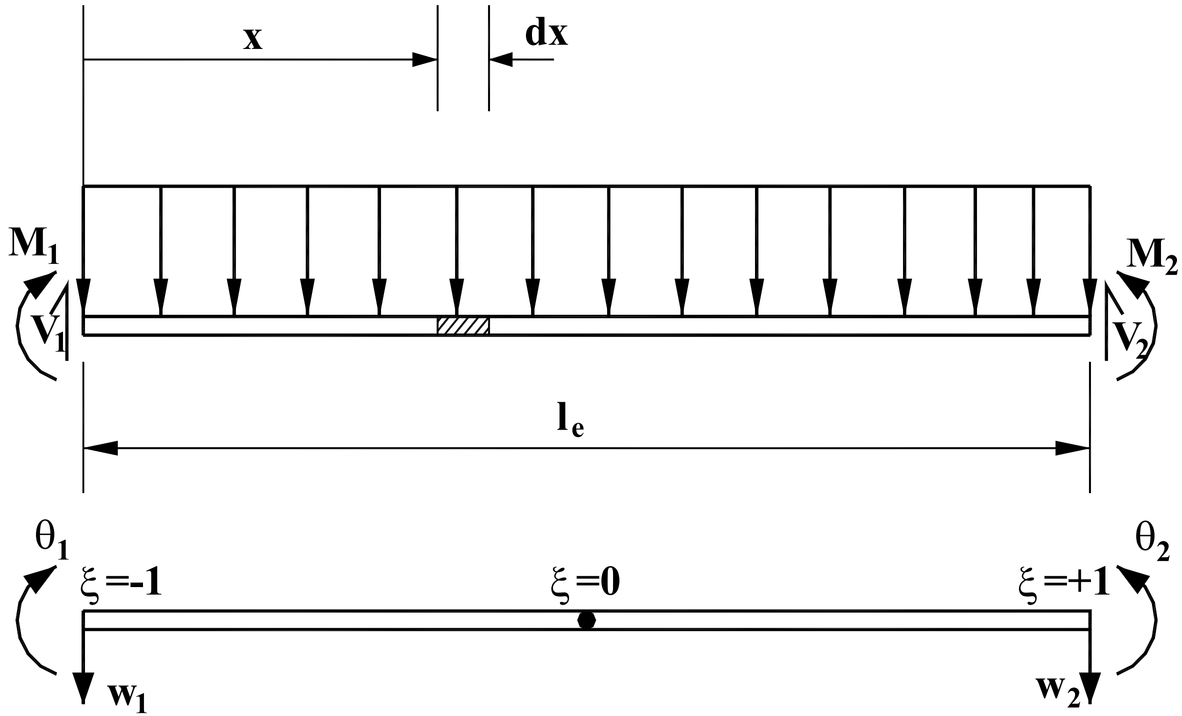
$$\boldsymbol{k}^e = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

Use Hermite or cubic shape function to relate displacements in the element and nodal displacements.


M1
M2
le
θ1
θ2
w1
w2

SOLUTION

HERMITE SHAPE FUNCTION



$$w(\xi) = H_1 w_1 + H_2 \theta_1 + H_3 w_2 + H_4 \theta_2$$

$$= H_1 w_1 + H_2 \left(\frac{dw}{d\xi}\right)_1 + H_3 w_2 + H_4 \left(\frac{dw}{d\xi}\right)_2$$

Note: Alternative way

HERMITE SHAPE FUNCTION

$$H_1 = \frac{1}{4}\left(2 - 3\xi + \xi^3\right)$$

Note: Alternative way

$$H_2 = \frac{1}{4}\left(1 - \xi - \xi^2 + \xi^3\right)$$

$$H_3 = \frac{1}{4}\left(2 + 3\xi - \xi^3\right)$$

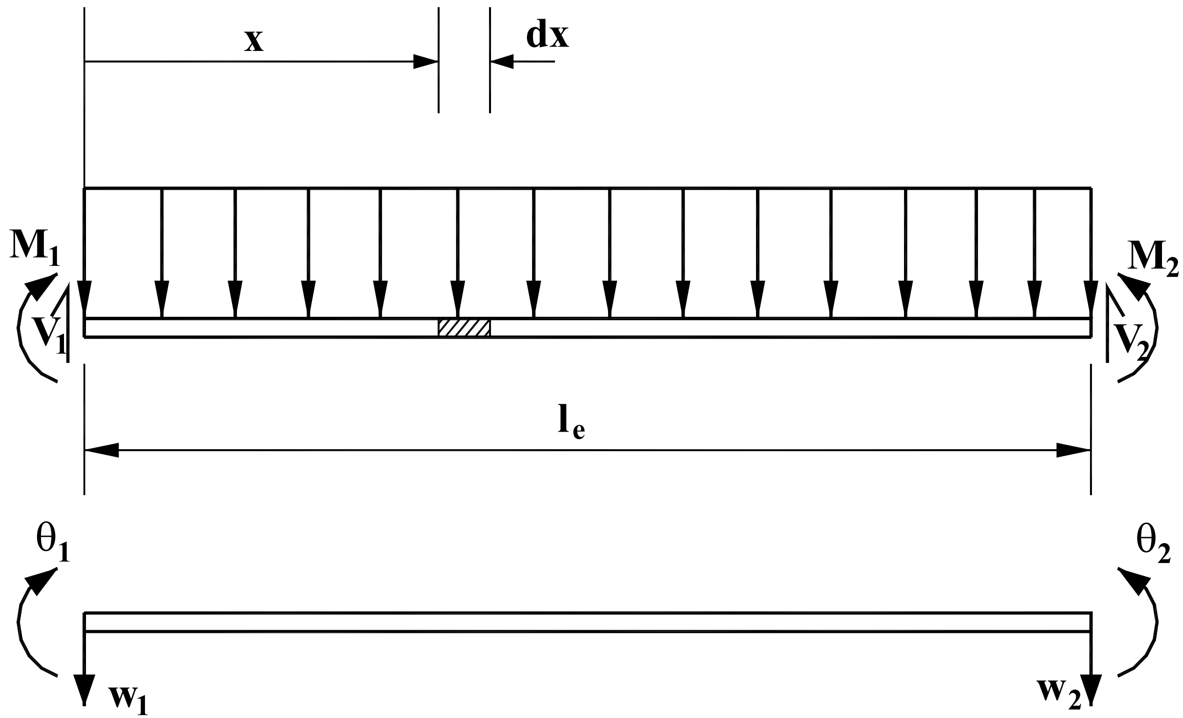
$$H_4 = \frac{1}{4}\left(-1 - \xi + \xi^2 + \xi^3\right)$$

COORDINATE TRANSFORMATION

$$x = \frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2}\xi$$

gives $\quad dx = \frac{l_e}{2} d\xi$

CUBIC SHAPE FUNCTION



$$w(x) = N_1 w_1 + N_2 \theta_1 + N_3 w_2 + N_4 \theta_2 = \boldsymbol{Nw}$$

$$= N_1 w_1 + N_2 \left(\frac{dw}{dx}\right)_1 + N_3 w_2 + N_4 \left(\frac{dw}{dx}\right)_2$$

CUBIC SHAPE FUNCTION

$$N_1 = \frac{1}{l_e^3}\left(l_e^3 - 3l_e x^2 + 2x^3\right)$$

$$N_2 = \frac{1}{l_e^2}\left(l_e^2 x - 2l_e x^2 + x^3\right)$$

$$N_3 = \frac{1}{l_e^3}\left(3l_e x^2 - 2x^3\right)$$

$$N_4 = \frac{1}{l_e^2}\left(x^3 - l_e x^2\right)$$

$$dU = \frac{1}{2}\frac{M^2}{EI}dx$$

$$\boldsymbol{D} = EI$$

but $$M = EI\frac{d^2w}{dx^2} = \boldsymbol{DA}w$$

$$\boldsymbol{A} = \frac{d^2}{dx^2}$$

$$dU = \frac{1}{2}\frac{M}{EI}Mdx$$

$$= \frac{1}{2EI}\left(EI\frac{d^2w}{dx^2}\right)Mdx$$

$$= \frac{1}{2}\left(\frac{d^2w}{dx^2}\right)Mdx$$

$$= \frac{1}{2}\left(\boldsymbol{A}w\right)^T Mdx$$

$$w(x) = \boldsymbol{Nw}$$

$$= \frac{1}{2}\left(\boldsymbol{ANw}\right)^T Mdx$$

$$dU = \frac{1}{2}\left(\boldsymbol{ANw}\right)^T \boldsymbol{DA}wdx$$

$$= \frac{1}{2}\left(\boldsymbol{ANw}\right)^T \boldsymbol{DANw}dx$$

$$= \frac{1}{2}\boldsymbol{w}^T\left(\boldsymbol{AN}\right)^T \boldsymbol{DANw}dx$$

The total strain energy of the element is thus

$$U = \frac{1}{2}\int_0^{l_e} \boldsymbol{w}^T \left(\boldsymbol{AN}\right)^T \boldsymbol{DANw}dx$$

or

$$U = \frac{1}{2}\int_0^{l_e} \boldsymbol{w}^T \boldsymbol{B}^T \boldsymbol{DBw}dx \qquad \text{where} \quad \boldsymbol{B} = \boldsymbol{AN}$$

$$= \frac{1}{2}\boldsymbol{w}^T \left(\int_0^{l_e} \boldsymbol{B}^T \boldsymbol{DB}dx\right)\boldsymbol{w}$$

EXTERNAL WORK

External work (element dx), $dW = qwdx$

Total external work over element, $W = \int_0^{l_e} qw^T dx$

$$= \int_0^{l_e} q\left(\boldsymbol{N}\boldsymbol{w}\right)^T dx$$

$$= \int_0^{l_e} q\boldsymbol{w}^T \boldsymbol{N}^T dx$$

$$= q\boldsymbol{w}^T \int_0^{l_e} \boldsymbol{N}^T dx$$

$$= \boldsymbol{w}^T q \int_0^{l_e} \boldsymbol{N}^T dx$$

POTENTIAL ENERGY APPROACH

$$\Pi = U - W$$