

Potential Energy Approach for Beam

$$dU = \frac{1}{2} \frac{M^2}{EI} dx$$

$$\text{But } \frac{d^2 w}{dx^2} = \frac{M}{EI}$$

$$U = \int dU$$

$$= \int_0^L \frac{1}{2} \frac{M^2}{EI} dx$$

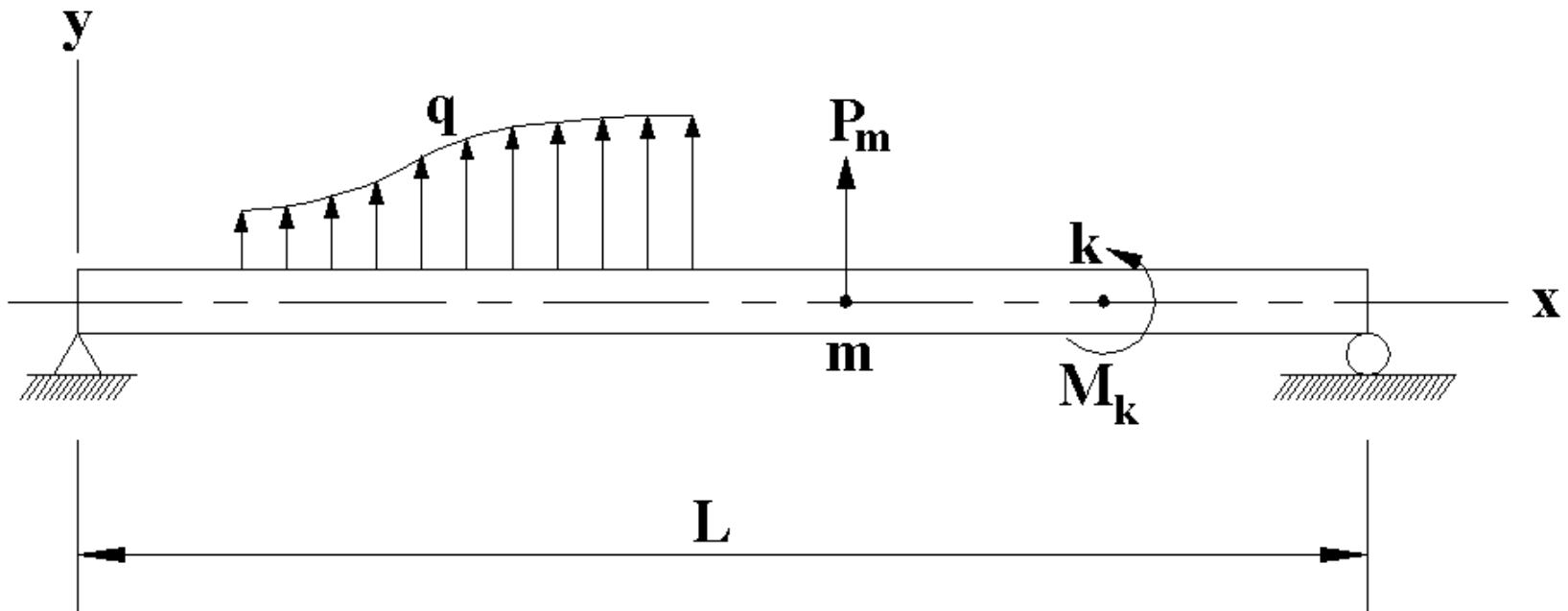
$$= \int_0^L \frac{1}{2EI} M^2 dx$$

$$= \int_0^L \frac{1}{2EI} \left(EI \frac{d^2 w}{dx^2} \right)^2 dx = \frac{1}{2} \int_0^L EI \left(\frac{d^2 w}{dx^2} \right)^2 dx$$

The potential energy of the beam is then given by

$$\Pi = \frac{1}{2} \int_0^L EI \left(\frac{d^2 w}{dx^2} \right)^2 dx - \int_0^L q w dx - \sum_m P_m w_m - \sum_k M_k w_k'$$

Where q is the distributed load per unit length, P_m is the point load at point m , M_k is the moment of the couple applied at point k , w_m is the deflection at point m , w_k' is the slope at point k .



Element Stiffness Matrix

Hermite shape functions:

$$\begin{aligned}w(\xi) &= H_1 w_1 + H_2 \theta_1 + H_3 w_2 + H_4 \theta_2 \\ &= H_1 w_1 + H_2 \left(\frac{dw}{d\xi} \right)_1 + H_3 w_2 + H_4 \left(\frac{dw}{d\xi} \right)_2\end{aligned}$$

$$H_1 = \frac{1}{4} (2 - 3\xi + \xi^3)$$

$$H_2 = \frac{1}{4} (1 - \xi - \xi^2 + \xi^3)$$

$$H_3 = \frac{1}{4} (2 + 3\xi - \xi^3) \quad (E)$$

$$H_4 = \frac{1}{4} (-1 - \xi + \xi^2 + \xi^3)$$

Coordinates transform:

$$\begin{aligned}x &= \frac{1-\xi}{2}x_1 + \frac{1+\xi}{2}x_2 \\ &= \frac{x_1+x_2}{2} + \frac{x_2-x_1}{2}\xi\end{aligned}$$

$$dx = \frac{x_2-x_1}{2}d\xi = \frac{l_e}{2}d\xi$$

Applying the chain rule:

$$\begin{aligned}\frac{dw}{d\xi} &= \frac{dw}{dx} \frac{dx}{d\xi} \\ &= \frac{l_e}{2} \frac{dw}{dx}\end{aligned}\quad (A)$$

$$\text{Then } w(\xi) = H_1u_1 + \frac{l_e}{2}H_2u_2 + H_3u_3 + \frac{l_e}{2}H_4u_4$$

which may be denoted as $w = \mathbf{H}\mathbf{u}$

$$\text{where } \mathbf{H} = \begin{bmatrix} H_1 & \frac{l_e}{2}H_2 & H_3 & \frac{l_e}{2}H_4 \end{bmatrix} \quad (F)$$

$$U_e = \frac{1}{2}EI \int_e \left(\frac{d^2 w}{dx^2} \right)^2 dx \quad (B)$$

From Eq. (A),

$$\frac{dw}{dx} = \frac{2}{l_e} \frac{dw}{d\xi} \quad \text{and} \quad \frac{d^2 w}{dx^2} = \frac{4}{l_e^2} \frac{d^2 w}{d\xi^2}$$

Then, substituting $w = \mathbf{H}\mathbf{u}$, we obtain

$$\begin{aligned} \frac{d^2 w}{dx^2} &= \frac{4}{l_e^2} \frac{d^2}{d\xi^2} (\mathbf{H}\mathbf{u}) \\ &= \frac{4}{l_e^2} \left(\frac{d^2 \mathbf{H}}{d\xi^2} \right) \mathbf{u} \end{aligned}$$

$$\begin{aligned}
\left(\frac{d^2 w}{dx^2}\right)^2 &= \frac{4}{l_e^2} \left(\frac{d^2 \mathbf{H}}{d\xi^2}\right) \mathbf{u} \frac{4}{l_e^2} \left(\frac{d^2 \mathbf{H}}{d\xi^2}\right) \mathbf{u} \\
&= \frac{16}{l_e^4} \mathbf{u}^T \left(\frac{d^2 \mathbf{H}}{d\xi^2}\right)^T \left(\frac{d^2 \mathbf{H}}{d\xi^2}\right) \mathbf{u} \quad \left| \begin{array}{l} \text{Note: } \mathbf{AB}=\mathbf{B}^T\mathbf{A}^T \\ \mathbf{A}=1 \times 4 \text{ matrix, } \mathbf{B}=4 \times 1 \text{ matrix} \end{array} \right. \\
&= \mathbf{u}^T \frac{16}{l_e^4} \left(\frac{d^2 \mathbf{H}}{d\xi^2}\right)^T \left(\frac{d^2 \mathbf{H}}{d\xi^2}\right) \mathbf{u} \quad (C)
\end{aligned}$$

where

$$\left(\frac{d^2 \mathbf{H}}{d\xi^2}\right) = \begin{bmatrix} \frac{3}{2} \xi & \frac{-1+3\xi}{2} \frac{l_e}{2} & -\frac{3}{2} \xi & \frac{1+3\xi}{2} \frac{l_e}{2} \end{bmatrix} \quad (D)$$

On substituting $dx=(l_e/2)d\xi$ and Eqs. (C) and (D) in Eq. (B), we get

$$U_e = \frac{1}{2} \mathbf{u}^T \frac{8EI}{l_e^3} \int_{-1}^{+1} \begin{bmatrix} \frac{9}{4} \xi^2 & \frac{3}{8} \xi (-1+3\xi) l_e & -\frac{9}{4} \xi^2 & \frac{3}{8} \xi (1+3\xi) l_e \\ \left(\frac{-1+3\xi}{4} \right)^2 l_e^2 & -\frac{3}{8} \xi (-1+3\xi) l_e & \frac{-1+9\xi^2}{16} l_e^2 & \\ \text{Symmetric} & \frac{9}{4} \xi^2 & -\frac{3}{8} \xi (1+3\xi) l_e & \\ & & \left(\frac{1+3\xi}{4} \right)^2 l_e^2 & \end{bmatrix} d\xi \mathbf{u}$$

Each term in the matrix needs to be integrated. Note that

$$\int_{-1}^{+1} \xi^2 d\xi = \frac{2}{3} \quad \int_{-1}^{+1} \xi d\xi = 0 \quad \int_{-1}^{+1} d\xi = 2$$

This results in the element strain energy given by

$$U_e = \frac{1}{2} \mathbf{u}^T \mathbf{k}^e \mathbf{u}$$

where the element stiffness matrix is

$$\mathbf{k}^e = \frac{EI}{l_e^3} \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

which is symmetric.

Load Vector

We assume that the distributed load is uniform over the element:

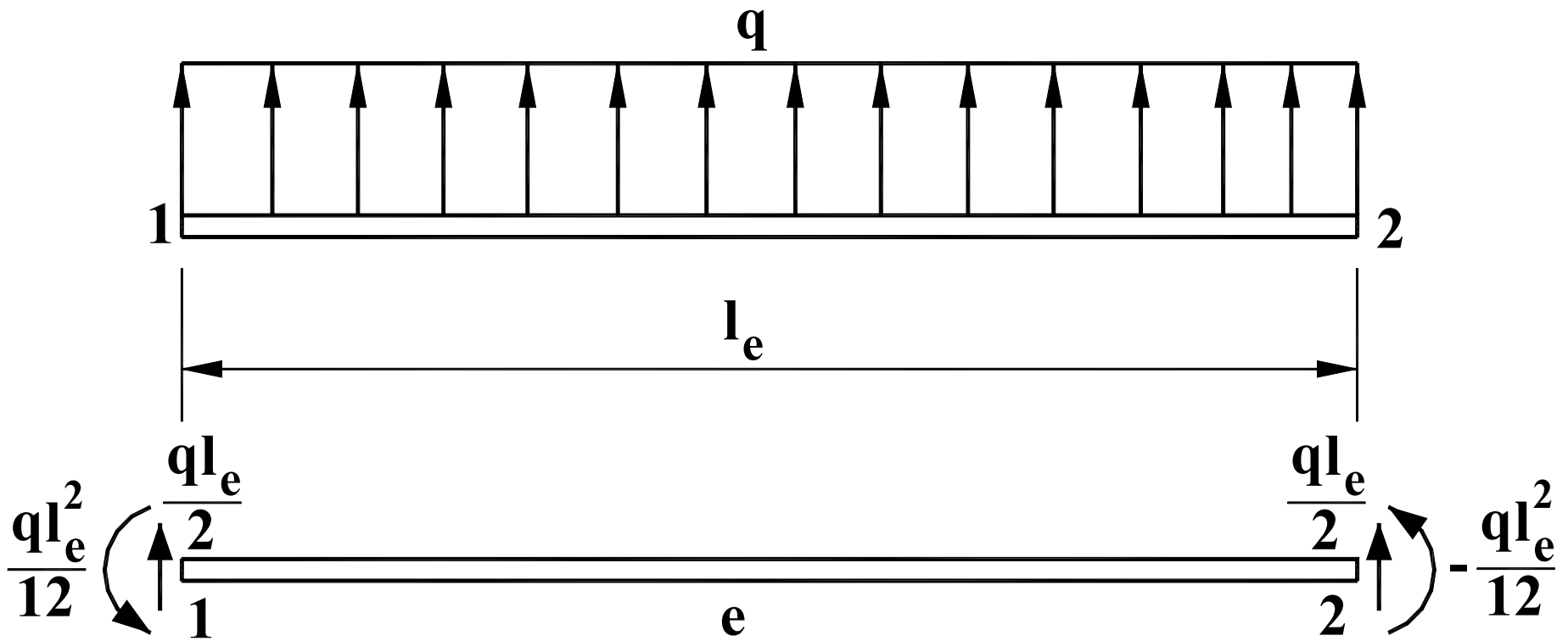
$$\begin{aligned}\int_{l_e} q w dx &= \int_{l_e} q (\mathbf{H} \mathbf{u}) dx \\ &= \left(q \int_{l_e} \mathbf{H} dx \right) \mathbf{u} \\ &= \left(q \int_{l_e} \mathbf{H} \frac{l_e}{2} d\xi \right) \mathbf{u} \\ &= \left(\frac{q l_e}{2} \int_{-1}^{+1} \mathbf{H} d\xi \right) \mathbf{u}\end{aligned}$$

On substituting for \mathbf{H} from Eqs. (E) and (F) and integrating, we obtain

$$\int_{l_e} q w dx = (\mathbf{f}^e)^T \mathbf{u}$$

where

$$\mathbf{f}^e = \left\{ \begin{array}{c} \frac{ql_e}{2} \\ \frac{ql_e^2}{12} \\ \frac{ql_e}{2} \\ -\frac{ql_e^2}{12} \end{array} \right\}$$



Distributed load on an element