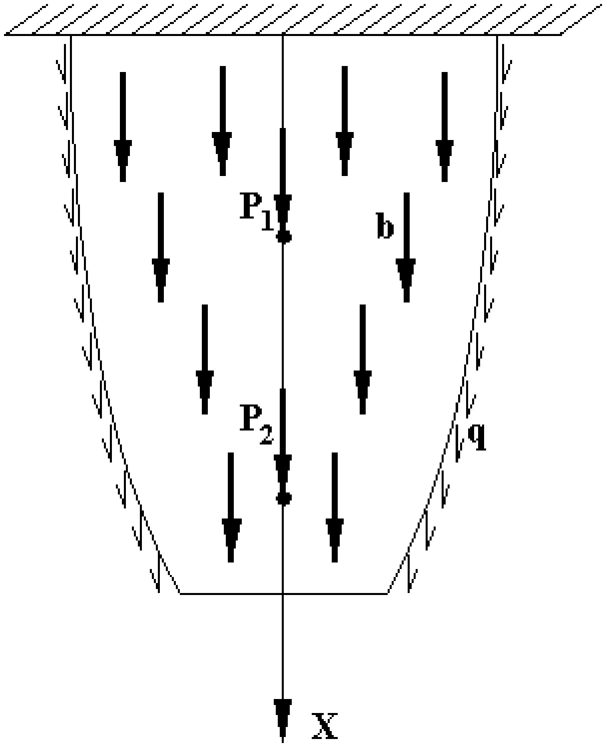
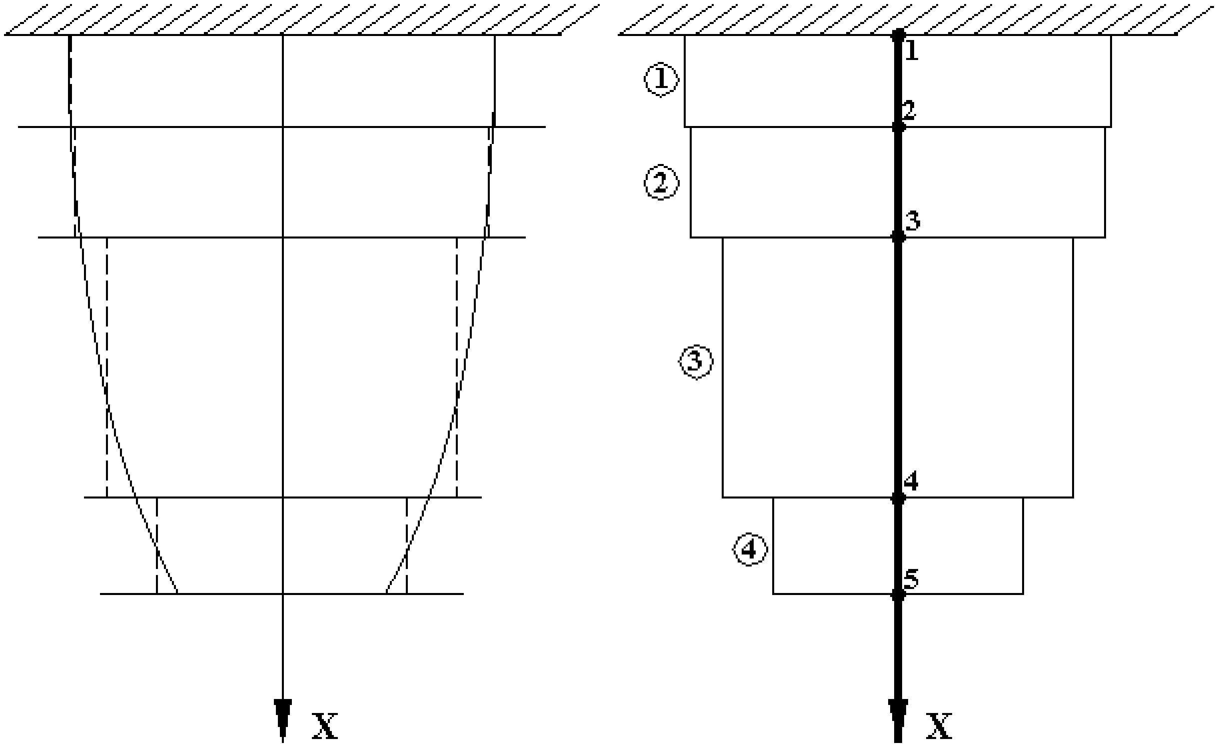
Element Stiffness Matrix for 1-D structure (bar structure) by Potential-Energy Approach



One-dimensional bar loaded by traction, body and point loads



Finite element modeling of a bar

$$\Pi = \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma}^T \boldsymbol{\varepsilon} dV - \int_{\Omega} \boldsymbol{d}^T \boldsymbol{b} dV - \int_{\Gamma} \boldsymbol{d}^T \boldsymbol{q} dS$$

For 1-D structure:

$$\Pi = \frac{1}{2}\int_{L} \sigma^{\mathrm{T}} \varepsilon A dx - \int_{L} d^T b A dx - \int_{L} d^T q dx - \sum_{i} u_i P_i$$

P_i represents a point load acting at point i, u_i is the x displacement at that point.

Since the continuum has been discretised into finite elements, thus

$$\Pi = \sum_{e} \frac{1}{2}\int_{e} \sigma^{\mathrm{T}} \varepsilon A dx - \sum_{e}\int_{e} d^T b A dx - \sum_{e}\int_{e} d^T q dx - \sum_{e} Q_i P_i$$

$$\Pi = \sum_{e} U_e - \sum_{e}\int_{e} d^T b A dx - \sum_{e}\int_{e} d^T q dx - \sum_{e} Q_i P_i$$

Element Stiffness Matrix

Consider Strain Energy term $U_e = \frac{1}{2}\int_e \sigma^{\mathrm{T}} \varepsilon A dx$

$\varepsilon = \mathbf{Bu}$

nodal displacement

$\sigma = \mathbf{D}\varepsilon = \mathrm{E}\varepsilon = \mathrm{E}\mathbf{Bu}$

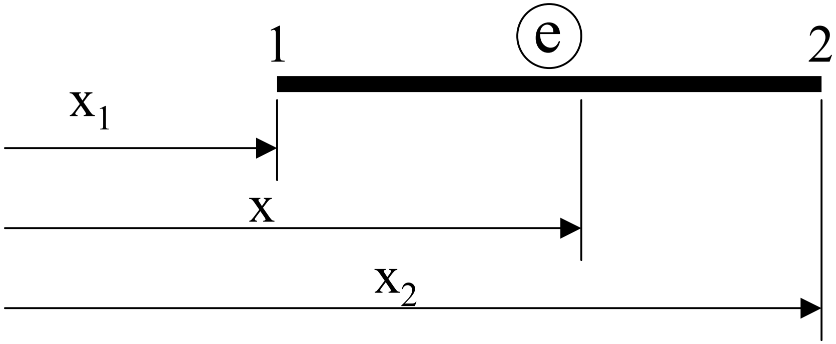
$$U_e = \frac{1}{2}\int_e \left(E\mathbf{Bu}\right)^{\mathrm{T}} \left(\mathbf{Bu}\right) A dx$$

$$= \frac{1}{2}\int_e \left(\mathbf{Bu}\right)^{\mathrm{T}} E^T \left(\mathbf{Bu}\right) A dx$$

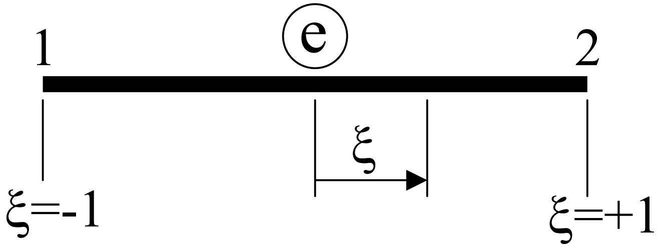
$$= \frac{1}{2}\int_e \mathbf{u}^{\mathrm{T}} \mathbf{B}^T E \left(\mathbf{Bu}\right) A dx$$

$$= \frac{1}{2}\mathbf{u}^T \left[\underbrace{\int_e \mathbf{B}^T E \mathbf{B} A dx}_{\text{Element stiffness matrix}} \right] \mathbf{u} \qquad \text{(A)}$$

Element stiffness matrix



Local Coordinate:



$$\xi = \frac{2}{x_2 - x_1}\left(x - x_1\right) - 1$$

$$d = N_1 u_1 + N_2 u_2$$

where

$$N_1\left(\xi\right) = \frac{1-\xi}{2} \qquad N_2\left(\xi\right) = \frac{1+\xi}{2}$$

Also $x = N_1 x_1 + N_2 x_2$

d=**Nu**

where

$$\mathbf{N} = \begin{bmatrix} N_1 & N_2 \end{bmatrix}$$

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Referring Eq. (A):

A=A$_e$ (Constant within an element)

$$\mathbf{B} = L\mathbf{N}$$

$$= \frac{d}{dx}\mathbf{N} = \frac{1}{\mathrm{x}_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix} \longrightarrow \text{Constant within an element}$$

Further

$$dx = \frac{x_2 - x_1}{2} d\xi = \frac{l_e}{2} d\xi$$

$$U_e = \frac{1}{2}\mathbf{u}^T \left[\int_e \mathbf{B}^T E \mathbf{B} A dx \right] \mathbf{u}$$

$$= \frac{1}{2}\mathbf{u}^T \mathbf{B}^T E_e \mathbf{B} A_e \left[\int_e dx \right] \mathbf{u}$$

$$= \frac{1}{2}\mathbf{u}^T \mathbf{B}^T E_e \mathbf{B} A_e \left[\frac{l_e}{2} \int_{-1}^{1} d\xi \right] \mathbf{u}$$

Element stiffness matrix

$$
\begin{aligned}
U_e &= \frac{1}{2}\mathbf{u}^T\left[A_e\frac{l_e}{2}E_e\mathbf{B}^T\mathbf{B}\int_{-1}^{1}d\xi\right]\mathbf{u}\\
&= \frac{1}{2}\mathbf{u}^T\left[A_e\frac{l_e}{2}E_e\frac{1}{l_\mathrm{e}}\begin{Bmatrix}-1\\ 1\end{Bmatrix}\frac{1}{l_\mathrm{e}}\begin{bmatrix}-1 & 1\end{bmatrix}(2)\right]\mathbf{u}\\
&= \frac{1}{2}\mathbf{u}^T\left[A_e l_e E_e\frac{1}{l_e^2}\begin{Bmatrix}-1\\ 1\end{Bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix}\right]\mathbf{u}\\
&= \frac{1}{2}\mathbf{u}^T\left[\frac{A_e E_e}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right]\mathbf{u} = \frac{1}{2}\mathbf{u}^T\mathbf{k}^\mathrm{e}\mathbf{u}
\end{aligned}
$$

Element stiffness matrix: $\mathbf{k}^e = \dfrac{A_e E_e}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

Force Terms – Body Force

Body force per
unit volume

$$\int_e d^T bAdx = A_e b\int_e d^T dx = A_e b\int_e \left(d\right)dx$$

$$= A_e b\int_e \left(N_1u_1 + N_2u_2\right)dx$$

$$= \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{Bmatrix} A_e b\int_e N_1 dx \\ A_e b\int_e N_2 dx \end{Bmatrix}$$

$$= \mathbf{u}^T \begin{Bmatrix} A_e b\int_e N_1 dx \\ A_e b\int_e N_2 dx \end{Bmatrix}$$

But

$$\int_e N_1 dx = \int_{-1}^{1} \left(\frac{1-\xi}{2} \right) \frac{l_e}{2} d\xi = \frac{l_e}{2} \int_{-1}^{1} \left(\frac{1-\xi}{2} \right) d\xi = \frac{l_e}{2}$$

$$\int_e N_2 dx = \int_{-1}^{1} \left(\frac{1+\xi}{2} \right) \frac{l_e}{2} d\xi = \frac{l_e}{2} \int_{-1}^{1} \left(\frac{1+\xi}{2} \right) d\xi = \frac{l_e}{2}$$

Thus

$$\int_e d^T bA dx = \mathbf{u}^T \begin{Bmatrix} A_e b \dfrac{l_e}{2} \\ A_e b \dfrac{l_e}{2} \end{Bmatrix} = \mathbf{u}^T \frac{A_e}{2} l_e b \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \mathbf{u}^T \mathbf{b}^e$$

Where $\mathbf{b}^e = \dfrac{A_e l_e b}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

Force Terms – Traction Force

$$\int_e d^T q dx = \int_e (d) q dx$$

$$= \int_e (N_1 u_1 + N_2 u_2) q dx$$

Since the traction force q is constant within the element, we have

$$\int_e d^T q dx = [u_1 \quad u_2] \begin{Bmatrix} q\int_e N_1 dx \\ q\int_e N_2 dx \end{Bmatrix}$$

$$= \mathbf{u}^T \begin{Bmatrix} q\dfrac{l_e}{2} \\ q\dfrac{l_e}{2} \end{Bmatrix} = \mathbf{u}^T \mathbf{q}^e \qquad \text{where } \mathbf{q}^e = \frac{q l_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Potential Energy of a bar element

$$\Pi_e = \frac{1}{2}\mathbf{u}^T\mathbf{k}^e\mathbf{u} - \mathbf{u}^T\mathbf{b}^e - \mathbf{u}^T\mathbf{q}^e$$

Applying Principle of Minimum Potential Energy, we get an equilibrium equation as follow

$$\mathbf{k}^e\mathbf{u} - \mathbf{b}^e - \mathbf{q}^e = \mathbf{0}$$

or

$$\mathbf{k}^e\mathbf{u} = \mathbf{b}^e + \mathbf{q}^e$$