BOUNDARY ELEMENT METHOD

by

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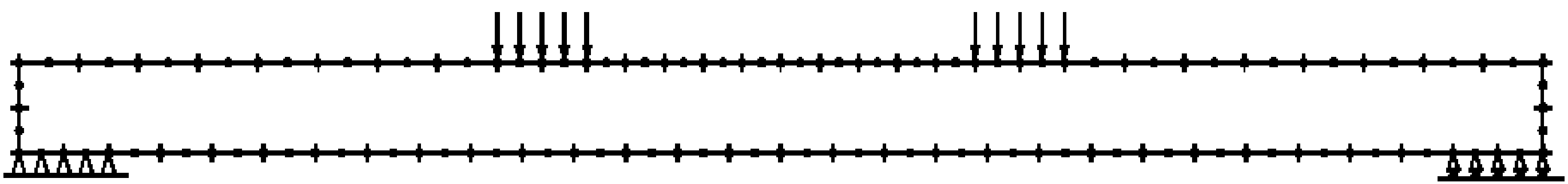
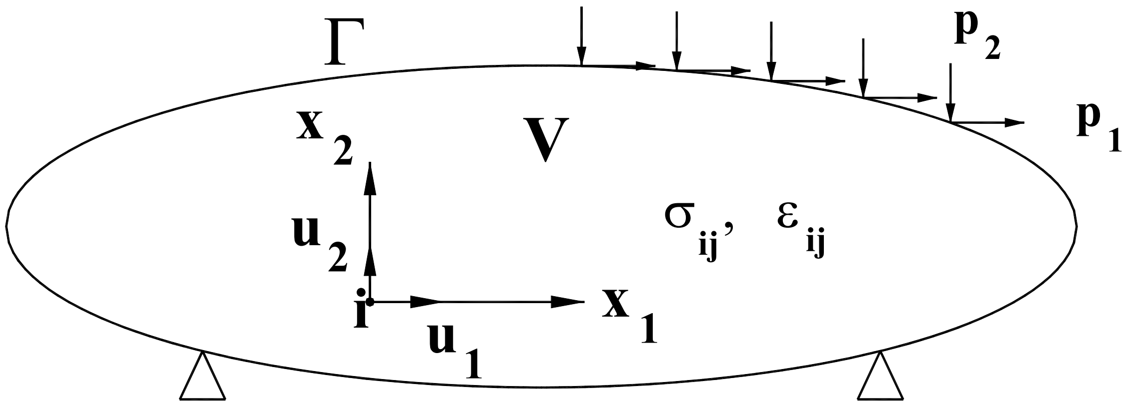
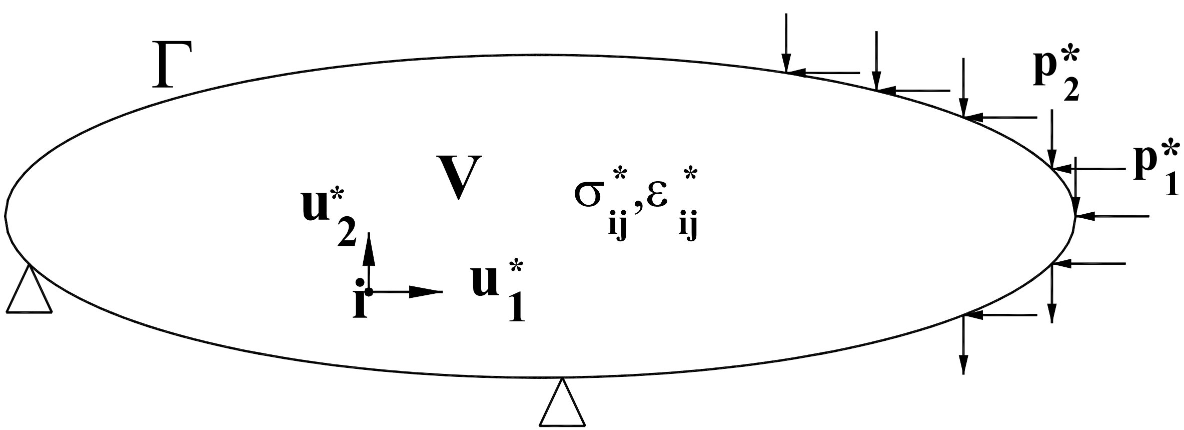


VIRTUAL WORK METHOD



(a) Actual System



(b) Virtual System

External Virtual Work = Internal Virtual Work

$$\int_S p_i u_i^* dS + \int_V b_i u_i^* dV = \int_V \sigma_{ij} \varepsilon_{ij}^* dV$$

VIRTUAL WORK METHOD (CONTINUED)

- The material properties for actual and virtual systems should equal
- Actual and Virtual systems should be in equilibrium state
- The geometrical region for actual and virtual systems should equal

BETTI RECIPROCAL THEOREM

$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ (B1) where C_{ijkl} = elastic constant. The i, j, k, l subscript varies from 1 to 3.

Note: $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$ (B2)

Substituting eq. B1 into virtual work (W) expression gives:

$$W = \int_V \varepsilon_{ij}^{*}\sigma_{ij}dV = \int_V \varepsilon_{ij}^{*}\left(C_{ijkl}\varepsilon_{kl}\right)dV \quad \text{(B3)}$$

Rearrange eq. B3 gives:

$$\int_V \varepsilon_{ij}^{*}\sigma_{ij}dV = \int_V (C_{ijkl}\varepsilon_{ij}^{*})\varepsilon_{kl}dV \quad \text{(B4)}$$

Applying eq. B2 into eq. B4, eq. B4 becomes:

$$\int_V \varepsilon_{ij}^{*}\sigma_{ij}dV = \int_V (C_{klij}\varepsilon_{ij}^{*})\varepsilon_{kl}dV \quad \text{(B5)}$$

BETTI RECIPROCAL THEOREM (CONTINUED)

The stress-strain relationship for virtual system:

$$\sigma^*_{kl} = C_{klij}\varepsilon^*_{ij} \qquad \text{(B6)}$$

Substituting eq. B6 into eq. B5 gives:

$$\int_V \varepsilon^*_{ij}\sigma_{ij}dV = \int_V \sigma^*_{kl}\varepsilon_{kl}dV \longleftarrow \text{Betti reciprocal theorem}$$

or

$$\int_V \sigma^*_{ij}\varepsilon_{ij}dV = \int_V \sigma_{ij}\varepsilon^*_{ij}dV \qquad (4.1)$$

Note: Betti reciprocal theorem is true if all elastic parameters for actual and virtual systems are equal

VERIFICATION OF SOMIGLIANA IDENTITY

but

$$2\varepsilon^*_{ij} = \frac{\partial(u^*_i)}{\partial x_j} + \frac{\partial(u^*_j)}{\partial x_i} \qquad (4.3)$$

Substituting eq. 4.3 into eq. 4.1 gives:

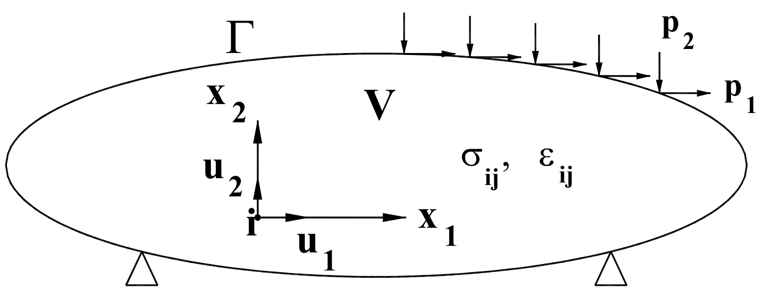
$$\int_V \sigma^*_{ij}\varepsilon_{ij}dV = \int_V \sigma_{ij}\frac{1}{2}\left(\frac{\partial(u^*_i)}{\partial x_j} + \frac{\partial(u^*_j)}{\partial x_i}\right)dV$$

$$= \frac{1}{2}\int_V\left(\sigma_{ij}u^*_{i,j} + \sigma_{ij}u^*_{j,i}\right)dV$$

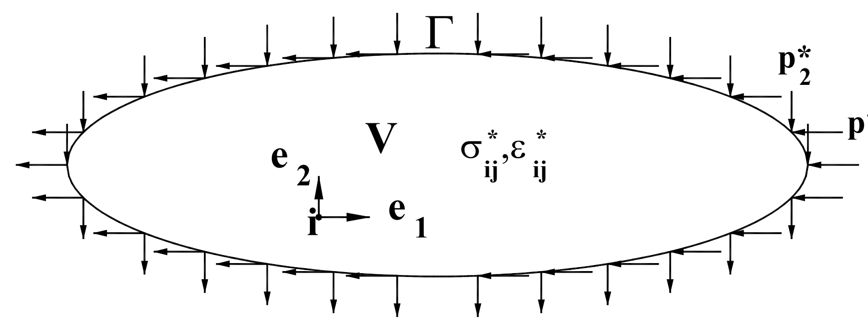
$$= \int_V \sigma_{ij}u^*_{i,j}dV$$

or $$\int_V \sigma^*_{ij}\varepsilon_{ij}dV = \int_V u^*_{i,j}\sigma_{ij}dV \qquad (4.4)$$

VERIFICATION OF SOMIGLIANA IDENTITY USING VIRTUAL WORK METHOD



(a) Actual system



(b) Virtual system

$$\int_S p_i u_i^* dS = \int_V \sigma_{ij} \varepsilon_{ij}^* dV \qquad \text{(C1)}$$

VERIFICATION OF SOMIGLIANA IDENTITY USING VIRTUAL WORK METHOD (CONTINUED)

Substituting eq. 4.3 into eq. C1 :

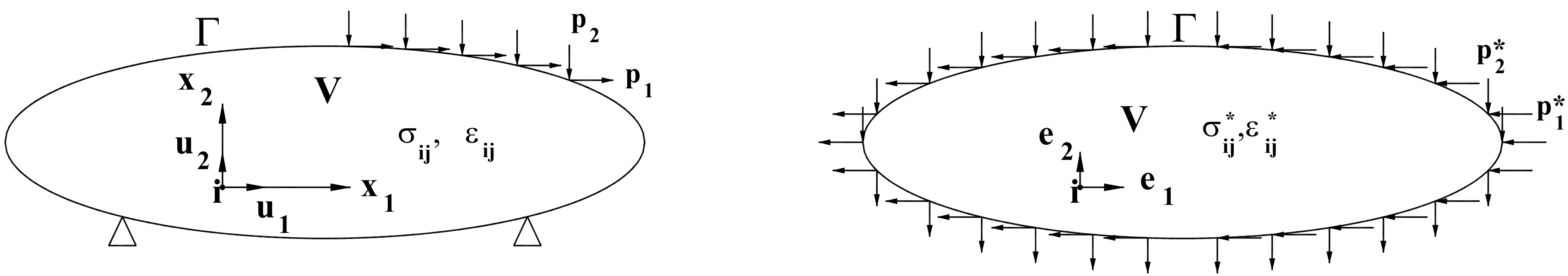
$$\int_S p_i u_i^* dS = \int_V \sigma_{ij} \frac{1}{2}\left(\frac{\partial (u_i^*)}{\partial x_j} + \frac{\partial (u_j^*)}{\partial x_i} \right) dV$$

$$= \frac{1}{2}\int_V \left(\sigma_{ij} u^*_{i,j} + \sigma_{ij} u^*_{j,i} \right) dV$$

$$= \int_V \sigma_{ij} u^*_{i,j} dV$$

$$\int_S p_i u_i^* dS = \int_V \sigma_{ij} u^*_{i,j} dV \qquad \text{(C2)}$$

VERIFICATION OF SOMIGLIANA IDENTITY USING VIRTUAL WORK METHOD (CONTINUED)



(a) Actual system **(b) Virtual system**

Virtual work method (Reverse observation):

$$\int_S p_i^* u_i dS + e_k u_k^i = \int_V \sigma_{ij}^* \varepsilon_{ij} dV \qquad \text{(C3)}$$

VERIFICATION OF SOMIGLIANA IDENTITY USING VIRTUAL WORK METHOD (CONTINUED)

Substituting eq. 4.4 into eq. C3 gives:

$$\int_S p_i^* u_i dS + e_k u_k^i = \int_V u_{i,j}^* \sigma_{ij} dV \qquad \text{(C4)}$$

Substituting eq. C2 into eq. C4:

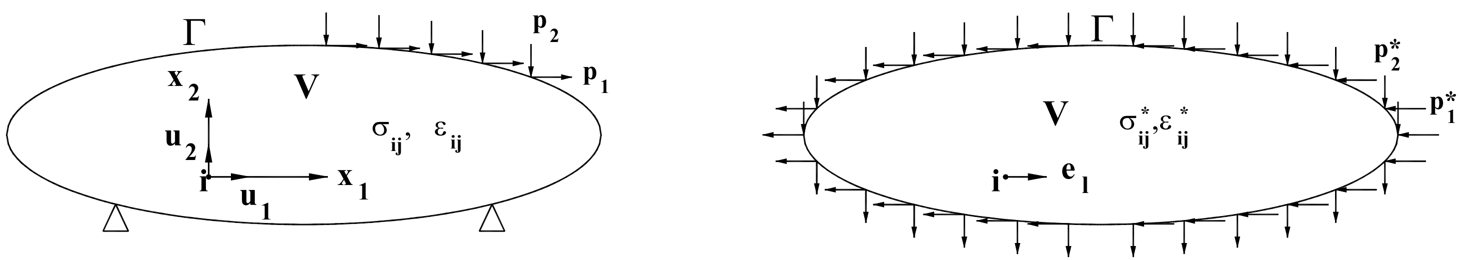
$$\int_S p_i^* u_i dS + e_k u_k^i = \int_S p_i u_i^* dS$$

or

$$e_k u_k^i = \int_\Gamma u_k^* p_k d\Gamma - \int_\Gamma p_k^* u_k d\Gamma \qquad (4.5)$$

SOMIGLIANA IDENTITY

By considering one unit load e_l^i instead of two unit loads acting at point 'i', equation 4.5 becomes:



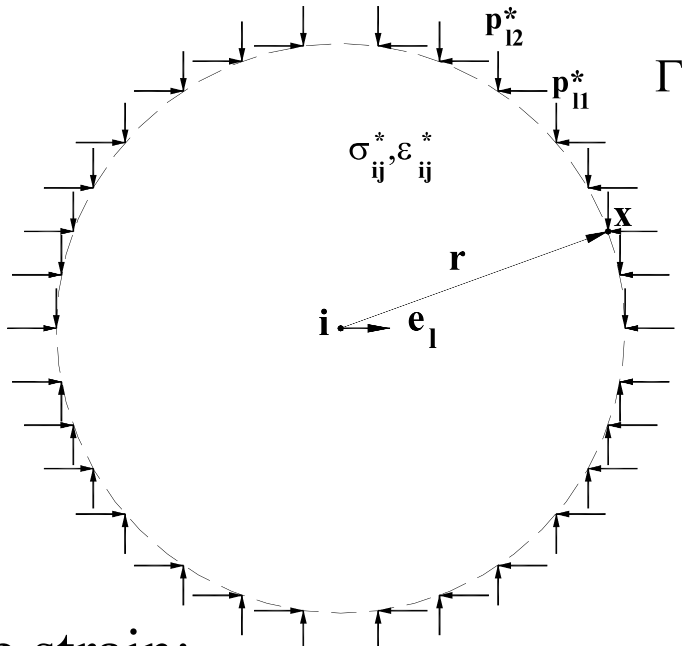
(a) Actual system

(b) Virtual system

Somigliana Identity

$$u_l^i = \int_\Gamma u_{lk}^* p_k \, d\Gamma - \int_\Gamma p_{lk}^* u_k \, d\Gamma \qquad (4.6)$$

VIRTUAL SOLUTION FOR ISOTROPIC MATERIAL (2D)



$\Gamma \to \infty$

$V \to \infty$

$$r = \left(r_j r_j\right)^{1/2}$$

$$r_j = x_j(x) - x_j(i)$$

$$r_{,j} = \frac{\partial r}{\partial x_j(x)} = \frac{r_j}{r}$$

i=load point
x=field point

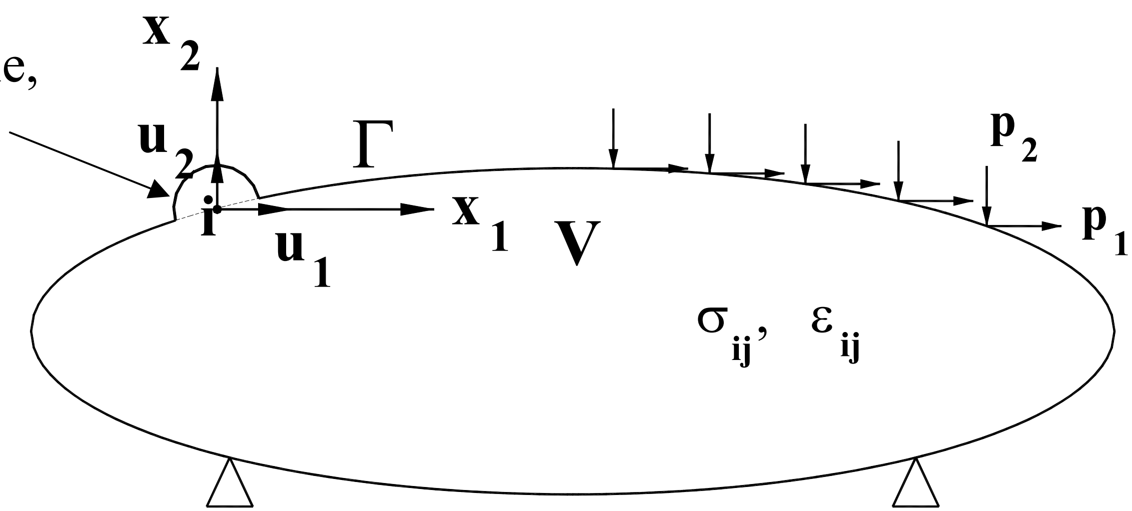
Plane strain:

$$u^*_{lk}(i,x) = \frac{-1}{8\pi(1-\nu)G}\left\{(3-4\nu)\ln(r)\delta_{lk} - r_{,l}r_{,k}\right\}$$

$$p^*_{lk}(i,x) = \frac{-1}{4\pi(1-\nu)r}\left\{\left[(1-2\nu)\delta_{lk} + 2r_{,l}r_{,k}\right]\frac{\partial r}{\partial n} - (1-2\nu)\left(r_{,l}n_k - r_{,k}n_l\right)\right\}$$

Plane stress: $\nu \to \dfrac{\nu}{(1+\nu)}$

BOUNDARY ELEMENT FORMULATION



$$u_l^i = \int_\Gamma u_{lk}^* p_k \, d\Gamma - \int_\Gamma p_{lk}^* u_k \, d\Gamma \qquad (4.6)$$

By using Theory of Limit:

$$c_{lk}^i u_k^i + \int_\Gamma p_{lk}^* u_k \, d\Gamma = \int_\Gamma u_{lk}^* p_k \, d\Gamma$$

(Boundary element formulation)

Refer Brebbia, C.A. dan Dominguez, J. (1990). "Boundary Elements: An Introductory Course." Computational Mechanics Publications.

BOUNDARY ELEMENT FORMULATION (CONTINUED)

$$c^i_{lk} u^i_k + \int_\Gamma p^*_{lk} u_k \, d\Gamma = \int_\Gamma u^*_{lk} p_k \, d\Gamma$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^i \begin{Bmatrix} u^i_1 \\ u^i_2 \end{Bmatrix} + \int_\Gamma \begin{bmatrix} p^*_{11} & p^*_{12} \\ p^*_{21} & p^*_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} d\Gamma = \int_\Gamma \begin{bmatrix} u^*_{11} & u^*_{12} \\ u^*_{21} & u^*_{22} \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} d\Gamma$$

$$\mathbf{C}^i \mathbf{u}^i + \int_\Gamma \mathbf{p}^* \mathbf{u} \, d\Gamma = \int_\Gamma \mathbf{u}^* \mathbf{p} \, d\Gamma \longleftarrow \text{(Boundary element formulation)}$$

where:

$$\mathbf{C}^i = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^i; \ \mathbf{p}^* = \begin{bmatrix} p^*_{11} & p^*_{12} \\ p^*_{21} & p^*_{22} \end{bmatrix}; \ \mathbf{u}^* = \begin{bmatrix} u^*_{11} & u^*_{12} \\ u^*_{21} & u^*_{22} \end{bmatrix}$$

$$\mathbf{u}^i = \begin{Bmatrix} u^i_1 \\ u^i_2 \end{Bmatrix}; \ \mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}; \ \mathbf{p} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$