An Axiomatization for State-Dependent Preferences and Subjective Expected Utility

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Preliminary and Incomplete - Do Not Quote

1 The Model

1.1 States

There is a (non-empty) set of states of the world Ω.

1.2 Simple Probabilities

Let $Q := \{q : \Omega \to \mathbf{R}_+ | \sum_{\omega \in \Omega} q(\omega) = 1\}$. For every $q \in Q$, let $C(q) := \{\omega \in \Omega | q(\omega) > 0\}$. The set of simple probabilities on Ω is defined by $\Delta(\Omega) := \{q \in Q | \exists r \in \mathbf{R} : \#[C(q)] \leq r\}$. A simple probability $q \in \Delta(\Omega)$ assigns a positive probability to a finite subset of states in Ω. For any state ω, the probability $i \in \Delta(\Omega)$ for which $i(\omega) = 1$ is denoted i_ω.

1.3 Consequences

For every state $\omega \in \Omega$ there is an associated (non-empty) set of consequences $Y(\omega)$.

1.4 Acts

The set of acts is defined as the product space $Y := \prod_{\omega \in \Omega} Y(\omega)$. Each act hence associates with every state $\omega \in \Omega$ a consequence $y(\omega) \in Y(\Omega)$.

1.5 Hypothetical Lotteries

The set of hypothetical lotteries is defined as $\Delta(\Omega) \times Y$. Each hypothetical lottery $(q, y) \in \Delta(\Omega) \times Y$ hence consists of a simple probability q on the state space and an act y. The intended interpretation of hypothetical lottery (q, y) is that it realizes consequence $y(\omega) \in Y(\omega)$ with probability $q(\omega)$. Two hypothetical lotteries (q, y_1) and (q, y_2) that a) assign identical probabilities to the states ; and b) assigns the same consequences to all states with positive probabilities $[y_1(C(q)) = y_2(C(q))]$ are to be regarded as equivalent.

1.6 Introspective Preferences over Hypothetical Lotteries

There is a (binary) preference relation $\succeq$ on the set of Hypothetical lotteries $\Delta(\Omega) \times Y$.

1.7 Preferences over Acts

There is a (binary) preference relation $\succeq^*$ on the set of acts Y.

1.8 Discussion of the Various Elements of the Model

A decision problem in the presence of risk or uncertainty is fundamentally different from a decision problem when neither is present. When neither risk nor uncertainty is present, a decision maker knows what the consequence of each of his available acts will be. Risk and uncertainty both entail a lack of knowledge on the part of the decision maker. This lack of knowledge is modelled here by introducing a set of states of the world. A state of the world has the property that if the decision maker knows the true state of the world, he also knows, for each and every act available to him, the consequence that would result. The set of states of the world is to be understood to include all combinations of consequences associated with the available acts that the decision maker could conceive off. In order to avoid putting unnecessary restrictions on the consequence spaces, we have chosen to allow the consequence space to be different in different states of the world.

The distinction between decision making under risk and decision making under uncertainty is usually characterized in the literature as a distinction between decision making in the presence of known objective probabilities and decision making when there is no known objective probability. In the present authors view, all real decision making really falls into the second category. That is, there is really no such thing as an objective probability. We will, however, argue that decision making under uncertainty is intimately tied to what decision making would look like if objective probabilities existed. This is the reason for the presence of two different preference relations in the present model. The first preference relation, $\succeq$, represents preferences over objective lotteries, the second preferences relation $\succeq^*$ represents preferences over acts. In the next couple of sections, we state axioms for the preference relation over objective lotteries, axioms for the preference relation over acts, and two axioms that apply to the two preference relations jointly.

Another important distinction between this paper and the literatures standard approach is that the space of lotteries includes only lotteries that assigns a positive probability to no more than one consequence in each state of the world. As a result of this, the space of lotteries is, in general, a non-convex space. If the standard approach in the literature would have been followed, we would have needed to take, as the space of lotteries, the set of all simple probabilities defined on $\cup_{\omega\in\Omega}(\{\omega\} \cup Y(\omega))$. In our view, the approach taken here is more natural.

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2 Rational Decision Making in the Presence of Risk

2.1 Axioms for Rational Decision Making in the Presence of Risk

Axiom (A2: Completeness). *For every set of elements (q, y) and (q', y') in $\Delta(\Omega) \times Y$, it follows that either $[(q, y) \succeq (q', y')]$ or $[(q', y') \succeq (q, y)]$.*

This axiom states that rational decision makers should be able to decide if (q, y) is preferred to (q', y') or (q', y') is preferred to (q, y).

Axiom (A3: Transitivity). *For every set of elements (q, y), (q', y'), and (q'', y'') in $\Delta(\Omega) \times Y$, $[(q, y) \succeq (q', y'), (q', y') \succeq (q'', y'')]$ implies $[(q, y) \succeq (q'', y'')]$.*

Axiom (A1: Reflexivity). *For every $q \in \Delta(\Omega)$, and every set of acts y, y' in Y such that $[y(C(q)) = y'(C(q))]$ it follows that $[(q, y) \sim (q, y')]$.*

This axioms states that rational decision makers should be indifferent between equivalent lotteries. If $\#[\Omega] = 1$ or $\#[Y] = 1$, then this axiom is implied by the first axiom.

Axiom (A4: Archimedean Axiom). *Let $y_1, y_2 \in Y$. If $q_1, q_2, q_3 \in \Delta(\Omega)$ such that $(q_1, y_1) \succ (q_2, y_2) \succ (q_3, y_1)$ then there exists $\alpha, \beta \in (0, 1)$ such that $(\alpha q_1 + (1-\alpha)q_3, y_1) \succ (q_2, y_2)$, and $(q_2, y_2) \succ (\beta q_1 + (1-\beta)q_3, y_1)$.*

Axiom (A5: Constrained Independence). *Let $(q_1, y_1) \sim (q_2, y_2)$. Then for all $q, q' \in \Delta(\Omega)$ and any $\alpha \in (0, 1)$, $[(q, y_1) \succeq (q', y_2)] \Leftrightarrow [(\alpha q + (1-\alpha)q_1, y_1) \succeq (\alpha q' + (1-\alpha)q_2, y_2)]$.*

Axiom (A7: Non-Degeneracy). *For every $\omega \in \Omega$, there exists $y, y' \in Y$ such that $(i_\omega, y) \succ (i_\omega, y')$.*

Axiom (A6: Closedness). *Let $\omega \in \Omega$ and $(q^*, y^*) \in \Delta(\Omega) \times Y$. Fix $y(\Omega \backslash \{\omega\}) \in \prod_{\omega' \in \Omega \backslash \{\omega\}} Y(\omega')$. Then the sets*

$$\{y(\omega) \in Y(\omega) | (i_\omega, y) \succeq (q^*, y^*)\}$$
$$\text{and}$$
$$\{y(\omega) \in Y(\omega) | (q^*, y^*) \succeq (i_\omega, y)\}$$

are both closed in $Y(\omega)$.

2.2 Representation of Preferences by an Expected Utility Function

Definition 1. A function $U : \Delta(\Omega) \times Y \to \mathbf{R}$ is a utility function representation of the preference relation $\succeq$ if for any pair of elements $(q, y), (q', y') \in \Delta(\Omega) \times Y$,

$$[(q, y) \succeq (q', y')] \Leftrightarrow [U(q, y) \geq U(q', y')]$$

Definition 2. A function $U : \Delta(\Omega) \times Y \to \mathbf{R}$ is an expected utility function representation of the preference relation $\succeq$ if (i) it is a utility function representation of $\succeq$; and (ii) there exists a function $u : \cup_{\omega \in \Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ such that for every $(q, y) \in \Delta(\Omega) \times Y$,

$$U(q, y) = \sum_{\omega \in C(q)} q(\omega) u(\omega, y(\omega))$$

Fact 1. If $\#[\Omega] = 1$, then an expected utility function representation of $\succeq$ exists whenever there exists a utility function representation of $\succeq$.

Theorem 1. *If $\#[\Omega] \geq 2$, then preference relation $\succeq$ satisfies axioms $A1 - A5$ if and only if there exists a function $u : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ such that*

$$[(q, y) \succeq (q', y')] \Leftrightarrow [\sum_{\omega\in C(q)} q(\omega)u(\omega, y(\omega)) \geq \sum_{\omega\in C(q')} q'(\omega)u(\omega, y'(\omega))].$$

The function $U : \Delta(\Omega) \times Y \to \mathbf{R}$ defined by $U(q, y) = \sum_{\omega\in C(q)} q(\omega)u(\omega, y(\omega))$ has the property that for any $(q, y), (q', y') \in \Delta(\Omega) \times Y$ for which $(q, y) \succ (q', y')$ it follows that U is onto the interval $[U(q', y'), U(q, y)]$. Moreover,

(i) if there does not exist distinct elements ω_1^*, ω_2^* of Ω and y^* in Y such that $[(q, y) \not\sim (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$ then another function $v : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ has the same properties if an only if there is a, b $(b > 0)$ such that $v = a + bu$; and

(ii) if there does exist distinct elements ω_1^*, ω_2^* of Ω and y^* in Y such that $[(q, y) \not\sim (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$ then another function $v : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ has the same properties if and only if there is $a, b_1, b_2 (b_1 > 0, b_2 > 0)$ such that:

$$v(\omega, y(\omega)) = \begin{cases} a + b_1[u(\omega, y(\omega)) - u(\omega_2^*, y^*(\omega^*))] \text{ if } (i_\omega, y) \succeq (i_{\omega_2^*}, y^*) \\ a + b_2[u(\omega, y(\omega)) - u(\omega_2^*, y^*(\omega^*))] \text{ otherwise} \end{cases}$$

Corollary 1. *If $\#[\Omega] \geq 2$, then preference relation $\succeq$ satisfies axioms $A1 - A5$ and $A7$ if and only if there exists a function $u : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$, nonconstant for every ω, such that*

$$[(q, y) \succeq (q', y')] \Leftrightarrow [\sum_{\omega\in C(q)} q(\omega)u(\omega, y(\omega)) \geq \sum_{\omega\in C(q')} q'(\omega)u(\omega, y'(\omega))].$$

The function $U : \Delta(\Omega) \times Y \to \mathbf{R}$ defined by $U(q, y) = \sum_{\omega\in C(q)} q(\omega)u(\omega, y(\omega))$ has the property that for any $(q, y), (q', y') \in \Delta(\Omega) \times Y$ for which $(q, y) \succ (q', y')$ it follows that U is onto the interval $[U(q', y'), U(q, y)]$. Moreover, another function $v : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ has the same properties if an only if there is a, b $(b > 0)$ such that $v = a + bu$.

Theorem 2. *If for every $\omega \in \Omega$, $Y(\omega)$ is a connected subset of $\mathbf{R}^{l(\omega)}$, then preference relation $\succeq$ satisfies axioms $A1 - A6$ if and only if there exists a continuous function $u : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ such that*

$$[(q, y) \succeq (q', y')] \Leftrightarrow [\sum_{\omega\in C(q)} q(\omega)u(\omega, y(\omega)) \geq \sum_{\omega\in C(q')} q'(\omega)u(\omega, y'(\omega))].$$

Moreover,

(i) if $\#\Omega \geq 2$ and there does not exist distinct elements ω_1^*, ω_2^* of Ω and y^* in Y such that $[(q, y) \not\sim (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$ then another function $v : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ has the same properties if an only if there is a, b $(b > 0)$ such that $v = a + bu$; and

(ii) if $\#\Omega \geq 2$ and there does exist distinct elements ω_1^*, ω_2^* of Ω and y^* in Y such that $[(q, y) \not\sim (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$ then another function $v : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ has the same properties if and only if there is $a, b_1, b_2 (b_1 > 0, b_2 > 0)$ such that:

$$v(\omega, y(\omega)) = \begin{cases} a + b_1[u(\omega, y(\omega)) - u(\omega_2^*, y^*(\omega^*))] \text{ if } (i_\omega, y) \succeq (i_{\omega_2^*}, y^*) \\ a + b_2[u(\omega, y(\omega)) - u(\omega_2^*, y^*(\omega^*))] \text{ otherwise} \end{cases}$$

Corollary 2. *If for every* $\omega \in \Omega$, $Y(\omega)$ *is a connected subset of* $\mathbf{R}^{l(\omega)}$, *then preference relation* $\succeq$ *satisfies axioms* $A1 - A7$ *if and only if there exists a continuous function, non-constant for every* ω, $u : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ *such that*

$$[(q, y) \succeq (q', y')] \Leftrightarrow [\sum_{\omega \in C(q)} q(\omega)u(\omega, y(\omega)) \geq \sum_{\omega \in C(q')} q'(\omega)u(\omega, y'(\omega))].$$

Moreover, if $\#[\Omega] \geq 2$ *then another function* $v : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ *has the same properties if an only if there is* a, b $(b > 0)$ *such that* $v = a + bu$.

2.3 Experiments and Rational Decision Making Under Risk

Let X be a (non-empty) set of feasible outcomes of an experiment. Let $\Omega_X := X \times \Omega$ and $Y_X := \prod_{x\in X} \prod_{\omega\in\Omega} Y(\omega)$. In an analogous manner to how $\Delta(\Omega)$ was defined in section 1, we can define $\Delta(\Omega_X)$. We will now consider a (binary) preference relation $\succeq^X$ defined on $\Delta(\Omega_X) \times Y_X$. In addition to imposing subsets of the Axioms of the previous section, it will now also be reasonable to impose the following additional axioms.

Axiom (A8: Data Independence)**.** *Let* $\omega \in \Omega$ *be given. Then for any* $x, x' \in X$, $[y_X(x, \omega) = y'_X(x', \omega)] \Rightarrow [(i_{(x,\omega)}, y_X) \sim^X (i_{(x',\omega)}, y'_X)]$

Definition 3. An act $y_X \in Y_X$ is data constant if there exists an act $y \in Y$ such that $y_X(x, .) = y$ for all $x \in X$.

For any probability $q_X \in \Delta(\Omega_X)$, denote by $q^*(.|q_X)$ the unique element of $\Delta(\Omega)$ defined by $q^*(\omega|q_X) = \sum_{x\in X} q_X(x, \omega)$.

Axiom (A9:)**.** *Let* q_X, q'_X *be elements of* $\Delta(\Omega_X)$, *and let* y, y' *be elements of* Y *with associated constant acts* $y_X, y'_X \in Y_X$. *Then* $[(q_X, y_X) \succeq^X (q'_X, y'_X)] \Leftrightarrow [(q^*(.|q_X), y) \succeq (q^*(.|q'_X), y')]$.

3 Subjective Probability and Rational Decision Making Under Uncertainty

3.1 Axioms for Rational Decision Making Under Uncertainty

The preference relation $\succeq^*$ will be connected to the preference relation $\succeq$ through two axioms.

Axiom (B1: Completeness)**.** *For every pair of elements* y *and* y' *in* Y, *it follows that either* $[y \succeq^* y']$ *or* $[y' \succeq^* y]$.

Axiom (B2: Transitivity)**.** *For every set of elements* y, y', *and* y'' *in* Y, $[y \succeq^* y', y' \succeq^* y'']$ *implies* $[y \succeq^* y'']$.

Axiom (B3: Closedness)**.** *For any* $y^* \in Y$, *the sets*

$$\{y \in Y | y \succeq^* y^*\}$$
and
$$\{y \in Y | y^* \succeq^* y\}$$

are both closed in Y.

Definition 4. Let E be a non-empty subset of Ω, and let $y(E), y'(E) \in \prod_{\omega \in E} Y(\omega)$. Then $y(E) \succeq^*_E y'(E)$ if and only if for every set of acts $y^*, y^{**} \in Y$ such that $y^*(E) = y(E)$, $y^{**}(E) = y'(E)$, and $y^*(\Omega \setminus E) = y^{**}(\Omega \setminus E)$, it follows that $y^* \succeq^* y^{**}$.

Axiom (B4: Separability)**.** *For every non-empty subset E of Ω and every $y(E), y'(E) \in \prod_{\omega \in E} Y(\omega)$, either $y(E) \succeq^*_E y'(E)$ or $y'(E) \succeq^*_E y(E)$.*

Definition 5. An event $E \subset \Omega$ is null if there does not exist $y(E), y'(E) \in \prod_{\omega \in E} Y(\omega)$ such that $y(E) \succ^*_E y'(E)$.

Axiom (B5:)**.** *For every state ω for which $\{\omega\}$ is non-null, and any $y, y' \in Y$,*

$$[y(\omega) \succeq^*_{\{\omega\}} y'(\omega)] \Leftrightarrow [(i_\omega, y) \succeq (i_\omega, y')]$$

Axiom (B6: Non-Degeneracy)**.** *There exists $y, y' \in Y$ such that $y \succ^* y'$.*

Definition 6. Let E be a subset of Ω and let $q \in \Delta(\Omega)$ satisfy the property that $\sum_{\omega \in E} q(\omega) > 0$. Then $(q|E)$ is defined as the element of $\Delta(\Omega)$ for which:

$$(q|E)(\omega) = \begin{cases} \frac{q(\omega)}{\sum_{\omega' \in E} q(\omega')} & \text{if } \omega \in E \\ 0 & \text{otherwise} \end{cases}$$

Definition 7. Let E be a subset of Ω and let $q \in \Delta(\Omega)$ satisfy the property that $\sum_{\omega \in E} q(\omega) > 0$. Then for any $y, y' \in Y$,

$$[y(E) \succeq_{(q|E)} y'(E)] \Leftrightarrow [((q|E), y) \succeq ((q|E), y')].$$

Axiom (B7:)**.** *Let E, E' be two disjoint nonempty subset of Ω such that:*

(i) $[y^*(E) \succeq^*_E y^{**}(E)] \Leftrightarrow [y^*(E) \succeq_{(q|E)} y^{**}(E)]$*; and*

(ii) $[y^*(E') \succeq^*_{E'} y^{**}(E')] \Leftrightarrow [y^*(E) \succeq_{(q|E')} y^{**}(E')]$.
Then (iii) below implies (iv).

(iii) There exists $y, y' \in Y$ such that
a) $y(E) \succ^*_E y'(E)$*;*
b) $y'(E') \succ^*_{E'} y(E')$*;*
c) $y(E \cup E') \sim^*_{E \cup E'} y'(E \cup E')$*; and*
d) $y(E \cup E') \sim_{(q|E \cup E')} y'(E \cup E')$.

(iv) For every $y, y' \in Y$,

$$[y(E \cup E') \succeq^*_{E \cup E'} y'(E \cup E')] \Leftrightarrow [y(E \cup E') \succeq_{(q|E \cup E')} y'(E \cup E')].$$

3.2 Existence of a Unique Subjective Probability and an Expected Utility Function Representation of Preferences

Theorem 3. *If Y is a connected subset of $\mathbf{R}^l$, then $\succeq$ satisfies Axioms $A1 - A7$, and $\succeq^*$ satisfies axioms $B1 - B7$ if an only if there is a probability π in $\Delta(\Omega)$ and a continuous function, non-constant for every ω, $u : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ such that:*

(i) $[(q, y) \succeq (q', y')] \Leftrightarrow [\sum_{\omega\in C(q)} q(\omega)u(\omega, y(\omega)) \geq \sum_{\omega\in C(q')} q'(\omega)u(\omega, y'(\omega))]$; *and*

(ii) $[y \succeq^* y'] \Leftrightarrow [\sum_{\omega\in C(\pi)} \pi(\omega)u(\omega, y(\omega)) \geq \sum_{\omega\in C(\pi)} \pi(\omega)u(\omega, y'(\omega))]$.

Moreover, if $\#[\Omega] \geq 2$ then π' and v is another such pair if and only if $\pi' = \pi$ and there exists a, b $(b > 0)$ such that $v = a + bu$.

3.3 Experiments, Posterior Preferences Over Acts, the Existence of Posterior Probabilities, and Bayes' Rule

Definition 8. Let X be a set of possible outcomes of an experiment. For every $x \in X$, the posterior preference relation $(\succeq^* |x)$ is a (binary) preference relation on Y.

Theorem 4. *If Y is a connected subset of $\mathbf{R}^l$, then $\succeq$ satisfies Axioms $A1 - A7$, and $(\succeq^* |x)$ satisfies axioms $B1 - B7$ for every $x \in X$ if an only if (a) for every $x \in X$ there is a probability $\pi(.|x)$ in $\Delta(\Omega)$ and (b) there is a continuous function, non-constant for every ω, $u : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ and such that:*

(i) $[(q, y) \succeq (q', y')] \Leftrightarrow [\sum_{\omega\in C(q)} q(\omega)u(\omega, y(\omega)) \geq \sum_{\omega\in C(q')} q'(\omega)u(\omega, y'(\omega))]$; *and*

(ii) $\forall x \in X : [y(\succeq^* |x)y'] \Leftrightarrow [\sum_{\omega\in C(\pi(.|x))} \pi(\omega|x)u(\omega, y(\omega)) \geq \sum_{\omega\in C(\pi(.|x))} \pi(\omega|x)u(\omega, y'(\omega))]$.

Moreover, if $\#[\Omega] \geq 2$ then v is another such function if and only if there exists a, b $(b > 0)$ such that $v = a + bu$ and for every $x \in X$, $\pi'(.|x)$ is another such probability if and only if $\pi'(.|x) = \pi(.|x)$.

4 Proofs

4.1 Theorem 3

Claim 1. *Let E be a non-empty subset of Ω and let y^* and $y(\Omega \setminus E)$ be elements of Y and $\prod_{\omega\in(\Omega\setminus E)} Y(\omega)$ respectively. Then the sets $\{y(E) \in Y(E)|y \succeq^* y^*\}$ and $\{y(E) \in Y(E)|y^* \succeq^* y\}$ are both closed sets.*

Proof of Claim 16: (i) Let $\{y_k(E)\}_{k=1}^{\infty}$ be a convergent sequence of elements of $\{y(E) \in Y(E)|y \succeq^* y^*\}$. Then $\{(y_k(E), y(\Omega \setminus E))\}_{k=1}^{\infty}$ is a convergent sequence of elements in $\{y \in Y|y \succ^* y^*\}$. From Axiom B3, it follows that $\overline{y} = \lim_{k\to\infty}(y_k(E), y(\Omega\setminus E)) \in \{y \in Y|y \succ^* y^*\}$.

But then $\overline{y}_k \in \{y(E) \in Y(E)|y \succeq^* y^*\}$ and hence $\{y(E) \in Y(E)|y \succeq^* y^*\}$ is closed as required.

(ii) Let $\{y_k(E)\}_{k=1}^{\infty}$ be a convergent sequence of elements of $\{y(E) \in Y(E)|y^* \succeq^* y\}$. Then $\{(y_k(E), y(\Omega \setminus E))\}_{k=1}^{\infty}$ is a convergent sequence of elements in $\{y \in Y|y^* \succ^* y\}$. From Axiom B3, it follows that $\overline{y} = \lim_{k\to\infty}(y_k(E), y(\Omega \setminus E)) \in \{y \in Y|y^* \succ^* y\}$. But then $\overline{y}_k \in \{y(E) \in Y(E)|y^* \succeq^* y\}$ and hence $\{y(E) \in Y(E)|y^* \succeq^* y\}$ is closed as required. Q.E.D.

Claim 2. *Let E and E' be two disjoint non-empty subsets of Ω. Let $y^*(E \cup E')$ and $y(E')$ be in $\prod_{\omega \in E \cup E'} Y(\omega)$ and $\prod_{\omega \in E'} Y(\omega)$ respectively and let $\overline{y}(E), \underline{y}(E) \in \prod_{\omega \in E} Y(E)$ satisfy $(\overline{y}(E), y(E')) \succeq^*_{E\cup E'} y^*(E \cup E') \succeq^*_{E\cup E'} (\underline{y}(E), y(E'))$. Then there exists $y(E) \in Y(E)$ such that $y(E \cup E') \sim^*_{E\cup E'} y^*(E \cup E')$.*

Proof of Claim 17:

(i) If $(\overline{y}(E), y(E')) \sim^*_{E\cup E'} (y^*(E \cup E')$, then we can set $y(E) = \overline{y}(E)$ and we are done.
(ii) If $(\underline{y}(E), y(E')) \sim^*_{E\cup E'} (y^*(E \cup E')$, then we can set $y(E) = \underline{y}(E)$ and we are done.

(iii) $(\overline{y}(E), y(E')) \succ^*_{E\cup E'} (y^*(E \cup E') \succ^*_{E\cup E'} (\underline{y}(E), y(E'))$. Suppose there does not exist $y(E) \in Y(E)$ such that $(y(E), y(E')) \sim^*_{E\cup E'} (y^*(E\cup E')$. Then $\{y(E) \in Y(E)|(y(E), y(E')) \succ^*_{E\cup E'} y^*(E\cup E')\} = \Omega \setminus \{y(E) \in Y(E)|(y^*(E\cup E') \succeq^*_{E\cup E'} (y(E), y(E'))\}$ and contains $\overline{y}(E)$. Hence it follows from claim 16 that $\{y(E) \in Y(E)|(y(E), y(E')) \succ^*_{E\cup E'} (y^*(E \cup E')\}$ is a non-empty open set. Likewise, $\{y(E) \in Y(E)|y^*(E \cup E') \succ^*_{E\cup E'} (y(E), y(E'))\} = \Omega \setminus \{y(E) \in Y(E)|(y(E), y(E')) \succeq^*_{E\cup E'} y^*(E \cup E')\}$ and contains $\underline{y}(E)$. Hence, it follows from claim 16 that $\{y(E) \in Y(E)|y^*(E \cup E') \succ^*_{E\cup E'} (y(E), y(E'))\}$ is a non-empty open set. We note further that $\{y(E) \in Y(E)|(y(E), y(E')) \succ^*_{E\cup E'} y^*(E \cup E')\} \cap \{y(E) \in Y(E)|y^*(E \cup E') \succ^*_{E\cup E'} (y(E), y(E'))\} = \emptyset$ and $\{y(E) \in Y(E)|(y(E), y(E')) \succ^*_{E\cup E'} y^*(E \cup E')\} \cup \{y(E) \in Y(E)|y^*(E \cup E') \succ^*_{E\cup E'} (y(E), y(E'))\} = Y(E)$. Hence $\{y(E) \in Y(E)|(y(E), y(E')) \succ^*_{E\cup E'} y^*(E\cup E')\}$ and $\{y(E) \in Y(E)|y^*(E\cup E') \succ^*_{E\cup E'} (y(E), y(E'))\}$ forms a separation for $Y(E)$. But Y connected implies $Y(E)$ connected, a contradiction. Q.E.D.

Claim 3. *Let E and E' be two disjoint non-empty subsets of Ω. Let $\overline{y}(E), \underline{y}(E) \in \prod_{\omega \in E} Y(\omega)$, $\overline{y}(E'), \underline{y}(E') \in \prod_{\omega \in E'} Y(\omega)$, and $q_1, q_2 \in \Delta(\Omega)$ satisfy:*

(i) $\overline{y}(E) \succ^*_E \underline{y}(E)$;

(ii) $\overline{y}(E') \succ^*_{E'} \underline{y}(E')$;

(iii) $[y(E) \succeq^*_E y'(E)] \Leftrightarrow [y(E) \succeq_{q_1|E} y'(E)]$; and

(iv) $[y(E') \succeq^*_{E'} y'(E')] \Leftrightarrow [y(E') \succeq_{q_2|E'} y'(E')]$.

Then there exists $q \in \Delta(\Omega)$ such that $[y(E \cup E') \succeq^*_{E\cup E'} y'(E \cup E')] \Leftrightarrow [y(E \cup E') \succeq_{q|E\cup E'} y'(E \cup E')]$

Proof of Claim 18: Assume without loss of generality that

$$(\overline{y}(E), \underline{y}(E')) \succeq^*_{E\cup E'} (\underline{y}(E), \overline{y}(E')) \succ^*_{E\cup E'} (\underline{y}(E), \underline{y}(E')).$$

From Claim 17, it follows that there exists $\overline{y}^*(E) \in Y(E)$ such that

$$(\overline{y}^*(E), \underline{y}(E')) \sim^*_{E\cup E'} (\underline{y}(E), \overline{y}(E')) \succ^*_{E\cup E'} (\underline{y}(E), \underline{y}(E')).$$

We note that $\overline{y}^*(E) \succ^*_E \underline{y}(E)$. Now, define q_1^* and q_2^* respectively by,

$$q_1^*(\omega) := \begin{cases} \frac{q_1(\omega)}{\sum_{\omega'} q_1(\omega')} & \text{if } \omega \in E \\ 0 & \text{otherwise} \end{cases}$$

and,

$$q_2^*(\omega) := \begin{cases} \frac{q_2(\omega)}{\sum_{\omega'} q_2(\omega')} & \text{if } \omega \in E' \\ 0 & \text{otherwise.} \end{cases}$$

Let $y(\Omega \setminus (E \cup E')) \in \prod_{\omega\in\Omega\setminus(E\cup E')} Y(\omega)$. Then we have,

$$\begin{array}{rcl} (q_1^*, \overline{y}^*(E), \underline{y}(E'), y(\Omega \setminus (E \cup E'))) & \succ & (q_1^*, \underline{y}(E), \overline{y}(E'), y(\Omega \setminus (E \cup E'))) \\ & \Downarrow & \\ \sum_{\omega\in C(q_1^*)} q_1^*(\omega)u(\omega, \overline{y}^*(\omega)) & > & \sum_{\omega\in C(q_1^*)} q_1^*(\omega)u(\omega, \underline{y}(\omega)). \end{array}$$

Likewise, we have

$$\begin{array}{rcl} (q_2^*, \underline{y}(E), \overline{y}(E'), y(\Omega \setminus (E \cup E'))) & \succ & (q_2^*, \overline{y}^*(E), \underline{y}(E'), y(\Omega \setminus (E \cup E'))) \\ & \Downarrow & \\ \sum_{\omega\in C(q_2^*)} q_2^*(\omega)u(\omega, \overline{y}(\omega)) & > & \sum_{\omega\in C(q_2^*)} q_2^*(\omega)u(\omega, \underline{y}(\omega)). \end{array}$$

Now, let

$$\alpha^* := \frac{\sum_{\omega\in C(q_2^*)} q_2^*(\omega)[u(\omega, \overline{y}(\omega)) - u(\omega, \underline{y}(\omega))]}{[\sum_{\omega\in C(q_1^*)} q_1^*(\omega)[u(\omega, \overline{y}^*(\omega)) - u(\omega, \underline{y}(\omega))]] + [\sum_{\omega\in C(q_2^*)} q_2^*(\omega)[u(\omega, \overline{y}(\omega)) - u(\omega, \underline{y}(\omega))]]}.$$

We note that $0 < \alpha^* < 1$. Let

$$\beta = [\sum_{\omega\in C(q_1^*)} q_1^*(\omega)[u(\omega, \overline{y}^*(\omega)) - u(\omega, \underline{y}(\omega))]] + [\sum_{\omega\in C(q_2^*)} q_2^*(\omega)[u(\omega, \overline{y}(\omega)) - u(\omega, \underline{y}(\omega))]].$$

Now, define $q := \alpha^* q_1^* + (1 - \alpha^*)q_2^*$ and denote by y_1^* and y_2^* respectively the elements of Y for which,

$$y_1^*(\omega) = \begin{cases} \overline{y}^*(\omega) & \text{if } \omega \in E \\ \underline{y}(\omega) & \text{if } \omega \in E' \\ y(\omega) & \text{otherwise} \end{cases}$$

and

$$y_2^*(\omega) = \begin{cases} \overline{y}(\omega) & \text{if } \omega \in E' \\ \underline{y}(\omega) & \text{if } \omega \in E \\ y(\omega) & \text{otherwise} \end{cases}$$

Then we have,

$$
\begin{aligned}
\textstyle\sum_{\omega\in C(q)} q(\omega)[u(\omega,y_1^*(\omega))-u(\omega,y_2^*(\omega))] &= \textstyle\sum_{\omega\in C(q_1^*)} \alpha^* q_1^*(\omega)[u(\omega,\overline{y}^*(\omega))-u(\omega,\underline{y}(\omega))] \\
&\quad +\textstyle\sum_{\omega\in C(q_2^*)}(1-\alpha^*)q_2^*(\omega)[u(\omega,\underline{y}(\omega))-u(\omega,\overline{y}(\omega))] \\
&= \tfrac{1}{\beta}[[\textstyle\sum_{\omega\in C(q_2^*)} q_2^*(\omega)[u(\omega,\overline{y}(\omega))-u(\omega,\underline{y}(\omega))]] \\
&\quad *[\textstyle\sum_{\omega\in C(q_1^*)} q_1^*(\omega)[u(\omega,\overline{y}^*(\omega))-u(\omega,\underline{y}(\omega))]] \\
&\quad +[\textstyle\sum_{\omega\in C(q_1^*)} q_1^*(\omega)[u(\omega,\overline{y}^*(\omega))-u(\omega,\underline{y}(\omega))]] \\
&\quad *[\textstyle\sum_{\omega\in C(q_2^*)} q_2^*(\omega)[u(\omega,\underline{y}(\omega))-u(\omega,\overline{y}(\omega))]]] \\
&= 0.
\end{aligned}
$$

Rearranging, we have

$$
\begin{array}{rcl}
\sum_{\omega\in C(q)} q(\omega)u(\omega,y_1^*(\omega)) & = & \sum_{\omega\in C(q)} q(\omega)u(\omega,y_2^*(\omega)) \\
& \Downarrow & \\
(q,y_1^*) & \sim & (q,y_2^*) \\
& \Downarrow & \\
(q,\overline{y}^*(E),\underline{y}(E'),y(\Omega\setminus(E\cup E'))) & \sim & (q,\underline{y}(E),\overline{y}(E'),y(\Omega\setminus(E\cup E'))) \\
& \Downarrow & \\
(q,\overline{y}^*(E),\underline{y}(E')) & \sim_{q|(E\cup E')} & (q,\underline{y}(E),\overline{y}(E')).
\end{array}
$$

From Axiom $B7$ it then follows that for all $y(E\cup E'), y'(E\cup E') \in \prod_{\omega\in E\cup E'} Y(\omega)$, $[y(E\cup E') \succeq^*_{E\cup E'} y'(E\cup E')] \Leftrightarrow [y(E\cup E') \succeq_{q|E\cup E'} y'(E\cup E')]$ as required.
Q.E.D.

Claim 4. *Let E, E' be any two distinct null events. Then $E\cup E'$ is null.*

Proof of Claim 19: Let $y(E), y(E') \in \prod_{\omega\in E}$, $y(E'), y'(E') \in \prod_{\omega\in E'}$, and $y(\Omega\setminus(E\cup E')) \in \prod_{\omega\in\Omega\setminus(E\cup E')} Y(\omega)$. Then we have,

$$
\begin{aligned}
(y(E),y(E'),y(\Omega\setminus(E\cup E'))) &\sim^* (y'(E),y(E'),y(\Omega\setminus(E\cup E'))) \\
&\sim^* (y'(E),y'(E'),y(\Omega\setminus(E\cup E')))
\end{aligned}
$$

But then it follows from Axiom $B4$ that $(y(E),y(E')) \sim^*_{E\cup E'} (y'(E),y'(E'))$ as required.
Q.E.D.

Claim 5. *Let E, E' be distinct events where E' is null. Then*

$$
\begin{array}{rcl}
(y(E),y(E')) & \succeq^*_{E\cup E'} & (y'(E),y'(E')) \\
& \Updownarrow & \\
y(E) & \succeq^*_E & y'(E)
\end{array}
$$

Proof of Claim 20: Let $y(\Omega\setminus(E\cup E))\prod_{\omega\in\Omega\setminus(E\cup E')} Y(\omega)$. First note that

$$
\begin{array}{rcl}
y(E) & \succeq^*_E & y'(E) \\
& \Updownarrow & \\
(y(E),y(E'),y(\Omega\setminus(E\cup E'))) & \succeq^* & (y'(E),y(E'),y(\Omega\setminus(E\cup E')))
\end{array}
$$

But $(y'(E), y'(E'), y(\Omega \setminus (E \cup E'))) \sim^* (y'(E), y'(E'), y(\Omega \setminus (E \cup E')))$. Hence it follows from completeness and transitivity that

$$\begin{array}{rcl} y(E) & \succeq^*_E & y'(E) \\ & \Updownarrow & \\ (y(E), y(E'), y(\Omega \setminus (E \cup E'))) & \succeq^* & (y'(E), y'(E'), y(\Omega \setminus (E \cup E'))) \end{array}$$

But then it follows from axiom $B4$ that

$$\begin{array}{rcl} y(E) & \succeq^*_E & y'(E) \\ & \Updownarrow & \\ (y(E), y(E')) & \succeq^*_{E \cup E'} & (y'(E), y'(E')) \end{array}$$

as required.

Q.E.D.

Fact 2. $\# [\Omega] \leq l$.

Claim 6. *Let* $E^* := \{\omega \in \Omega | \exists y(\omega), y'(\omega) \in Y(\omega) : y(\omega) \succ^*_{\{\omega\}} y'(\omega)\}$ *and let* $y(\Omega \setminus E^*), y'(\Omega \setminus E^*) \in \prod_{\omega \in \Omega \setminus E^*} Y(\omega)$. *Then* E^* *is non-empty and* $[y(E^*) \succeq^*_{E^*} y'(E^*)] \Leftrightarrow [(y(E^*), y(\Omega \setminus E^*)) \succeq^* (y'(E^*), y'(\Omega \setminus E^*))]$.

Proof of Claim 21: Step 1: Show that he Claim holds if $\#[\Omega \setminus E^*] = 0$.

If $\#[\Omega \setminus E^*] = 0$ it follows that $\Omega = E^*$. By definition $\#[\Omega] \geq 1$ and from Axiom $B6$, it follows that there exists $y, y' \in Y$ such that $y \succ^* y'$. But then $y \succ^*_{E^*} y'$ and E^* is null as required.

Step 2: Show that $\Omega \setminus E^*$ is null whenever $\#[\Omega \setminus E^*] \geq 1$.

From Fact 2, it follows that $\#[\Omega \setminus E^*]$ is finite. Order the elements in $\Omega \setminus E^*$ as $(\omega_1, \omega_2, \ldots, \omega_K)$. Define $E_1 := \{\omega_1\}$, and E_k ($k \geq 2$) by $E_k = E_{k-1} \cup \{\omega_k\}$. We note that for each k, $\{\omega_k\}$ is null and hence E_1 is null. From Claim 19, it follows that E_k is null whenever E_{k-1} is null. It follows by induction that E_k is null for all k. We now simply note that $\Omega \setminus E^* = E_K$ and hence $\Omega \setminus E^*$, as required, is null.

Step 3: Show that the claim is true if $\#[\Omega \setminus E^*] \geq 1$.

Step 3A: Show that $\#[E^*] \geq 1$.

Suppose $\#[E^*] = 0$. Then $\Omega = \Omega \setminus E^*$. Then it follows from step 2 that Ω is null. By Axiom $B6$, there exists $y, y' \in Y$ such that $y \succ^* y'$. Then $y \succ^*_\Omega y'$ and Ω is non-null, a contradiction.

Step 3B: Show that $[y \succeq^* y'] \Leftrightarrow [y(E^*) \succeq^*_{E^*} y'(E^*)]$.

We know from step 2 that $\Omega \setminus E^*$ is null. Then our conclusion follows from Claim 20.

Q.E.D.

Claim 7. *Let* $\omega \in \Omega$ *satisfy the property that* $\{\omega\}$ *is non-null and let* $E \subset \Omega$ *satisfy the property that* $\omega \in E$. *Then* E *is non-null.*

Proof of Claim 22: Since $\{\omega\}$ is non-null, there exists $\overline{y}(\{\omega\}), \underline{y}(\{\omega\}) \in Y(\omega)$ such that $\overline{y}(\{\omega\}) \succ^*_{\{\omega\}} \underline{y}(\{\omega\})$. Then there exist $y(\Omega \setminus \{\omega\}) \in \prod_{\omega' \in \Omega \setminus \{\omega\}}$ such that $(\overline{y}(\{\omega\}), y(\Omega \setminus \{\omega\})) \succ^* (\underline{y}(\{\omega\}), y(\Omega \setminus \{\omega\}))$. But then it follows that $(\overline{y}(\{\omega\}), y(E \setminus \{\omega\})) \succ^*_E (\underline{y}(\{\omega\}), y(E \setminus \{\omega\}))$ and hence E is non-null. Q.E.D.

Proof of Theorem 3: Let $E^* := \{\omega \in \Omega | \exists \overline{y}(\omega), \underline{y}(\omega) \in Y(\omega) : \overline{y}(\omega) \succ^*_{\{\omega\}} \underline{y}(\omega)\}$. From Claim 21, we know that E^* is non-empty.

Step 1: Show that there exists $q \in \Delta(\Omega)$ such that $[y(E^*) \succeq^*_{E^*} y'(E^*)] \Leftrightarrow [y(E^*) \succeq_{q|E^*} y'(E^*)]$.

Order the elements of E^* as $\{\omega_1, \omega_2, \ldots, \omega_K\}$. For $k = 1, 2, \ldots, K$ define $E_k := \{\omega_{k'} \in E^* | k' \leq k\}$. We claim that for each $k = 1, 2, \ldots, K$, there exists $q_k \in \Delta(\Omega)$ such that $[y(E_k) \succeq^*_{E_k} y'(E_k)] \Leftrightarrow [y(E_k) \succeq_{q_k|E_k} y'(E_k)]$. Indeed, if $k = 1$, we can set $q_k = i_{\omega_1}$ and the conclusion then follows from Axiom $B5$ since $q_1|\{\omega_1\} = i_{\omega_1}$ and $[y(\{\omega_1\}) \succeq_{i_{\omega_1}|\{\omega_1\}} y'(\{\omega_1\})] \Leftrightarrow [(i_{\omega_1}, y) \succeq (i_{\omega_1}, y')] \Leftrightarrow [y(\omega_1) \succeq^*_{\{\omega_1\}} y'(\omega_1)]$. Suppose now that the property holds for $k-1$ where $k \geq 2$. Since E_{k-1} has ω_1 as an element, it follows from Claim 22 that E_{k-1} is non-null. Hence, there exist $\overline{y}^*(E_{k-1}), \underline{y}(E_{k-1}) \in \prod_{\omega \in E_{k-1}} Y(\omega)$ such that $\overline{y}^*(E_{k-1}) \succ^*_{E_{k-1}} \underline{y}(E_{k-1})$. Likewise, since $\{\omega_k\}$ is non-null, we know that there exists $\overline{y}^*(\omega_k), \underline{y}(\omega_k) \in Y(\omega_k)$ such that $\overline{y}^*(\{\omega_k\}) \succ^*_{\{\omega_k\}} \underline{y}(\{\omega_k\})$. Under the present assumptions, we have $[y(E_{k-1}) \succeq^*_{E_{k-1}} y'(E_{k-1})] \Leftrightarrow [y(E_{k-1}) \succeq_{q_{k-1}|E_{k-1}} y'(E_{k-1})]$ and $[y(\{\omega_k\}) \succeq^*_{\{\omega_k\}} y'(\{\omega_k\})] \Leftrightarrow [y(\{\omega_k\}) \succeq_{i_{\omega_k}|\{\omega_k\}} y'(\{\omega_k\})]$. Then it follows from Claim 18 that there exists $q_k \in \Delta(\Omega)$ such that $[y(E_k) \succeq^*_{E_k} y'(E_k)] \Leftrightarrow [y(E_k) \succeq_{q_k|E_k} y'(E_k)]$. Hence, by induction, the desired property hold for every $k = 1, 2, \ldots, K$. Using the fact that $E^* = E_K$ hence concludes step 1.

Step 2: Show that Axioms $A1 - A7$ and $B1 - B7$ together implies the existence of the desired probability $\pi \in \Delta(\Omega)$ and function u.

From Theorem 2, we know that there exists a continuous function $u : \cup_{\omega \in \Omega}(\{\omega\} \times Y(\omega)) \rightarrow \mathbf{R}$ such that

$$[(q, y) \succeq (q', y')] \Leftrightarrow [\sum_{\omega \in C(q)} q(\omega) u(\omega, y(\omega)) \geq \sum_{\omega \in C(q')} q'(\omega) u(\omega, y'(\omega))].$$

From step 1, we know that there exists $q \in \Delta(\Omega)$ such that $[y(E^*) \succeq^*_{E^*} y'(E^*)] \Leftrightarrow [y(E^*) \succeq_{q|E^*} y'(E^*)]$. Letting $\pi = q|E^*$, we hence have $[y(E^*) \succeq^*_{E^*} y'(E^*)] \Leftrightarrow [(\pi, y(E^*)) \succeq (\pi, y'(E^*))]$. From Claim 21, we know that $[y(E^*) \succeq^*_{E^*} y'(E^*)] \Leftrightarrow [(y(E^*), y(\Omega \setminus E^*)) \succeq^* (y'(E^*), y'(\Omega \setminus E^*))]$. Combining these, we have $[(y(E^*), y(\Omega \setminus E^*)) \succeq^* (y'(E^*), y'(\Omega \setminus E^*))] \Leftrightarrow [(\pi, y(E^*)) \succeq (\pi, y'(E^*))]$. Hence it follows from above that $[(y(E^*), y(\Omega \setminus E^*)) \succeq^* (y'(E^*), y'(\Omega \setminus E^*))] \Leftrightarrow [\sum_{\omega \in C(\pi)} \pi(\omega) u(\omega, y(\omega)) \geq \sum_{\omega \in C(\pi)} \pi(\omega) u(\omega, y'(\omega))]$. Finally, using axiom $A7$ we note that $u(\omega, .)$ must be non-constant for every ω as required.

Step 3: Suppose $\#[\Omega] \geq 2$. Show that for any a, b ($b > 0$), $\pi' = \pi$, and $v = a + bu$ also satisfies the required properties.

From Theorem 2, we know that v is a continuous function with the property that

$$[(q,y) \succeq (q',y')] \Leftrightarrow [\sum_{\omega \in C(q)} q(\omega)v(\omega, y(\omega)) \geq \sum_{\omega \in C(q')} q'(\omega)v(\omega, y'(\omega))].$$

Clearly, it is also non-constant for every ω. Since $\pi' = \pi$, we have

$$\begin{array}{rcl} \sum_{\omega \in C(\pi)} \pi(\omega)u(\omega, y(\omega)) & \geq & \sum_{\omega \in C(\pi)} \pi(\omega)u(\omega, y'(\omega)) \\ & \Updownarrow & \\ \sum_{\omega \in C(\pi')} \pi'(\omega)v(\omega, y(\omega)) & \geq & \sum_{\omega \in C(\pi')} \pi'(\omega)v(\omega, y'(\omega)) \end{array}$$

We also have,

$$\begin{array}{rcl} (y(E^*), y(\Omega \setminus E^*)) & \succeq^* & (y'(E^*), y'(\Omega \setminus E^*)) \\ & \Updownarrow & \\ \sum_{\omega \in C(\pi)} \pi(\omega)u(\omega, y(\omega)) & \geq & \sum_{\omega \in C(\pi)} \pi(\omega)u(\omega, y'(\omega)) \end{array}$$

Combining these, we have

$$\begin{array}{rcl} (y(E^*), y(\Omega \setminus E^*)) & \succeq^* & (y'(E^*), y'(\Omega \setminus E^*)) \\ & \Updownarrow & \\ \sum_{\omega \in C(\pi')} \pi'(\omega)v(\omega, y(\omega)) & \geq & \sum_{\omega \in C(\pi')} \pi'(\omega)v(\omega, y'(\omega)) \end{array}$$

as required.

Step 4: Let $\omega \in \Omega$. Show that $\pi(\omega) = 0$ if an only if $\{\omega\}$ is null.

Step 5: Suppose $\#[\Omega] \geq 2$. Show that another probability π' and function v satisfies the same properties only if $\pi' = \pi$ and there exists a, b ($b > 0$) such that $v = a + bu$.

Suppose probability π' and function v satisfies the same properties. Then it follows from Theorem 2 that there must exist a, b ($b > 0$) such that $v = a + bu$. Suppose that $\pi' \neq \pi$. Then there exists $\omega_1, \omega_2 \in \Omega$ such that $\pi(\omega_1) > \pi'(\omega_1)$ and $\pi(\omega_2) < \pi'(\omega_2)$. From step 4, it follows that both $\{\omega_1\}$ and $\{\omega_2\}$ are non-null and that hence $\pi'(\omega_1) > 0$ and $\pi(\omega_2) > 0$. From axiom $A7$, it follows that there exists $\overline{y}(\omega_1), \underline{y}(\omega_1) \in Y(\omega_1)$ such that $u(\omega_1, \overline{y}(\omega_1)) > u(\omega_1, \underline{y}(\omega_1))$. Likewise, it also follows that there exists $\overline{y}(\omega_2), \underline{y}(\omega_2) \in Y(\omega_2)$ such that $u(\omega_2, \overline{y}(\omega_2)) > u(\omega_2, \underline{y}(\omega_2))$. Assume without loss of generality that $\pi(\omega_1)(u(\omega_1, \overline{y}(\omega_1)) - u(\omega_1, \underline{y}(\omega_1))) \geq \pi(\omega_2)(u(\omega_2, \overline{y}(\omega_2)) - u(\omega_2, \underline{y}(\omega_2))) > 0$. It is well known that the connectedness of $Y(\omega_1)$ combined with the continuity of $u(\omega_1, .)$ implies that $u(\omega_1, .)$ is onto the interval $[u(\omega_1, \underline{y}(\omega_1)), u(\omega_1, \overline{y}(\omega_1))]$.

Hence there exists $y^*(\omega_1) \in Y(\omega_1)$ such that

$$\pi(\omega_1)(u(\omega_1, y^*(\omega_1)) - u(\omega_1, \underline{y}(\omega_1))) = \pi(\omega_2)(u(\omega_2, \overline{y}(\omega_2)) - u(\omega_2, \underline{y}(\omega_2))) > 0.$$

Using this property, we have:

$$
\begin{array}{rcl}
\frac{\pi(\omega_1)}{\pi(\omega_2)} & = & \frac{(u(\omega_2,\overline{y}(\omega_2))-u(\omega_2,\underline{y}(\omega_2)))}{(u(\omega_1,y^*(\omega_1))-u(\omega_1,\underline{y}(\omega_1)))} \\
& \Updownarrow & \\
\pi(\omega_1)u(\omega_1,y^*(\omega_1))+\pi(\omega_2)u(\omega_2,\underline{y}(\omega_2)) & = & \pi(\omega_1)u(\omega_1,\underline{y}(\omega_1)))+\pi(\omega_2)u(\omega_2,\overline{y}(\omega_2)) \\
& \Updownarrow & \\
(y^*(\omega_1),\underline{y}(\omega_2)) & \sim^*_{\{\omega_1,\omega_2\}} & (\underline{y}(\omega_1),\overline{y}(\omega_2)) \\
& \Updownarrow & \\
\pi'(\omega_1)v(\omega_1,y^*(\omega_1))+\pi'(\omega_2)v(\omega_2,\underline{y}(\omega_2)) & = & \pi'(\omega_1)v(\omega_1,\underline{y}(\omega_1)))+\pi'(\omega_2)v(\omega_2,\overline{y}(\omega_2)) \\
& \Updownarrow & \\
\pi'(\omega_1)u(\omega_1,y^*(\omega_1))+\pi'(\omega_2)u(\omega_2,\underline{y}(\omega_2)) & = & \pi'(\omega_1)u(\omega_1,\underline{y}(\omega_1)))+\pi'(\omega_2)u(\omega_2,\overline{y}(\omega_2)) \\
& \Updownarrow & \\
\frac{\pi'(\omega_1)}{\pi'(\omega_2)} & = & \frac{(u(\omega_2,\overline{y}(\omega_2))-u(\omega_2,\underline{y}(\omega_2)))}{(u(\omega_1,y^*(\omega_1))-u(\omega_1,\underline{y}(\omega_1)))}
\end{array}
$$

Hence we can conclude that $\frac{\pi(\omega_1)}{\pi(\omega_2)} = \frac{\pi'(\omega_1)}{\pi'(\omega_2)}$. We note that the same thing must hold for every pair of non-null states. From this it must then then in turn follow that $\pi' = \pi$ as required.

Step 6: Show that the existence of both a probability and a continuous expected utility function implies axioms $A1 - A7$ and $B1 - B7$.