INCREASING RETURNS IN NON-SIDE-PAYMENT GAMES

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Abstract

We formulate an increasing returns condition for a marginal worth vector. This condition is shown to guarantee that a marginal worth vector is in the core under standard assumptions. For games that satisfy an additional negative slopedness assumption, this is shown to be a necessary and sufficient condition. An increasing returns condition for non-side-payment games that is weaker than ordinal convexity is also formulated. For games that satisfy the negative slopedness assumption, this increasing returns condition is shown to imply that (i) every marginal worth vector is in the core; and (ii) that the core is a von Neumann-Morgenstern solution.

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1 Introduction

For side-payment games, both convexity and increasing returns with respect to the coalition size have been extensively analyzed. Shapley (1971) demonstrated that convexity and increasing returns with respect to the coalition size are equivalent and that these conditions imply that all marginal worth vectors are in the core. Ichiishi (1981) proved that if all the marginal worth vectors of a side-payment game are in the core, then the game is convex and hence satisfies increasing returns with respect to the coalition size.

The notion of convexity was extended to non-side-payment games by Vilkov (1977) in the form of ordinal convexity. He demonstrated that under certain regularity conditions, ordinal convexity implies non-emptiness of the core. Peleg (1982) later removed some of the regularity conditions in this theorem and, after first extending the notion of marginal worth vectors to the non-side-payment game, Ichiishi (1993) demonstrated that for an ordinal convex game, all the marginal worth vectors are in the core. Unlike the side-payment game, the converse of this last result does not hold.

Properties of the core has been examined for both side-payment and non-side-payment games. Sharkey (1982) demonstrated that the core of a convex side-payment game is large. This result was later extended to the non-side-payment game by Ichiishi (1990) who demonstrated that the core of an ordinal convex game is large. For side-payment-games, the von Neumann-Mogenstern solution was proposed as a solution concept by von Neumann and Morgenstern (1944). Aumann and Peleg (1960) extended the von Neumann-Mogenstern solution concept to the non-side-payment game and Peleg (1986) demonstrated that the core of an ordinal convex game is a von Neumann-Mogenstern solution.

In this paper, we first formulate an increasing returns condition for a marginal worth vector. Under standard assumptions, we demonstrate that a marginal worth vector satisfying this condition, if well defined, is in the core. Under an additional negative slopedness assumption, it is demonstrated that this becomes a necessary and sufficient condition.

Comments from David Schmeidler on an earlier draft of this paper led the present author to formulate an extension of the increasing returns condition for side-payment games to the non-side-payment game. This is stated in terms of the feasible utility allocations in the game and is weaker than ordinal convexity. For games that satisfy the negative slopedness assumption, we establish two results. The first states that every marginal worth vector is in the core, and the second states that the core is a von Neumann-Morgenstern solution. Our proof of the last result is different from that of Peleg. In addition to the results we establish, the paper also contains several examples of games that are used to compare ordinal convexity with our increasing returns conditions.

In section 2, we present our model and main results. Section 3 contains our proofs.

2 Model and Main Results

Let N be a finite set of players and let $\mathcal{N} = 2^N \setminus \{\emptyset\}$ be the family of all nonempty coalitions. A **non-side-payment game** is a nonempty-valued correspondence $V : \mathcal{N} \to \mathbf{R}^N$ such that for every $S \in \mathcal{N}$,

$$[u, v \in \mathbf{R}^N,\ \forall j \in S:\ u_j = v_j] \Rightarrow [u \in V(S) \text{ iff } v \in V(S)].$$

The **core** of a non-side-payment game V is the set $C(V)$ of all $u \in \mathbf{R}^N$ such that [1]

$$u \in V(N), \ \textit{and}$$
$$\neg \exists S \in \mathcal{N} : \exists u' \in V(S) : u' \gg u$$

We provide some additional definitions in order to be able to state the definition of marginal worth vectors due to Ichiishi (1993). Let $n := \#N$. Denote by G_n the family of all linear orders on N, each identified with a bijection from N to $\{1, 2, \ldots, n\}$. Given $\sigma \in G_n$, the **marginal worth vector** $a^\sigma(V)$ is inductively defined as follows: Let P_j^σ be the set of all players who precede j with respect to σ. Assume that $a_i^\sigma(V)$ has been defined for every $i \in P_j^\sigma$. Then $a_j^\sigma(V)$ is defined by:

$$a_j^\sigma(V) := \max \left\{ u_j \in \mathbf{R} \;\middle|\; \begin{array}{l} u \in V(P_j^\sigma \cup \{j\}) \\ u_i = a_i^\sigma(V) \text{ for all } i \in P_j^\sigma \end{array} \right\}.$$

Without some specific assumptions on the non-side-payment game, a marginal worth vector may not be well defined. In all of our Theorems, however, the assumptions therein guarantee that the relevant marginal worth vectors are well defined. Given $\sigma \in G_n$, if $j \in A \subset P_j^\sigma \cup \{j\}$, the **marginal worth of player** j **to coalition** A is defined by:

$$a_j^{\sigma,A}(V) := \sup \left\{ u_j \in \mathbf{R} \;\middle|\; \begin{array}{l} u \in V(A) \\ u_i = a_i^\sigma(V) \text{ for all } i \in A \setminus \{j\} \end{array} \right\}.$$

Note that the use of the supremum rather than the maximum means that the marginal worth is well defined even if the set over which the supremum is taken is empty. Also note that it is clear from this definition that $a_j^{\sigma, P_j^\sigma \cup \{j\}}(V) = a_j^\sigma(V)$.

We are now ready to introduce our condition on a marginal worth vector. Let $\sigma \in G_n$. **Marginal worth vector** $a^\sigma(V)$ **satisfies increasing returns with respect to the coalition size** if for every player j and every $A \subset P_j^\sigma \cup \{j\}$ it follows that:

$$a_j^\sigma(V) \geq a_j^{\sigma,A}(V),$$

that is, for every $A \subset P_j^\sigma \cup \{j\}$ it is true that given the constraint that all the other members of the coalitions have to be given their marginal worth utility allocations, the maximum utility that can be given to player j by coalition $P_j^\sigma \cup \{j\}$ is at least as high as the maximum utility that can be given to player j by coalition A. Recall that a side-payment game is a function $v : \mathcal{N} \to \mathbf{R}$. For a side-payment game, it is easy to verify that a marginal worth vector associated with a linear order σ satisfies increasing returns with respect to the coalition size if and only if for player j and every $A \subset P_j^\sigma \cup \{j\}$, it is true that:

$$v(P_j^\sigma \cup \{j\}) - v(P_j^\sigma) \geq v(A) - \sum_{i \in A \setminus \{j\}} (v(P_i^\sigma \cup \{i\}) - v(P_i^\sigma))$$

It is straightforward to verify that if a side-payment game satisfies increasing returns with respect to the coalition size, then every marginal worth vector must satisfy increasing returns with respect to the coalition size.

[1]The convention $\gg, >, \geq$ is used in order to compare vectors.

We will introduce some assumptions that will be used throughout the remainder of the paper. In order to do so, let V be a non-side-payment game and define $b \in \mathbf{R}^N$ by: $b_j := \sup\{u_j \in \mathbf{R} \mid u \in V(\{j\})\}$, for every $j \in N$. The four assumptions are then stated as:

(1) $V(S) - \mathbf{R}^N_+ = V(S)$ *for every* $S \in \mathcal{N}$;
(2) *There exists* $M \in \mathbf{R}$ *such that for every* $S \in \mathcal{N}$, $[u \in V(S), u \geq b]$ *implies* $[u_j < M$ *for every* $j \in S]$*;*
(3) $V(S)$ *is closed in* $\mathbf{R}^N$ *for every* $S \in \mathcal{N}$; and
(4) $[x, x' \in V(S), x_S \geq b_S, x'_S > x_S]$ *implies* $[\exists x'' \in V(S) : x'' \gg x]$ *for all* $S \in \mathcal{N}$.

The first three assumptions are standard in the literature and will not be commented on here. The fourth assumption in essence implies that the utility frontier has a negative slope. Our first Theorem now states that, under assumptions (1)-(3), a well defined marginal worth vector is in the core if it satisfies increasing returns with respect to the coalition size.

THEOREM 1 *Let* V *be a non-side-payment game. Assume (1)-(3). Choose any* $\sigma \in G_n$ *for which the associated marginal worth vector* $a^\sigma(V)$ *is well defined. Then* $a^\sigma(V)$ *is in the core if it well defined and satisfies increasing returns with respect to the coalition size.*

It is a clear implication of this result that if every marginal worth vector satisfies increasing returns with respect to the coalition size, then all the marginal worth vectors are in the core of game V.

The following Theorem demonstrates that if an additional assumption is satisfied, the converse to Theorem 1 also holds true. That is, a marginal worth vector is in the core if and only if it satisfies increasing returns with respect to the coalition size.

THEOREM 2 *Let* V *be a non-side-payment game. Assume (1)-(4). Choose any* $\sigma \in G)n$ *for which the associated marginal worth vector* $a^\sigma(V)$ *is well defined. Then* $a^\sigma(V)$ *is in the core of game* V *if and only if it satisfies increasing returns with respect to the coalition size.*

Again, an implication of this is the equivalence of all the marginal worth vectors being in the core to every marginal worth vector satisfying increasing returns with respect to the coalition size.

Because our increasing returns condition is defined in terms of the marginal worth vectors, it is difficult to give the condition a nice and clear interpretation. In particular, it may be difficult to make comparisons between this condition and ordinal convexity. Recall that game V is **ordinal convex** if:

$$\forall S, T \in \mathcal{N} : V(S) \cap V(T) \subset V(S \cap T) \cup V(S \cup T),$$

where $V(\emptyset) := \emptyset$. In order to facilitate a comparison between ordinal convexity and increasing returns, we provide the following extension of the increasing returns condition to the non-side-payment game. Game V is said to satisfy **increasing returns with respect to the coalition size**, if for any $A, B \subset N$ and any $j \in N$ such that $A \subset B \subset N \setminus \{j\}$, it follows that

$$V(A \cup \{j\}) \cap V(B) \subset V(A) \cup V(B \cup \{j\}).$$

Clearly, a game that satisfies ordinal convexity must satisfy increasing returns with respect to the coalition size. Unfortunately, increasing returns with respect to the coalition size fails to guarantee that all the marginal worth vector are in the core, even with the standard assumptions (1)-(3). The following Theorem illustrates that if assumption (4) also is satisfied, increasing returns with respect to the coalition size implies that all the marginal worth vectors are in the core.

THEOREM 3 *Let V be a non-side-payment game. Assume game V satisfies (1)-(4) and increasing returns with respect to the coalition size. Then every marginal worth vector $a^\sigma(V)$ is well defined, satisfies increasing returns with respect to the coalition size and is in the core of game V.*

A quick comparison to Ichiishi's (1993) result thus demonstrates that once assumption (4) is satisfied, the assumed ordinal convexity in that result can be relaxed to increasing returns with respect to the coalition size while still guaranteeing that all the marginal worth vectors are in the core.

We now introduce another cooperative solution concept. The core of game V is a **von Neumann-Morgenstern solution** if for every $u \in V(N)$ that is not in the core of game V, there exists a utility allocation u' in the core of game V satisfying the property that u' dominates u. That is, there is some coalition S for which $u' \in V(S)$, and $u'_s \gg u_S$. A result due to Peleg (1986) is that under assumptions (1)-(3) the core of an ordinal convex game is a von Neumann-Morgenstern solution. The following Theorem strengthens Peleg's result as follows:

THEOREM 4 *Let V be a non-side-payment game. Assume game V satisfies (1)-(4) and increasing returns with respect to the coalition size. Then the core of game V is a von Neumann-Morgenstern solution.*

A comparison to Peleg's (1986) result thus demonstrates that if assumption (4) is satisfied, ordinal convexity can be relaxed to increasing returns with respect to the coalition size.

Several examples are in order. Before presenting these, we recall that the core of game V is **large** if for every $u \in \mathbf{R}^N \setminus \cup_{S \in \mathcal{N}} \overset{\circ}{V}(S)$, there exists u' in the core of game V such that $u' \leq u$. As noted in the introduction, Ichiishi (1990) showed that the core of an ordinal convex game is large.

The first of our examples shows that if assumption (4) is removed, there is an ordinal convex game where at least one of the marginal worth vectors fails to satisfy increasing returns with respect to the coalition size. The second example is of a game that violates assumption (4), satisfies increasing returns with respect to the coalition size but has an empty core. Finally, the third example is of a game satisfying assumption (4) and increasing returns with respect to the coalition size but violating ordinal convexity. The core of this last game fails to be large.

EXAMPLE 1: A three player game that satisfies ordinal convexity but where at least one of the marginal worth vectors fails to satisfy increasing returns with respect to the coalition size. Define:

$$\begin{aligned}
V(\{j\}) &:= \{u \in \mathbf{R}^3 \mid u_j \leq 0\} \\
V(\{1,2\}) &:= \{u \in \mathbf{R}^3 \mid u_1 \leq 0, u_2 \leq 1\} \\
V(\{1,3\}) &:= \{u \in \mathbf{R}^3 \mid u_1 \leq 0, u_3 \leq 2\} \\
V(\{2,3\}) &:= \{u \in \mathbf{R}^3 \mid u_2 \leq 1, u_3 \leq 1\} \\
V(N) &:= \{u \in \mathbf{R}^3 \mid u_1 \leq 0, u_2 \leq 1, u_3 \leq 1\} \\
&\quad \bigcup \{u \in \mathbf{R}^3 \mid u_1 \leq 0, u_2 \leq 0, u_3 \leq 2\}
\end{aligned}$$

We leave verification of ordinal convexity as an exercise for the reader. To see that marginal worth vector $a^{\{1,2,3\}}(V)$ fails to satisfy increasing returns with respect to the coalition size, note that $a_3^{\{1,2,3\},\{1,2,3\}}(V) = 1 < 2 = a_3^{\{1,2,3\},\{1,3\}}(V)$. □

EXAMPLE 2: A four player game that violates assumption (4), satisfies increasing returns with respect to the coalition size, fails to satisfy ordinal convexity, and has an empty core. Define:

$$\begin{aligned}
V(\{j\}) &:= \{u \in \mathbf{R}^4 \mid u_j \leq 0\} \\
V(\{1,2\}) &:= \{u \in \mathbf{R}^4 \mid u_1 \leq 1, u_2 \leq 1\} \\
V(\{1,3\}) &:= \{u \in \mathbf{R}^4 \mid u_1 \leq 0, u_3 \leq 0\} \\
V(\{1,4\}) &:= \{u \in \mathbf{R}^4 \mid u_1 \leq 0, u_4 \leq 0\} \\
V(\{2,3\}) &:= \{u \in \mathbf{R}^4 \mid u_2 \leq 0, u_3 \leq 0\} \\
V(\{2,4\}) &:= \{u \in \mathbf{R}^4 \mid u_2 \leq 0, u_4 \leq 0\} \\
V(\{3,4\}) &:= \{u \in \mathbf{R}^4 \mid u_3 \leq 1, u_4 \leq 1\} \\
V(\{1,2,3\}) &:= \{u \in \mathbf{R}^4 \mid u_1 \leq 1, u_2 \leq 1, u_3 \leq 0\} \\
V(\{1,2,4\}) &:= \{u \in \mathbf{R}^4 \mid u_1 \leq 1, u_2 \leq 1, u_4 \leq 0\} \\
V(\{1,3,4\}) &:= \{u \in \mathbf{R}^4 \mid u_1 \leq 0, u_3 \leq 1, u_4 \leq 1\} \\
V(\{2,3,4\}) &:= \{u \in \mathbf{R}^4 \mid u_2 \leq 0, u_3 \leq 1, u_4 \leq 1\} \\
V(\{1,2,3,4\}) &:= \{u \in \mathbf{R}^4 \mid u_1 \leq 1, u_2 \leq 1, u_3 \leq 0, u_4 \leq 0\} \\
&\quad \bigcup \{u \in \mathbf{R}^4 \mid u_1 \leq 0, u_2 \leq 0, u_3 \leq 1, u_4 \leq 1\}
\end{aligned}$$

We leave verification of increasing returns with respect to the coalition size as an exercise for the reader. To see that the game fails to satisfy ordinal convexity, note that $(1,1,1,1) \in V(\{1,2\}) \cap V(\{3,4\})$ but not in $V(\{1,2,3,4\})$. To see that the core is empty, first note that if the core was non-empty, then either $(1,1,0,0)$, or $(0,0,1,1)$ would also be in the core. But $(1,1,0,0)$ is blocked by coalition $\{3,4\}$ and $(0,0,1,1)$ is blocked by coalition $\{1,2\}$. Hence the core must be empty. □

EXAMPLE 3: The following is an example of a game that satisfies assumptions (1)-(4) and increasing returns with respect to the coalition size but where the core fails to be large. Note that the game also fails to be ordinal convex. Define:

$$\begin{aligned}
V(\{j\}) &:= \{u \in \mathbf{R}^4 \mid u_j \leq 0\}, \text{ for all } j \in N \\
V(\{i,j\}) &:= \begin{cases} \{u \in \mathbf{R}^4 \mid \exists u' \in \mathbf{R}^4_+ : (u'_i + u'_j \leq 2, u' \geq u)\} & \text{if } \{i,j\} = \{1,2\},\{3,4\} \\ \{u \in \mathbf{R}^4 \mid u_i \leq 0, u_j \leq 0\} & \text{otherwise} \end{cases} \\
V(\{1,2,3\}) &:= \{u \in \mathbf{R}^4 \mid \exists u' \in \mathbf{R}^4_+ : (10u'_3 + u'_1 + u'_2 \leq 2, u' \geq u)\} \\
V(\{1,2,4\}) &:= \{u \in \mathbf{R}^4 \mid \exists u' \in \mathbf{R}^4_+ : (10u'_4 + u'_1 + u'_2 \leq 2, u' \geq u)\} \\
V(\{1,3,4\}) &:= \{u \in \mathbf{R}^4 \mid \exists u' \in \mathbf{R}^4_+ : (10u'_1 + u'_3 + u'_4 \leq 2, u' \geq u)\} \\
V(\{2,3,4\}) &:= \{u \in \mathbf{R}^4 \mid \exists u' \in \mathbf{R}^4_+ : (10u'_2 + u'_3 + u'_4 \leq 2, u' \geq u)\} \\
V(\{1,2,3,4\}) &:= \{u \in \mathbf{R}^4 \mid u_1 + u_2 + u_3 + u_4 \leq 28/12\} \\
&\quad \bigcup_{i \in N} \bigcup_{j \in N\setminus\{i\}} \bigcup_{k \in N\setminus\{i,j\}} \\
&\quad \bigcup_{l \in N\setminus\{i,j,k\}} \{u \in \mathbf{R}^4 \mid \exists u' \in \mathbf{R}^4_+ : (4u'_i + u'_j + u'_k + u'_l \leq 4, u' \geq u)\}
\end{aligned}$$

To see that the game fails to have a large core, use utility allocation $(1,1,1,1)$. □

3 Proofs

Proof of Theorem 1: Let $\sigma \in G_n$ be a linear order for which the marginal worth vector is well defined and satisfies increasing returns with respect to the coalition size. Suppose $a^\sigma(V)$ is not in the core. Then there exists $S \subset N$ and $u \in V(S)$ such that $u \gg a^\sigma(V)$. Choose any such u. Let $i^*(S)$ be the last player in S according to σ. By comprehensiveness $(a^\sigma_{N\setminus\{i^*(S)\}}(V), u_{i^*(S)}) \in V(S)$. By increasing returns relative to σ we have $u_{i^*(S)} > a^\sigma_{i^*(S)}(V) \geq a^{\sigma,S}_{i^*(S)}(V)$. But this contradicts the definition of $a^{\sigma,S}_{i^*(S)}(V)$. Hence $a^\sigma(V)$ must be in the core of game V. □

Proof of Theorem 2: The if part of our current Theorem is an immediate consequence of Theorem 1. To show the only if part of the Theorem, we suppose that $a^\sigma(V)$ is in the core of game V. Clearly, $a^\sigma_j(V) \geq b_j$ for all $j \in N$. Suppose $a^\sigma(V)$ does not satisfy increasing returns with respect to the coalition size. Then there exists $j \in N$ and $A \subset P^\sigma_j \cup \{j\}$ such that $a^\sigma_j(V) < a^{\sigma,A\cup\{j\}}_j(V)$. By assumption (4), there exists $u \in V(A \cup \{j\})$ such that $u \gg a^\sigma(V)$. But this contradicts $a^\sigma(V)$ being in the core. Hence marginal worth vector $a^\sigma(V)$ must satisfy increasing returns with respect to the coalition size. □

Proof of Theorem 3: Given Theorem 1, it suffices to show that the marginal worth vector associated with every linear order σ is well defined and satisfies increasing returns with respect to the coalition size. The marginal worth vector being well defined follows immediately from assumptions (1) through (3) combined with $V(P^\sigma_j) \cap V(\{j\}) \subset V(\emptyset) \cup V(P^\sigma_j \cup \{j\})$. Now, suppose the marginal worth vector did not satisfy increasing returns with respect to the coalition size. Then there exists a player i and a coalition $A \subset P_i^\sigma$ such that $a_i^{\sigma,A\cup\{i\}}(V) > a_i^\sigma(V)$. Let i^* be the first of all such players according to linear order σ. Clearly i^* is not the first player according to σ. Now, $V(\{i^*\}) \cap V(P_{i^*}^\sigma) \subset V(\emptyset) \cup V(P_{i^*}^\sigma \cup \{i^*\})$ implies that $a_{i^*}^\sigma(V) \geq b_{i^*}$. Choose any maximal $B \subset P_{i^*}^\sigma$ such that $a_{i^*}^{\sigma,B\cup\{i^*\}}(V) > a_{i^*}^\sigma(V)$. That is, choose any coalition B for which there does not exist another larger coalition containing B that also satisfies the desired property.

Clearly, at least one such coalition must exist. Since $a_{i^*}^\sigma(V) = a_{i^*}^{\sigma,P_{i^*}^\sigma\cup\{i^*\}}(V)$, it must be the case that $B \neq P_{i^*}^\sigma$. Because of how i^* was selected, it must also be the case that $(a_{i^*}^{\sigma,B\cup\{i^*\}}(V), a_{N\setminus\{i^*\}}^\sigma(V)) \in (V(B \cup \{i^*\}) \cap V(P_{i^*}^\sigma)) \setminus V(P_{i^*}^\sigma \cup \{i^*\})$.

Increasing returns with respect to the coalition size combined with the definition of i^* then implies that $(a_{i^*}^{\sigma,B\cup\{i^*\}}(V), a_{N\setminus\{i^*\}}^\sigma(V)) \in \partial V(B)$. We now consider two cases.

Case 1: $a_{P_{i^*}^\sigma\setminus B}^\sigma(V) = b_{P_{i^*}^\sigma\setminus B}$. If this is true, we have

$$\begin{aligned}(a_{i^*}^{\sigma,B\cup\{i^*\}}(V), a_{N\setminus\{i^*\}}^\sigma(V)) &\in V(B\cup\{i^*\}) \cap (\cap_{j\in P_{i^*}^\sigma\setminus B} V(\{j\})) \\ &= \cap_{j\in P_{i^*}^\sigma\setminus B}(V(B\cup\{i^*\}) \cap V(\{j\})) \\ &\subset \cap_{j\in P_{i^*}^\sigma\setminus B} V(B\cup\{j\}\cup\{i^*\})\end{aligned}$$

But $B \neq P_{i^*}^\sigma$ then implies that there exist $j \in P_{i^*}^\sigma\setminus B$ for which $(a_{i^*}^{\sigma,B\cup\{i^*\}}(V), a_{N\setminus\{i^*\}}^\sigma(V)) \in V(B\cup\{j\}\cup\{i^*\})$. This in turn implies that $a_{i^*}^{\sigma,B\cup\{j\}\cup\{i^*\}}(V) > a_{i^*}^\sigma(V)$ a contradiction to B being maximal.

Case 2: $a_{P_{i^*}^\sigma\setminus B}^\sigma(V) > b_{P_{i^*}^\sigma\setminus B}$. Then there exists a player j^* in $P_{i^*}^\sigma\setminus B$ for which $a_{j^*}^\sigma(V) > b_{j^*}$. Now, let ϵ_{i^*} and ϵ_{j^*} be any two strictly positive numbers satisfying the properties that $\epsilon_{i^*} < a_{i^*}^{\sigma,B\cup\{i^*\}}(V) - b_{i^*}$ and $\epsilon_{j^*} < a_{j^*}^\sigma(V) - b_{j^*}$.

Because of assumption (4) and the fact that $(a_{i^*}^{\sigma,B\cup\{i^*\}}(V), a_{N\setminus\{i^*\}}^\sigma(V)) \in V(B\cup\{i^*\}) \cap V(P_{i^*}^\sigma)$, there exist $x_\epsilon \in V(B\cup\{i^*\})$ and $x'_\epsilon \in V(P_{i^*}^\sigma)$ such that $x_\epsilon \gg (a_{i^*}^{\sigma,B\cup\{i^*\}}(V) - \epsilon_{i^*}, a_{N\setminus\{i^*\}}^\sigma(V))$ and $x'_\epsilon \gg (a_{i^*}^{\sigma,B\cup\{i^*\}}(V), a_{j^*}^\sigma(V) - \epsilon_{j^*}, a_{N\setminus\{i^*,j^*\}}^\sigma(V))$. For every player k, define $x''_{\epsilon,k} := min\{x_{\epsilon,k}, x'_{\epsilon,k}\}$. Then it is clear that $x'' \in V(B\cup\{i^*\}) \cap V(P_{i^*}^\sigma)$. It is also clear that $x''_{\epsilon,B} \gg a_B^\sigma(V))$ which implies that $x''_\epsilon \notin V(B)$. Hence it follows from increasing returns with respect to the coalition size that $x''_\epsilon \in V(P^\sigma \cup \{i^*\})$.

From this it is clear that there exists a sequence of such x''_ϵ 's that converges to $(a_{i^*}^{\sigma,B\cup\{i^*\}}(V), a_{N\setminus\{i^*\}}^\sigma(V))$. Since $V(P^\sigma \cup \{i^*\})$ is a closed set, it then follows that $(a_{i^*}^{\sigma,B\cup\{i^*\}}(V), a_{N\setminus\{i^*\}}^\sigma(V)) \in V(P_{i^*}^\sigma \cup \{i^*\})$. But this contradicts $a_{i^*}^{\sigma,B\cup\{i^*\}}(V) > a_{i^*}^\sigma(V)$. Hence the marginal worth vector associated with σ must satisfy increasing returns with respect to the coalition size and it thus follows from Theorem 1 that it is in the core. □

Proof of Theorem 4: Choose any utility allocation $u \in V(N)$ that is not in the core. Let S be any minimal coalition for which there exist $u' \in \partial V(S)$ satisfying the property that $u' >> u$. Consider any linear order σ satisfying the property that for every player $j \in S$, $P_j^\sigma \subset S$. Define $d^\sigma(u')$ as follows: For every player $j \in S$, set $d_j^\sigma(u') = u'_j$. For the players outside of coalition S, define $d_j^\sigma(u')$ inductively by setting

$$d_j^\sigma(u') := \max \left\{ u_j \in \mathbf{R} \,\middle|\, \begin{array}{l} u \in V(P_j^\sigma \cup \{j\}) \\ u_i = d_i^\sigma(u') \text{ for all } i \in P_j^\sigma \end{array} \right\}$$

We claim that this is well defined for every player j and that furthermore $d_j^\sigma(u') \geq b_j$ for every player j. If $j \in S$, this is obvious. Thus suppose $j \in N \setminus S$ and that $d_i^\sigma(u')$ is well defined for every player i in P_j^σ. Now simply note that $(d_{N\setminus\{j\}}^\sigma(u'), b_j) \in V(P_j^\sigma) \cap V(\{j\}) \subset V(P_j^\sigma \cup \{j\})$. This combined with assumptions (1)-(3) then implies that $d_j^\sigma(u')$ is well defined and individually rational for every player j.

Now, for every player j and every $A \subset P_j^\sigma \cup \{j\}$ such that $j \in A$, define

$$d_j^{\sigma,A}(u') := \sup \left\{ u_j \in \mathbf{R} \,\middle|\, \begin{array}{l} u \in V(A) \\ u_i = d_i^\sigma(u') \text{ for all } i \in A \setminus \{j\} \end{array} \right\}$$

The remainder of our proof will be divided into two steps.

Step 1: Show that for every player j and every coalition $A \subset P_j^\sigma$, $d_j^\sigma(u') \geq d_j^{\sigma,A\cup\{j\}}(u')$.

Suppose not. Then there exists a player i and a coalition $A \subset P_i^\sigma$ such that $d_i^{\sigma,A\cup\{i\}}(u') > d_i^\sigma(u')$. Let i^* be the first of all such players according to linear order σ. Suppose i^* was a member of coalition S. Then there exists a coalition $A \subset S$ for which $d_{i^*}^{\sigma,A\cup\{i^*\}}(u') > d_{i^*}^\sigma(u') \geq b_{i^*}$. But then there exists $u'' \in V(A \cup \{i^*\})$ satisfying the property that $u'' \gg u'$ a contradiction to the selection of S as a minimal blocking coalition. Thus it must be that $i^* \in N \setminus S$. Now, choose any maximal $B \subset P_{i^*}^\sigma$ such that $d_{i^*}^{\sigma,B\cup\{i^*\}}(u') > d_{i^*}^\sigma(u')$. That is, choose any coalition B for which there does not exist another larger coalition containing B that also satisfies the desired property. Clearly, at least one such coalition must exist. Since $d_{i^*}^\sigma(u') = d_{i^*}^{\sigma,P_{i^*}^\sigma\cup\{i^*\}}(u')$, it must be the case that $B \neq P_{i^*}^\sigma$. Because of how i^* was selected, it must also be the case that $(d_{i^*}^{\sigma,B\cup\{i^*\}}(u'), d_{N\setminus\{i^*\}}^\sigma(u')) \in (V(B\cup\{i^*\}) \cap V(P_{i^*}^\sigma)) \setminus V(P_{i^*}^\sigma \cup \{i^*\})$.

Increasing returns with respect to the coalition size combined with the definition of i^* then implies that $(d_{i^*}^{\sigma,B\cup\{i^*\}}(u'), d_{N\setminus\{i^*\}}^\sigma(u')) \in \partial V(B)$. We now consider two cases.

Case 1: $d_{P_{i^*}^\sigma\setminus B}^\sigma(u') = b_{P_{i^*}^\sigma\setminus B}$. If this is true, we have

$$\begin{aligned}
(d_{i^*}^{\sigma,B\cup\{i^*\}}(u'), d_{N\setminus\{i^*\}}^\sigma(u')) &\in V(B \cup \{i^*\}) \cap (\cap_{j\in P_{i^*}^\sigma\setminus B} V(\{j\})) \\
&= \cap_{j\in P_{i^*}^\sigma\setminus B}(V(B \cup \{i^*\}) \cap V(\{j\})) \\
&\subset \cap_{j\in P_{i^*}^\sigma\setminus B} V(B \cup \{j\} \cup \{i^*\})
\end{aligned}$$

But $B \neq P_{i^*}^\sigma$ then implies that there exist $j \in P_{i^*}^\sigma \setminus B$ for which $(d_{i^*}^{\sigma,B\cup\{i^*\}}(u'), d_{N\setminus\{i^*\}}^\sigma(u')) \in V(B \cup \{j\} \cup \{i^*\})$. This in turn implies that $d_{i^*}^{\sigma,B\cup\{j\}\cup\{i^*\}}(u') > d_{i^*}^\sigma(u')$ a contradiction to B being maximal.

Case 2: $d_{P_{i^*}^\sigma\setminus B}^\sigma(u') > b_{P_{i^*}^\sigma\setminus B}$. Then there exists a player j^* in $P_{i^*}^\sigma \setminus B$ for which $d_{j^*}^\sigma(u') > b_{j^*}$. Now, let ϵ_{i^*} and ϵ_{j^*} be any two strictly positive numbers satisfying the properties that $\epsilon_{i^*} < d_{i^*}^{\sigma,B\cup\{i^*\}}(u') - b_{i^*}$ and $\epsilon_{j^*} < d_{j^*}^\sigma(u') - b_{j^*}$.

Because of assumption (4) and the fact that $(d_{i^*}^{\sigma,B\cup\{i^*\}}(u'), d_{N\setminus\{i^*\}}^\sigma(u')) \in V(B \cup \{i^*\}) \cap V(P_{i^*}^\sigma)$, there exist $x_\epsilon \in V(B \cup \{i^*\})$ and $x'_\epsilon \in V(P_{i^*}^\sigma)$ such that $x_\epsilon \gg (d_{i^*}^{\sigma,B\cup\{i^*\}}(u') - \epsilon_{i^*}, d_{N\setminus\{i^*\}}^\sigma(u'))$ and $x'_\epsilon \gg (d_{i^*}^{\sigma,B\cup\{i^*\}}(u'), d_{j^*}^\sigma(u') - \epsilon_{j^*}, d_{N\setminus\{i^*,j^*\}}^\sigma(u'))$. For every player k, define $x''_{\epsilon,k} := min\{x_{\epsilon,k}, x'_{\epsilon,k}\}$. Then it is clear that $x'' \in V(B \cup \{i^*\}) \cap V(P_{i^*}^\sigma)$. It is also clear that

$x''_{\epsilon,B} \gg d^\sigma_B(u')$) which implies that $x''_\epsilon \notin V(B)$. Hence it follows from increasing returns with respect to the coalition size that $x''_\epsilon \in V(P^\sigma \cup \{i^*\})$.

From this it is clear that there exists a sequence of such x''_ϵ 's that converges to $(d^{\sigma,B\cup\{i^*\}}_{i^*}(u'), d^\sigma_{N\setminus\{i^*\}}(u'))$. Since $V(P^\sigma \cup \{i^*\})$ is a closed set, it then follows that $(d^{\sigma,B\cup\{i^*\}}_{i^*}(u'), d^\sigma_{N\setminus\{i^*\}}(u')) \in V(P^\sigma_{i^*} \cup \{i^*\})$. But this contradicts $d^{\sigma,B\cup\{i^*\}}_{i^*}(u') > d^\sigma_{i^*}(u')$.

This concludes step 1.

Step 2: Show $d^\sigma(u')$ is a member of the core

Suppose $d^\sigma(u')$ was not in the core. Then there exists $T \subset N$ and $u'' \in V(T)$ such that $u'' \gg d^\sigma(u')$. Choose any such u''. Let $i^*(T)$ be the last player in T according to σ. By comprehensiveness $(d^\sigma_{N\setminus\{i^*(T)\}}(u'), u''_{i^*(T)}) \in V(T)$. Because the condition we demonstrated above holds, we have $u''_{i^*(T)} > d^\sigma_{i^*(T)}(u') \geq d^{\sigma,T}_{i^*(T)}(u')$. But this contradicts the definition of $d^{\sigma,T}_{i^*(T)}(u')$. Hence $d^\sigma(u')$ must be in the core of game V. Clearly $d^\sigma(u')$ dominates u as required. □

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