A More Complete Extension of a Theorem of von Neumann and Morgenstern

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Preliminary and Incomplete - Do Not Quote

1 Introduction

In a recent paper, Karni and Safra (2000) extends the theory of von Neumann and Morgenstern by relaxing the mixture space structure of the decision set. The purpose of this paper is to offer a more complete extension along the lines of the standard axiomatization for decision sets with the mixture space structure. We do so by relaxing several assumptions made in the Karni and Safra paper. For the main result of the present paper, the state space is no longer required to be finite; the feasible outcome set in each state is no longer required to be compact; and the binary preference relation is no longer required to be continuous. We also strengthen one of the statements in the Karni and Safra paper by showing that the expected utility function representation is unique up to positive linear transformations.

The axiomatization used here follows very closely the standard one for decision sets with the mixture space structure. In fact four of the five axioms used for the main result in the present paper (Completeness, Transitivity, Constrained Archimedean, Constrained Independence) have corresponding axioms for such decision sets (Completeness, Transitivity, Archimedean, Independence). The key contribution of this paper lies in replacing the continuity assumption of Karni and Safra (2000) with the Constrained Archimedean axiom. Our Constrained Archimedean axiom is a weakened form of the standard Archimedean axiom used for decision sets with the mixture space structure. The final axiom (non-degeneracy for every state of the world) corresponds to the coordinate essentiality used by Karni and Safra. In a Lemma used to establish our main result, we show that if this last Axiom is removed, the existence of an expected utility function is still assured. In the same Lemma, we also show that even though uniqueness up to positive linear transformations can fail once the non-degeneracy axiom is removed, a property closely related to the uniqueness still holds.

In a second Theorem, we show that the expected utility function representation can be shown to be continuous if, in addition to the five axioms indicated above, an additional closedness axiom is added. This second Theorem is closer to the result obtained by Karni and Safra (2000). It requires that the state space is finite; and that the set of consequences in each state is (i) a subset of a finite dimensional Euclidean space; and (ii) the union of finitely many connected components.

2 The Model

2.1 States

There is a (non-empty) set of states of the world Ω.

2.2 Simple Probabilities

Let $Q := \{q : \Omega \to \mathbf{R}_+ | \sum_{\omega\in\Omega} q(\omega) = 1\}$. For every $q \in Q$, let $C(q) := \{\omega \in \Omega | q(\omega) > 0\}$. The set of simple probabilities on Ω is defined by $\Delta(\Omega) := \{q \in Q | \exists r \in \mathbf{R} : \#[C(q)] \leq r\}$. A simple probability $q \in \Delta(\Omega)$ assigns a positive probability to a finite subset of states in Ω. For any state ω, the probability $i \in \Delta(\Omega)$ for which $i(\omega) = 1$ is denoted i_ω.

2.3 Consequences

For every state $\omega \in \Omega$ there is an associated (non-empty) set of consequences $Y(\omega)$.

2.4 Acts

The set of acts is defined as the product space $Y := \prod_{\omega\in\Omega} Y(\omega)$. Each act hence associates with every state $\omega \in \Omega$ a consequence $y(\omega) \in Y(\Omega)$.

2.5 Lotteries

The set of lotteries is defined as $\Delta(\Omega) \times Y$. Each lottery $(q, y) \in \Delta(\Omega) \times Y$ hence consists of a simple probability q on the state space and an act y. The intended interpretation of lottery (q, y) is that it realizes consequence $y(\omega) \in Y(\omega)$ with probability $q(\omega)$. Lotteries (q, y_1) and (q, y_2) that a) assign identical probabilities to the states ; and b) assigns the same consequences to all states with positive probabilities $[y_1(C(q)) = y_2(C(q))]$ are to be regarded as equivalent. A further discussion of this is contained in the next subsection.

2.6 Preferences over Lotteries

There is a (binary) preference relation $\succeq$ on the set of lotteries $\Delta(\Omega)\times Y$. Let $q \in \Delta(\Omega)$. Then whenever $y_1(C(q)) = y_2(C(q))$, it will be assumed throughout that for any $(q', y') \in \Delta(\Omega)\times Y$, $[(q', y') \succeq (q, y_1)] \Leftrightarrow [(q', y') \succeq (q, y_2)]$.

3 Rational Decision Making under Risk

3.1 Axioms for Rational Decision Making under Risk

Axiom 1 (Completeness). *For every set of elements* (q, y) *and* (q', y') *in* $\Delta(\Omega) \times Y$*, it follows that either* $[(q, y) \succeq (q', y')]$ *or* $[(q', y') \succeq (q, y)]$.

This standard axiom states that rational decision makers should either rate (q, y) at least as highly as (q', y'), or (q', y') at least as highly as (q, y). Reflexivity is a related concept that is often discussed in the literature. As shown below, reflexivity is implied by Axiom 1.

Definition 1. The preference relation $\succeq$ is said to be reflexive if for every $q \in \Delta(\Omega)$, and every set of acts $y_1, y_2 \in Y$ for which $[y_1(C(q)) = y_2(C(q))]$ it follows that $[(q, y_1) \sim (q, y_2)]$.

Fact 1. Every preference relation satisfying axiom 1 is also reflexive.

Proof: Let $q \in \Delta(\Omega)$ and $y_1, y_2 \in Y$ satisfy the property that $[y_1(C(q)) = y_2(C(q))]$. From axiom 1 it follows that $(q, y_1) \succeq (q, y_1)$. Likewise, we have $(q, y_2) \succeq (q, y_2)$. By definition (see above), it then follows that $(q, y_1) \succeq (q, y_2)$ and $(q, y_2) \succeq (q, y_1)$. Hence $(q, y_1) \sim (q, y_2)$. Q.E.D.

Axiom 2 (Transitivity). *For every set of elements* (q, y)*,* (q', y')*, and* (q'', y'') *in* $\Delta(\Omega) \times Y$*,* $[(q, y) \succeq (q', y'), (q', y') \succeq (q'', y'')]$ *implies* $[(q, y) \succeq (q'', y'')]$.

This is a standard axiom that is needed in order to avoid circularity of reasoning.

Axiom 3 (Constrained Archimedean Axiom). *Let* $y_1, y_2 \in Y$*. If* $q_1, q_2, q_3 \in \Delta(\Omega)$ *such that* $(q_1, y_1) \succ (q_2, y_2) \succ (q_3, y_1)$ *then there exists* $\alpha, \beta \in (0, 1)$ *such that* $(\alpha q_1 + (1 - \alpha) q_3, y_1) \succ (q_2, y_2)$*, and* $(q_2, y_2) \succ (\beta q_1 + (1 - \beta) q_3, y_1)$.

This is a weakened form of the standard Archimedean axiom that is standard in axiomatizations for decision sets with the mixture space structure. Note that the axiom only is applied when the mixture operation is defined.

Axiom 4 (Constrained Independence). *Let* $(q_1, y_1) \sim (q_2, y_2)$*. Then for all* $q, q' \in \Delta(\Omega)$ *and any* $\alpha \in (0, 1)$*,* $[(q, y_1) \succeq (q', y_2)] \Leftrightarrow [(\alpha q + (1 - \alpha) q_1, y_1) \succeq (\alpha q' + (1 - \alpha) q_2, y_2)]$.

This is the axiom introduced by Karni and Safra (2000). Their remark 2 contains a brief discussion concerning the need for this axiom. Note that it is a modified form of the independence axiom that is standard for choice sets with the mixture space structure.

Axiom 5 (Non-Degeneracy). *For every* $\omega \in \Omega$*, there exists* $y, y' \in Y$ *such that* $(i_\omega, y) \succ (i_\omega, y')$.

This axiom is required in order to establish that the expected utility function is unique up to positive linear transformations. As our Lemma 1 illustrates, it is not needed to establish the existence of an expected utility function representation.

Our final axiom will only be applied when the set $Y(\omega)$ is a subset of a finite dimensional Euclidean space endowed with the usual topology. In those instances, it will also be assumed that Y is endowed with the product topology.

Axiom 6 (Closedness)**.** *Let* $\omega \in \Omega$ *and* $(q^*, y^*) \in \Delta(\Omega) \times Y$. *Fix* $y(\Omega \backslash \{\omega\}) \in \prod_{\omega' \in \Omega \backslash \{\omega\}} Y(\omega')$. *Then the sets*

$$\{y(\omega) \in Y(\omega) | (i_\omega, y) \succeq (q^*, y^*)\}$$

and

$$\{y(\omega) \in Y(\omega) | (q^*, y^*) \succeq (i_\omega, y)\}$$

are both closed in $Y(\omega)$.

3.2 Representation of Preferences by an Expected Utility Function

Definition 2. A function $U : \Delta(\Omega) \times Y \to \mathbf{R}$ is a utility function representation of the preference relation $\succeq$ if for any pair of elements $(q, y), (q', y') \in \Delta(\Omega) \times Y$,

$$[(q, y) \succeq (q', y')] \Leftrightarrow [U(q, y) \geq U(q', y')]$$

Definition 3. A function $U : \Delta(\Omega) \times Y \to \mathbf{R}$ is an expected utility function representation of the preference relation $\succeq$ if (i) it is a utility function representation of $\succeq$; and (ii) there exists a function $u : \cup_{\omega \in \Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ such that for every $(q, y) \in \Delta(\Omega) \times Y$,

$$U(q, y) = \sum_{\omega \in C(q)} q(\omega) u(\omega, y(\omega))$$

Fact 2. If $\#[\Omega] = 1$, then an expected utility function representation of $\succeq$ exists whenever there exists a utility function representation of $\succeq$.

Theorem 1 (Expected Utility Function Representation)**.** *If* $\#[\Omega] \geq 2$*, then preference relation* $\succeq$ *satisfies axioms* $A1 - A5$ *if and only if there exists a function* $u : \cup_{\omega \in \Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$*, nonconstant for every* ω*, such that*

$$[(q, y) \succeq (q', y')] \Leftrightarrow [\sum_{\omega \in C(q)} q(\omega) u(\omega, y(\omega)) \geq \sum_{\omega \in C(q')} q'(\omega) u(\omega, y'(\omega))].$$

Moreover, another function $v : \cup_{\omega \in \Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ *has the same properties if an only if there is* a, b $(b > 0)$ *such that* $v = a + bu$.

We now turn to several examples that illustrates the method of proof, the role of the non-degeneracy axiom, and an instance where preferences are representable by an expected utility function despite discontinuity of the preference relation.

Theorem 2 (Continuous Expected Utility Function Representation)**.** *Assume that* $2 \leq \#[\Omega] \leq M$*, and that for every* $\omega \in \Omega$*,* $Y(\omega)$ *is (i) a subset of a finite dimensional Euclidean space; and (ii) the union of finitely many connected components. Then preference relation* $\succeq$ *satisfies axioms* $A1 - A6$ *if and only if there exists a continuous function, non-constant for every* ω*,* $u : \cup_{\omega \in \Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ *such that*

$$[(q, y) \succeq (q', y')] \Leftrightarrow [\sum_{\omega \in C(q)} q(\omega) u(\omega, y(\omega)) \geq \sum_{\omega \in C(q')} q'(\omega) u(\omega, y'(\omega))].$$

Moreover, another function $v : \cup_{\omega \in \Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ *has the same properties if an only if there is* a, b $(b > 0)$ *such that* $v = a + bu$.

Lemma 1. *If $\#[\Omega] \geq 2$, then preference relation $\succeq$ satisfies axioms $A1 - A4$ if and only if there exists a function $u : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ such that*

$$[(q,y) \succeq (q',y')] \Leftrightarrow [\sum_{\omega\in C(q)} q(\omega)u(\omega,y(\omega)) \geq \sum_{\omega\in C(q')} q'(\omega)u(\omega,y'(\omega))].$$

The function $U : \Delta(\Omega) \times Y \to \mathbf{R}$ defined by $U(q,y) = \sum_{\omega\in C(q)} q(\omega)u(\omega,y(\omega))$ has the property that for any $(q,y),(q',y') \in \Delta(\Omega) \times Y$ for which $(q,y) \succ (q',y')$ it follows that U is onto the interval $[U(q',y'),U(q,y)]$. Moreover,

(i) if there does not exist distinct elements ω_1^*,ω_2^* of Ω and y^* in Y such that $[(q,y) \not\succ (i_{\omega_2^*},y^*)] \Rightarrow [q(\omega_1^*) > 0]$ then another function $v : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ has the same properties if an only if there is a,b $(b > 0)$ such that $v = a + bu$; and

(ii) if there does exist distinct elements ω_1^*,ω_2^* of Ω and y^* in Y such that $[(q,y) \not\succ (i_{\omega_2^*},y^*)] \Rightarrow [q(\omega_1^*) > 0]$ then another function $v : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ has the same properties if and only if there is $a,b_1,b_2(b_1 > 0, b_2 > 0)$ such that:

$$v(\omega,y(\omega)) = \begin{cases} a + b_1[u(\omega,y(\omega)) - u(\omega_2^*,y^*(\omega^*))] \text{ if } (i_\omega,y) \succeq (i_{\omega_2^*},y^*) \\ a + b_2[u(\omega,y(\omega)) - u(\omega_2^*,y^*(\omega^*))] \text{ otherwise} \end{cases}$$

Lemma 2. *Assume that $1 \leq \#[\Omega] \leq M$, and that for every $\omega \in \Omega$, $Y(\omega)$ is (i) a subset of a finite dimensional Euclidean space; and (ii) the union of finitely many connected components. Then preference relation $\succeq$ satisfies axioms $1-4$ and 6 if and only if there exists a continuous function $u : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ such that*

$$[(q,y) \succeq (q',y')] \Leftrightarrow [\sum_{\omega\in C(q)} q(\omega)u(\omega,y(\omega)) \geq \sum_{\omega\in C(q')} q'(\omega)u(\omega,y'(\omega))].$$

Moreover,

(i) if $\#[\Omega] \geq 2$ and there does not exist distinct elements ω_1^*,ω_2^* of Ω and y^* in Y such that $[(q,y) \not\succ (i_{\omega_2^*},y^*)] \Rightarrow [q(\omega_1^*) > 0]$ then another function $v : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ has the same properties if an only if there is a,b $(b > 0)$ such that $v = a + bu$; and

(ii) if $\#[\Omega] \geq 2$ and there does exist distinct elements ω_1^*,ω_2^* of Ω and y^* in Y such that $[(q,y) \not\succ (i_{\omega_2^*},y^*)] \Rightarrow [q(\omega_1^*) > 0]$ then another function $v : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ has the same properties if and only if there is $a,b_1,b_2(b_1 > 0, b_2 > 0)$ such that:

$$v(\omega,y(\omega)) = \begin{cases} a + b_1[u(\omega,y(\omega)) - u(\omega_2^*,y^*(\omega^*))] \text{ if } (i_\omega,y) \succeq (i_{\omega_2^*},y^*) \\ a + b_2[u(\omega,y(\omega)) - u(\omega_2^*,y^*(\omega^*))] \text{ otherwise} \end{cases}$$

Lemma 3 (Utility Function Representation)**.** *Let X be (i) a subset of a finite dimensional Euclidean space; and (ii) the union of finitely many connected components. If $\succeq$ is a (binary) preference relation on X, then $\succeq$ is complete, transitive, and closed if and only if it is representable by a continuous utility function.*

4 Proofs

4.1 Fact 2

Proof of Fact 2: Let U be a utility function representation of $\succeq$. Denote by ω^* the unique element of Ω. For every $(q,y) \in \Delta(\Omega) \times Y$, set $u(\omega^*,y_\omega) = U(q,y)$. Then for every $(q,y) \in$

$\Delta(\Omega) \times Y$, $U(q, y) = u(\omega^*, y_{\omega^*}) = \sum_{\omega \in \Omega} q(\omega) * u(\omega, y_\omega)$ demonstrating that U is an expected utility function representation. Q.E.D.

4.2 Theorem 1

Proof of Theorem 1: Given Lemma 1, it suffices to show that we are in part (i) of the moreover statement. But this follows immediately from Axiom 5. Q.E.D.

4.3 Theorem 2

Proof of Theorem 2: Given Lemma 2, it suffices to show that we are in part (i) of the moreover statement. But this follows immediately from Axiom 5. Q.E.D.

Hence we see that Lemma 2 is crucial in establishing Theorem 2.

4.4 Lemma 3

Fact 3 (Property for Unions of Connected Sets)**.** Let C_1 and C_2 be two non-empty connected subsets of a Euclidean space. If $C_1 \cup C_2$ is disconnected, it follows that every limit point for C_1 in $C_1 \cup C_2$ is in $C_1 \setminus C_2$.

Proof of Fact 3: Let x be a limit point for C_1. Suppose $x \in C_2$. Since $C_1 \cup C_2$ is disconnected, there exists a separation D_1, D_2 of $C_1 \cup C_2$. Suppose without loss of generality that $x \in D_2$. Since C_2 is connected, it must be that $C_2 \subset D_2$ (otherwise D_1,D_2 would form a separation for C_2). It follows that $D_1 \cap C_1 \neq \emptyset$ (otherwise D_1,D_2 could not form a separation for $C_1 \cup C_2$). Let $y \in D_1 \cap C_1$. Since C_1 is connected, we must have that $y \in D_1$ (otherwise D_1,D_2 would form a separation for C_1). It then follows that x is a limit point for D_1 contained in D_2, a contradiction to the definition of a separation. Q.E.D.

Fact 4. Let $\{C_k\}_{k \in \tau}$ be a finite collection of connected subsets of a Euclidean space with the property that no union of two distinct elements of the collection forms a connected set. Let $\{x_l\}_{l=1}^{\infty}$ be a sequence of points in $\cup_{k \in \tau} C_k$ that converges to a point $\overline{x} \in C_1$. Then there exists L such that $l \geq L$ implies $x_l \in C_1$.

Proof of Fact 4: Suppose not. Then infinitely many elements of the sequence $\{x_l\}_{l=1}^{\infty}$ are contained in $[\cup_{k \in \tau} C_k] \setminus C_1$. Since the collection τ is finite, there exists $k \neq 1$ such that infinitely many elements of the sequence $\{x_l\}_{l=1}^{\infty}$ are contained in C_k. But then it follows that $\overline{x}$ is a limit point for C_k contained in C_1, a contradiction to Fact 3. Q.E.D.

Fact 5. Let $\{C_k\}_{k \in \tau}$ be a finite collection of connected subsets of a Euclidean space with the property that no union of two distinct elements of the collection forms a connected set. Then a function U defined on $\cup_{k \in \tau} C_k$ is continuous if an only if it is continuous on every element C_k of the collection.

Proof of Fact 5: Clearly, continuity on $\cup_{k \in \tau} C_k$ implies continuity on every element C_k of the collection. Consider now any function U that is continuous on every element C_k of the collection. Let $\{x_l\}_{l=1}^{\infty}$ be a sequence converging to some point $\overline{x} \in \cup_{k \in \tau} C_k$. Suppose without loss of generality that $x \in C_1$. From Fact 4, we know that there exists L such that $l \geq L$ implies $x_l \in C_1$. Then it follows from the continuity on C_1 that $\lim_{l\to\infty} U(x_l) = U(\overline{x})$ as required for continuity. Q.E.D.

Fact 6. Let $\{C_k\}_{k\in\tau}$ be a finite collection of connected subsets of a Euclidean space with the property that no union of two distinct elements of the collection forms a connected set. Let $\succeq$ be a (binary) preference relation on $\cup_{k\in\tau}C_k$ satisfying completeness, transitivity, and closedness. Assume furthermore, that for every C_k in the collection, $[x_1, x_2 \in C_k] \Rightarrow [x_1 \sim x_2]$. Then there exists a continuous function $V : \cup_{k\in\tau}C_k \to \mathbf{R}$ that represents the preferences on $\cup_{k\in\tau}C_k$.

Proof of Fact 6: From every C_k in the collection, pick a single element x_k^*. Since the collection $\{x_k^*\}_{k\in\tau}$ is finite, it is well known that under the present assumptions, there exists a continuous function $V^* : \cup_{k\in\tau}\{x_k\} \to \mathbf{R}$ that represents the preferences on the restriction of the preference relation to $\cup_{k\in\tau}\{x_k\}$. From fact 3, it follows that we can define a unique function V, constant on every C_k, that corresponds to V^* (i.e., $x \in C_k \Rightarrow V(x) = V^*(x_k^*)$). We claim that $V : \cup_{k\in\tau}C_k \to \mathbf{R}$ is the required continuous function.

To see that V represents the preferences on $\cup_{k\in\tau}C_k$, consider any $x_1, x_2 \in \cup_{k\in\tau}C_k$. Suppose without loss of generality that $x_1 \in C_k$ and $x_2 \in C_{k'}$. (i) Whenever $x_1 \succeq x_2$, it follows that $x_k^* \sim x_1 \succeq x_2 \sim x_{k'}^*$. From completeness and transitivity it then follows that $x_k^* \succeq x_{k'}^*$ which in turn implies $V(x_1) = V(x_k^*) \geq V(x_{k'}^*) = V(x_2)$. Likewise, (ii) whenever $V(x_1) \geq V(x_2)$, it follows that $V(x_k^*) = V(x_1) \geq V(x_2) = V(X_{k'}^*$. This in turn implies $x_1 \sim x_k^* \succeq x_{k'}^* \sim x_2$. Again, from completeness and transitivity, it follows that $x_1 \succeq x_2$. Hence V represents preferences on $\cup_{k\in\tau}C_k$.

To see that V is continuous, note that V is constant on every C_k. Hence it follows from Fact 5 that V is continuous on $\cup_{k\in\tau}C_k$. Q.E.D.

We are now ready to turn to the proof of Lemma 3. The Proof will follow the method of proof of Debreu (1959) to establish the result when X is a connected set.

Proof of Lemma 3: Throughout, we will make use of a finite collection $\{C_k\}_{k\in\tau}$ of connected subsets of X whose union is X. If complete indifference holds on X, the preference relation is clearly complete, transitive and closed and any constant function (clearly continuous) represents the preferences. Hence this case is easily disposed off. For the rest of our proof, we therefore assume the existence of $x, x' \in X$ such that $x \succ x'$.

Our proof is divided into 4 steps. In step 1, we carefully construct a function V^* defined on X. In step 2, we show that the constructed function V^* is (i) continuous; and (ii) represents the preferences on each of the elements C_k of the finite collection. In step 3, we use V^* to construct a new function U that is both continuous and represents preferences on all of X. In Step 4 finally, we show that the existence of a continuous utility function implies completeness, transitivity, and closedness.

Step 1: Let $\{\tilde{C}_k\}_{k\in\tilde{\tau}}$ be a finite collection of connected subsets of X whose union is X. Merge any sets whose union is connected until no more distinct sets whose union is a connected set remains. It has been shown (see Berge [1963, p.72]) that in the resulting collection of sets $\{C_k\}_{k\in\tau}$, the merged sets must be disjoint. From Fact 4 it then follows that whenever $\{x_l\}_{l=1}^{\infty}$ is a sequence of points in $\cup_{k\in\tau}C_k$ that converges to a point $\overline{x} \in C_{k^*}$, it follows that there exists L such that $l \geq L$ implies $x_l \in C_{k^*}$. From completeness, it follows that we can partition τ into sets τ_1 and τ_2, where (i) $[k \in \tau_1] \Leftrightarrow [\neg\exists x, x' \in C_k : x \succ x']$; and (ii) $[k \in \tau_2] \Leftrightarrow [\exists x, x' \in C_k : x \succ x']$. From Fact 6, it follows that whenever τ_1 is nonempty, there exists a function V^* that represents the restriction of the preferences to $\cup_{k\in\tau_1} C_k$. Since any such function can obtain at most a finite number of values, the function must also obtain both its maximum and minimum on $\cup_{k\in\tau_1} C_k$. Denote by a^* and b^* the maximum and minimum values for V^* on $\cup_{k\in\tau_1} C_k$. Without loss of generality, we assume that V^* takes on only rational values. Now, let $a < b$ be any two real numbers satisfying the following additional properties whenever τ_1 is nonempty: (i) If $\cup_{k\in\tau_1} C_k$ contains an element that is a least preferred element for X, then $a = a^*$, otherwise $a < a^*$; and (ii) if $\cup_{k\in\tau_1} C_k$ contains a most preferred element for X, then $b = b^*$, otherwise $b^* < b$.

As noted by Debreu (1959) section 1.6, every subset X of a finite dimensional Euclidean space contains a countable set $\tilde{X}$ that is dense in X. Hence, we can select a countable subset $\tilde{X}$ of $\cup_{k\in\tau_2} C_k$ that is dense in $\cup_{k\in\tau_2} C_k$. Let D be such a set. Since D is dense in C_k for every $k \in \tau_2$, it follows that for every $x \in C_k$, there are points in D arbitrarily close to x. Due to Fact 4, it must also be that for every $k \in \tau_2$, $D \cap C_k$ is a countable set that contains points arbitrarily close to every element of C_k. Now, remove from D any elements that are maximal or minimal elements for one of the sets $\{C_k\}_{k\in\tau_2}$ and denote the remaining set D'. Since for every $k \in \tau_2$, each C_k is connected and contains points x, x' such that $x \succ x'$, it can easily be seen that it follows from ((2), p.57) of Debreu (1959) that for every $k \in \tau_2$, $D' \cap C_k$ is a nonempty countable set that contains no greatest and no least element. It can also easily be seen that for every $k \in \tau_2$, $D' \cap C_k$ must contain contain an infinite number of indifference classes.

We first extend the function V^* to $D' \cup (\cup_{k\in\tau_1} C_k)$. Since D' is countable, its elements can be ranked, $(z^1, z^2, \ldots, z^p, \ldots)$; this ranking is unrelated to the pre-ordering $\succeq$. Similarly, the set of rationals, Q', of the interval (a, b) is also countable, and its elements can be ranked $(r^1, r^2, \ldots, r^q, \ldots)$; this ranking is unrelated to the ordering $\geq$. The elements of D' will be considered in succession; with x^p will be associated an element $r^{q_p} \in Q'$ in such a way that the preordering is preserved, and every element of Q' is eventually taken. For every $k \in \tau_1$, let y_k be in C_k. Denote by K_1 then number of element in τ_1.

Consider z^1; the set D' is partitioned into the following sets: the finite number of indifference classes of $y_1, y_2, \ldots, y_K$ (the number of different sets so obtained is at most K_1), and the associated intervals that fall above, between, and below those indifference classes. Note that there are at most $K_1 + 1$ such intervals (see Debreu (1959)). If $z^1 \sim y_k$ for some k, set $V^*(x^1) = V^*(y_k$, which by assumption is a rational number. If z^1 is instead in one of the intervals, consider the corresponding interval in Q', and select in it the rational of least rank r^{q_1}; set $V^*(z^1) = r^{q_1}$.

Consider z^p; the set D' is partitioned into the following sets: the finite number of indifference classes of $y_1, y_2, \ldots, y_K$ (the number of different sets so obtained is at most K_1); the finite number of indifference classes of $z_1, z_2, \ldots, z_{p-1}$ (the number of different additional sets so obtained is at most $p - 1$); and the associated intervals that fall above, between, and below the indifference classes. There are at most $K + 1 + p$ such intervals(see Debreu (1959)). If $z^p \sim y_k$ for some k, set $V^*(x^1) = V^*(y_k$, which by assumption is a rational number. If $z^p \sim z_{p'}$ for some $p' < p$, set $V^*(z^p) = V^*(z_{p'}$. If z^p is instead in one of the intervals, consider the corresponding interval in Q', and select in it the rational of least rank r^{q_p}; set

$V^*(z^p) = r^{q_p}$. It is clear that the function V^* is increasing. It is easy to check that the least rank condition implies that every element of Q' is eventually taken (if not, at least one of the sets $D' \cap C_k$ would either contain a maximum, a minimum, or violate (2) in Debreu).

4.5 Lemma 2

In order to establish Lemma 2, we first establish two claims that will be used in the proof.

Claim 1. *Let $\omega \in \Omega$ and $y_1, y_2 \in Y$ satisfy $(i_\omega, y_1) \succ (i_\omega, y_2)$ and let $(q^*, y^*) \in \Delta(\Omega) \times Y$ satisfy $(i_\omega, y_1) \succ (q^*, y^*) \succ (i_\omega, y_2)$. Assume furthermore that there exists a connected subset of $Y(\omega)$, denoted $D(\omega, y_1, y_2)$, containing both $y_1(\omega)$ and $y_2(\omega)$. Then $\exists y'' \in D(\omega, y_1, y_2)$ such that $(i_\omega, y'') \sim (q^*, y^*)$.*

Proof of Claim 1: Fix $y(\Omega \setminus \{\omega\}) \in \prod_{\omega' \in \Omega \setminus \{\omega\}} Y(\omega')$. We note that it follows from Fact 1 that $(i_\omega, y_1(\omega), y(\Omega \setminus \{\omega'\})) \sim (i_\omega, y_1) \succ (q^*, y^*) \succ (i_\omega, y_2) \sim (i_\omega, y_2(\omega), y(\Omega \setminus \{\omega'\}))$. Denote by $D(\omega, y_1, y_2)$ a connected subset of $Y(\omega)$ containing both $y_1(\omega)$ and $y_2(\omega)$.

Let $W_1 := \{y(\omega) \in D(\omega, y_1, y_2) | (i_\omega, y) \succ (q^*, y^*)\}$, $W_2 := \{y(\omega) \in D(\omega, y_1, y_2) | (q^*, y^*) \succ (i_\omega, y)\}$, and $W_3 := \{y(\omega) \in D(\omega, y_1, y_2) | (q^*, y^*) \sim (i_\omega, y)\}$. By definition, these sets are disjoint. We note that W_1 and W_2 are nonempty since $y_1(\omega) \in W_1$ and $y_2(\omega) \in W_2$. From Axiom 1, it follows that $Y(\omega) = W_1 \cup W_2 \cup W_3$. It now suffices to show that $W_3 \neq \emptyset$.

Suppose $W_3 = \emptyset$. From Axiom 6, it follows that the sets

$$\begin{array}{rcl} D(\omega, y_1, y_2) \setminus W_1 & = & \{y(\omega) \in D(\omega, y_1, y_2) | (q^*, y^*) \succeq (i_\omega, y)\}; \text{ and} \\ D(\omega, y_1, y_2) \setminus W_2 & = & \{y(\omega) \in D(\omega, y_1, y_2) | (i_\omega, y) \succeq (q^*, y^*)\} \end{array}$$

are closed. But then we know that W_1 and W_2 are nonempty disjoint open sets such that $D(\omega, y_1, y_2) = W_1 \cup W_2$. Hence the pair of sets W_1 and W_2 form a separation for $D(\omega, y_1, y_2)$. But $D(\omega, y_1, y_2)$ is a connected sets. A contradiction. Hence $W_3 \neq \emptyset$ and there exists $y'' \in Y$ such that $(i_\omega, y'') \sim (q^*, y^*)$. Q.E.D.

Claim 2. *The function u given in Theorem 1 satisfies the property that for every $\omega \in \Omega$ and any $y_1, y_2 \in Y$ for which $(i_\omega, y_1) \succ (i_\omega, y_2)$, it follows that for every $\alpha \in [u(\omega, y_2(\omega)), u(\omega, y_1(\omega))]$ there exists $y^* \in Y$ such that $u(\omega, y^*(\omega)) = \alpha$.*

Proof of Claim 15: Let $\omega \in \Omega$ and $y_1, y_2 \in Y$ satisfy the property that $(i_\omega, y_1) \succ (i_\omega, y_2)$ and let $\alpha \in [u(\omega, y_2(\omega)), u(\omega, y_1(\omega))]$. (i) If $\alpha = u(\omega, y_1(\omega))$, then we can set $y^* = y_1$ and we are done since $\alpha = u(\omega, y^*(\omega))$. (ii) Likewise, if $\alpha = u(\omega, y_2(\omega))$, then we cans set $y^* = y_2$ and we are done since $\alpha = u(\omega, y^*(\omega))$. (iii) If $u(\omega, y_2(\omega)) < \alpha < u(\omega, y_1(\omega))$, we know that $u(\omega, y_1(\omega)) = \sum_{\omega' \in C(i_\omega)} i(\omega') u(\omega', y_1(\omega'))$ and that $u(\omega, y_2(\omega)) = \sum_{\omega' \in C(i_\omega)} i(\omega') u(\omega', y_2(\omega'))$. Hence $\alpha \in (\sum_{\omega' \in C(i_\omega)} i(\omega') u(\omega', y_2(\omega')), \sum_{\omega' \in C(i_\omega)} i(\omega') u(\omega', y_1(\omega')))$. But then we know from the onto property stated in Theorem 1 that there exists $(q^*, y^*) \in \Delta(\Omega) \times Y$ such that $\alpha = \sum_{\omega' \in C(q^*)} q^*(\omega') u(\omega', y^*(\omega'))$. From representation, it then follows that $(i_\omega, y_1) \succ (q^*, y^*) \succ (i_\omega, y_2)$. Then it in turn follows from Claim 14 that there exists $y'' \in Y$ such that $(i_\omega, y'') \sim (q^*, y^*)$. But then we must have

$$\begin{array}{rcl} u(\omega, y''(\omega)) & = & \sum_{\omega' \in C(i_\omega)} u(\omega', y''(\omega')) \\ & = & \sum_{\omega' \in C(q^*)} u(\omega', y^*(\omega')) \\ & = & \alpha \end{array}$$

as required.

Q.E.D.

will make use of Lemma 1, Debreu's (1959) classical result on the representability of an ordering, and the following two Claims.

Proof of Lemma 2: We divide our argument into three steps. In step 1, we show that the Axioms implies the existence of a continuous expected utility representation. In step 2, we show that a continuous expected utility representation implies the Axioms. In order to complete the proof we then show, in step 3, that the moreover statement holds.

Step 1: Show that the Axioms implies existence of a continuous expected utility function.

For each $\omega \in \Omega$, define a preference relation $\succeq_\omega$ on Y such that $[y \succeq_\omega y'] \Leftrightarrow [(i_\omega, y) \succeq (i_\omega, y')]$. Since $\succeq$ satisfies axioms $A2, A4$, and $A6$ it follows that $\succeq_\omega$ is complete, transitive, and closed. From Debreu's result, it then follows that there exists an continuous function $v_\omega : Y \to \mathbf{R}$ such that $[v_\omega(y) \geq v_\omega(y')] \Leftrightarrow [y \succeq_\omega y']$. The rest of step 1 will be broken into two parts. In step 1A, we show that the result holds when $\#[\Omega] = 1$ and in step 1B, we show that it holds when $\#[\Omega] \geq 2$.

Step 1A: Show that a continuous expected utility function exists if $\#[\Omega] = 1$.

Denote by ω^* the unique element of Ω and by i_{ω^*} the unique element of $\Delta(\Omega)$. Define a function $u : \{\omega^*\} \times Y(\omega^*) \to \mathbf{R}$ by $u(\omega^*, y) = v_{\omega^*}(y)$. Since v is continuous, it follows that u is continuous. Then it follows that the function $U : \Delta(\Omega) \times Y$ defined by $U(i_{\omega^*}, y) = \sum_{\omega \in C(i_{\omega^*})} i_{\omega^*}(\omega) u(\omega, y(\omega))$ also is continuous as required. Now, we also have

$$\begin{array}{rcl}
(i_{\omega^*}, y) & \succeq & (i_{\omega^*}, y') \\
 & \Updownarrow & \\
y & \succeq_{\omega^*} & y' \\
 & \Updownarrow & \\
v_{\omega^*}(y) & \geq & v_{\omega^*}(y') \\
 & \Updownarrow & \\
u(\omega^*, y) & \geq & u(\omega^*, y') \\
 & \Updownarrow & \\
\sum_{\omega' \in C(i_{\omega^*})} i_{\omega^*}(\omega') u(\omega', y) & \geq & \sum_{\omega' \in C(i_{\omega^*})} i_{\omega^*}(\omega') u(\omega', y')
\end{array}$$

as required. Hence the desired continuous expected utility function exists.

Step 2: Show that a continuous expected utility function exists if $\#[\Omega] \geq 2$.

Let $u : \cup_{\omega \in \Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ be the function given in Theorem 1. It suffices to show that for every $\omega \in \Omega$, $u(\omega, .)$ is continuous.

For each $\omega \in \Omega$ define $A(\omega) := \{r \in \mathbf{R} | \exists y \in Y : v_\omega(y) = r\}$. We note that for every $y \in Y$, there exists uniquely $r^*_y \in \mathbf{R}$ such $u(\omega, y(\omega)) = r^*_y$. Furthermore, whenever $v_\omega(y) = v_\omega(y')$ we know that $r^*_y = r^*_{y'}$. We can hence define uniquely a function $w_\omega : A(\omega) \to \mathbf{R}$ such that $w_\omega(r) = r^*$ if and only if there exists $y \in Y$ such that $v_\omega(y) = r$ and $u(\omega, y(\omega)) = r^*$. We claim that for every ω, the function w_ω is continuous.

Indeed, let $\{r_k\}_{k=1}^{\infty}$ be a sequence of elements of $A(\omega)$ such that $r_k \to \overline{r}$ where $\overline{r} \in A(\omega)$. It suffices for us to show that $\lim_{k\to\infty} w_\omega(r_k) = w_\omega(\overline{r})$. If $\lim_{k\to\infty} w_\omega(r_k) \neq w_\omega(\overline{r})$, there are two possible cases:

(i) There exists $r^* < w_\omega(\overline{r})$ such that for infinitely many elements of the sequence $\{w_\omega(r_k)\}_{k=1}^{\infty}$, it follows that $w_\omega(r_k) \leq r^*$. We note that this implies the existence of $\overline{y} \in Y$ and for every k, $y_k \in Y$ such that for infinitely many element of the sequence $\{w_\omega(r_k)\}_{k=1}^{\infty}$, we have $w_\omega(r_k) = u(\omega, y_k(\omega)) \leq r^* < u(\omega, \overline{y}(\omega)) = w_\omega(\overline{r})$. From claim 15 it follows that there exists $y^* \in Y$ such that $u(\omega, y^*(\omega)) = r^*$. But then we must have that for infinitely many element of the sequence $\{r_k\}_{k=1}^{\infty}$, $r_k = v_\omega(y_k) \leq v_\omega(y^*) < v_\omega(\overline{y}) = \overline{r}$. But then it can not be that $r_k \to \overline{r}$, a contradiction.

(ii) There exists $r^* > w_\omega(\overline{r})$ such that for infinitely many elements of the sequence $\{w_\omega(r_k)\}_{k=1}^{\infty}$, it follows that $w_\omega(r_k) \geq r^*$. We note that this implies the existence of $\overline{y} \in Y$ and for every k, $y_k \in Y$ such that for infinitely many element of the sequence $\{w_\omega(r_k)\}_{k=1}^{\infty}$, we have $w_\omega(r_k) = u(\omega, y_k(\omega)) \geq r^* > u(\omega, \overline{y}(\omega)) = w_\omega(\overline{r})$. From claim 15 it follows that there exists $y^* \in Y$ such that $u(\omega, y^*(\omega)) = r^*$. But then we must have that for infinitely many element of the sequence $\{r_k\}_{k=1}^{\infty}$, $r_k = v_\omega(y_k) \geq v_\omega(y^*) > v_\omega(\overline{y}) = \overline{r}$. But then it can not be that $r_k \to \overline{r}$, a contradiction.

Since neither (i) nor (ii) can hold, we can hence conclude that $\lim_{k\to\infty} w_\omega(r_k) = w_\omega(\overline{r})$ which in turn implies that for every ω, the associated function w_ω is continuous. We now simply note that for every $\omega \in \Omega$, we have that for every $u(\omega, y(\omega)) = w_\omega(v_\omega(y))$. Since both v_ω and w_ω are continuous functions, it follows that $u(\omega, .)$ also is continuous as required.

Step 2: Show that the existence of a continuous expected utility function implies Axioms A1-A6.

Step2A: Show that if $\#\Omega = 1$, then every preference relation satisfying axioms $A2 - A3$ also satisfies axioms $A1$, $A4$, and $A5$.

Suppose $\#\Omega = 1$ and denote by ω^* the unique element of Ω. We note that $[y(\omega^*) = y'(\omega^*)] \Rightarrow [y = y']$. But then it follows from $A2$ and $A3$ that $(i_{\omega^*}, y) \sim (i_{\omega^*}, y')$ so $A1$ holds. We also not that $\#\Delta(\Omega) = 1$. But then it again follows from $A2$ and $A3$ that there does not exist $y_1 \in Y$ and $q_1, q_3 \in \Delta(\Omega)$ such that $(q_1, y_1) \succ (q_3, y_3)$. Hence $A4$ must be satisfied. Finally, let $(q_1, y_1), (q_2, y_2) \in \Delta(\Omega) \times Y$ satisfy $(q_1, y_1) \sim (q_2, y_2)$. Since $\#\Delta(\Omega) = 1$, it follows that for every $q^*, q^{**} \in \Delta(\Omega)$, $(q^*, y_1) = (q_1, y_1)$ and $(q^{**}, y_2) = (q_2, y_2)$ and it hence follows that $(q^*, y_1) \sim (q^{**}, y_2)$. Consider any $q, q' \in \Delta(\Omega)$ and $\alpha \in (0, 1)$. Then we must have both $(q, y_1) \sim (q', y')$ and $(\alpha q + (1-\alpha) q_1, y_1) \sim (\alpha q' + (1-\alpha) q_2, y_2)$. Hence $A5$ is satisfied.

Step 2B: Show that if the preference relation in representable by a continuous utility function representation it follows that axioms $A2, A3$, and $A6$ are satisfied.

(i) Note that for any pair $(q, y), (q', y') \in \Delta(\Omega) \times Y$, either $\sum_{\omega \in C(q)} q(\omega) u(\omega, y(\omega)) \geq \sum_{\omega \in C(q')} q'(\omega) u(\omega, y'(\omega))$, or $\sum_{\omega \in C(q')} q'(\omega) u(\omega, y'(\omega)) \geq \sum_{\omega \in C(q)} q(\omega) u(\omega, y(\omega))$. By representation, this implies $(q, y) \succeq (q', y')$ or $(q', y') \succeq (q, y)$. Hence $A2$ holds.

(ii) Let $(q, y), (q', y'), (q'', y'') \in \Delta(\Omega) \times Y$ satisfy the properties that $(q, y) \succeq (q', y)$ and $(q', y') \succeq (q'', y'')$. By representation, it follows that $\sum_{\omega \in C(q)} q(\omega) u(\omega, y(\omega)) \geq \sum_{\omega \in C(q')} q'(\omega) u(\omega, y'(\omega))$ and $\sum_{\omega \in C(q')} q'(\omega) u(\omega, y'(\omega)) \geq \sum_{\omega \in C(q'')} q''(\omega) u(\omega, y''(\omega))$. By the properties of the real number line, it follows that $\sum_{\omega \in C(q)} q(\omega) u(\omega, y(\omega)) \geq \sum_{\omega \in C(q'')} q''(\omega) u(\omega, y''(\omega))$. By representation, it then follows that $(q, y) \succeq (q'', y'')$. Hence $A3$ holds.

(iiia) Let $\{y_k\}_{k=1}^{\infty}$ be any sequence of elements in the set $\{y \in Y | (i_\omega, y) \succeq (q^*, y^*)\}$ such that $y_k \longrightarrow \overline{y}$. From representation, we know that for every k, we have $\sum_{\omega' \in \Omega} i_\omega(\omega') u(\omega', y_k(\omega')) \geq \sum_{\omega' \in \Omega} q^*(\omega') u(\omega', y^*(\omega'))$. From continuity, it then follows that $\sum_{\omega' \in \Omega} i_\omega(\omega') u(\omega', \overline{y}(\omega')) \geq \sum_{\omega' \in \Omega} q^*(\omega') u(\omega', y^*(\omega'))$. Then it in turn follows from representation that $(i_\omega, \overline{y}) \succeq (q^*, y^*)$. Hence the set $\{y \in Y | (i_\omega, y) \succeq (q^*, y^*)\}$ is closed.

(iiia) Let $\{y_k\}_{k=1}^{\infty}$ be any sequence of elements in the set $\{y \in Y | (q^*, y^*) \succeq (i_\omega, y)\}$ such that $y_k \longrightarrow \overline{y}$. From representation, we know that for every k, we have $\sum_{\omega' \in \Omega} i_\omega(\omega') u(\omega', y_k(\omega')) \leq \sum_{\omega' \in \Omega} q^*(\omega') u(\omega', y^*(\omega'))$. From continuity, it then follows that $\sum_{\omega' \in \Omega} i_\omega(\omega') u(\omega', \overline{y}(\omega')) \leq \sum_{\omega' \in \Omega} q^*(\omega') u(\omega', y^*(\omega'))$. Then it in turn follows from representation that $(q^*, y^*) \succeq (i_\omega, \overline{y})$. Hence the set $\{y \in Y | (q^*, y^*) \succeq (i_\omega, y)\}$ is closed.

We can then conclude from steps (i)-(iii) that axioms $A2, A3$, and $A6$ all hold.

Step 2C: Show that axioms $A1, A4$, and $A5$ all hold.

If $\#\Omega = 1$, this follows from step 2A. If $\#\Omega \geq 2$, then it follows from Theorem 1.

Step 3: Show that the moreover remark holds.

This is an immediate consequence of Theorem 1.

Q.E.D.

4.6 Lemma 1

4.6.1 Some Preliminary Results

Lemma 4 (Mixture Monotonicity)**.** *Let $y \in Y$ be fixed. Then for any $q_1, q_2 \in \Delta(\Omega)$, and any $\alpha, \beta \in [0,1]$, where $(q_1, y) \succ (q_2, y)$ and $\alpha < \beta$, $(\beta q_1 + (1-\beta) q_2, y) \succ (\alpha q_1 + (1-\alpha) q_2, y)$.*

Proof of Lemma 4: (i) Suppose $\alpha = 0$. Note that $q_1 = \beta q_1 + (1-\beta) q_1$ and $q_2 = \beta q_2 + (1-\beta) q_2$ obviously. Now, by $A5$, $[(q_1, y) \succ (q_2, y)] \Rightarrow [(\beta q_1 + (1-\beta) q_1, y) \succ (\beta q_1 + (1-\beta) q_2, y)]$ as we have βq_1 on both sides. But by $A5$ again, $(\beta q_1 + (1-\beta) q_2, y) \succ (\beta q_2 + (1-\beta) q_2, y)$ as we now have $(1-\beta)$ in common on both sides. But note that this implies $(\beta q_1 + (1-\beta) q_2, y) \succ (q_2, y) = (\alpha q_1 + (1-\alpha) q_2, y)$ when $\alpha = 0$, and we are done. (ii) Suppose $\alpha > 0$. Now recall from (i) that $(\beta q_1 + (1-\beta) q_2, y) \succ (q_2, y)$. Thus defining $q_3 = \beta q_1 + (1-\beta) q_2$, then $(q_3, y) \succ (q_2, y)$. Now, define $\gamma = \frac{\alpha}{\beta}$. Then $(\gamma q_3 + (1-\gamma) q_3, y) \succ (q_2, y)$. But as $(q_3, y) \succ (q_2, y)$, then by $A5$, $(\gamma q_3 + (1-\gamma) q_3, y) \succ (\gamma q_3 + (1-\gamma) q_2, y)$ where γq_3 is in common on both sides. Or, by definition of q_3, $(\gamma q_3 + (1-\gamma) q_3, y) \succ (\gamma(\beta q_1 + (1-\beta) q_2) + (1-\gamma) q_2, y)$. Then rearranging, $(\gamma q_3 + (1-\gamma) q_3, y) \succ_{E'}^{i,t^i} (\gamma \beta q_1 + (1-\beta\gamma) q_2, y)$. But by the definition of γ, $\gamma\beta = \alpha$, thus $(\gamma q_3 + (1-\gamma) q_3, y) \succ (\alpha q_1 + (1-\alpha) q_2, y)$. But as $q_3 = \gamma q_3 + (1-\gamma) q_3 = \beta q_1 + (1-\beta) q_2$ by definition, then $(\beta q_1 + (1-\beta) q_2, y) \succ (\alpha q_1 + (1-\alpha) q_2, y)$. Q.E.D.

Lemma 5 (Unique Solvability)**.** *Let $y_1, y_2 \in Y$. If $q_1, q_2, q_3 \in \Delta(\Omega)$ such that $(q_1, y_1) \succeq (q_2, y_2) \succeq (q_3, y_1)$ and $(q_1, y_1) \succ (q_3, y_1)$ then there exists a unique $\alpha^* \in [0,1]$ such that $(q_2, y_2) \sim (\alpha^* q_1 + (1-\alpha^*) q_3, y_1)$.*

Proof of Lemma 5: (i) If $(q_1, y_1) \sim (q_2, y_2)$, then $\alpha^* = 1$ and we are done. (ii) If $(q_2, y_2) \sim (q_3, y_1)$, then $\alpha^* = 0$ and we are done. (iii) if $(q_1, y_1) \succ (q_2, y_2) \succ (q_3, y_1)$, then define the set $Q = \{\alpha \in [0,1] | (q_2, y_2) \succeq (\alpha q_1 + (1-\alpha) q_3, y_1)\}$. This set is nonempty because $\alpha = 0$ is an element of it and it is bounded above by $\alpha = 1$. Thus there is a supremum of Q.

Let $\alpha^* := \sup Q$. We claim that $(q_2, y_2) \sim (\alpha^* q_1 + (1-\alpha^*)q_3, y_1)$. Indeed, suppose not. Since $(q_2, y_2) \succeq (\alpha q_1 + (1-\alpha)q_3, y_1)$, we must then have $(q_1, y_1) \succ (q_2, y_2) \succ (\alpha^* q_1 + (1-\alpha^*)q_3, y_1)$. Then by $A4$, there is a $\beta \in (0,1)$ such that $(q_2, y_2) \succ (\beta(\alpha^* q_1 + (1-\alpha^*)q_3) + (1-\beta)q_1, y_1)$. Or rearranging, $(q_2, y_2) \succ ((1-\beta(1-\alpha^*))q_1 + \beta(1-\alpha^*)q_3, y_1)$. But as $\beta(1-\alpha^*) < (1-\alpha^*)$, then $(1-\beta(1-\alpha^*)) > \alpha^*$. But then α^* is not a supremum of Q. A contradiction. Consequently, it must be that $(q_2, y_2) \sim (\alpha^* q_1 + (1-\alpha^*)q_3, y_1)$. Finally, by Lemma 1, α^* is unique. Q.E.D.

Let A be a subset of $\Delta(\Omega) \times Y$.

Lemma 6 (Order-Preservation). *If a function f represents the preferences on A and is affine, then $g = a + bf$ where $b > 0$ also (i) represents the preferences on A and (ii) is affine. Furthermore, if the function f is onto the interval $[f(q_2, y_2), f(q_1, y_1)]$ then (iii) the function g is onto the interval $[g(q_2, y_2), g(q_1, y_1)]$.*

Proof of Lemma 6: For any $(q, y), (q', y') \in A$, $[(q, y) \succeq (q', y')] \Leftrightarrow [f(q, y) \geq f(q', y')]$ by the representation of f. Thus, if $b > 0$, then this implies that $[a + bf(q, y) \geq a + bf(q', y')] \Leftrightarrow [(q, y) \succeq (q', y')]$. Thus $[g(q, y) \geq g(q', y')] \Leftrightarrow [(q, y) \succeq (q', y')]$ by definition. (ii) As f is affine, it follows that for any $(q, y), (q', y) \in A$, $f(\alpha q + (1-\alpha)q', y) = \alpha f(q, y) + (1-\alpha)f(q', y)$ whenever $(\alpha q + (1-\alpha)q', y) \in A$. Now, by definition, $g(\alpha q + (1-\alpha)q', y) = a + bf(\alpha q + (1-\alpha)q', y) = a + b[\alpha f(q, y) + (1-\alpha)f(q', y)] = \alpha a + (1-\alpha)a + b\alpha f(q, y) + b(1-\alpha)f(q', y) = \alpha[a + bf(q, y)] + (a-\alpha)[a + bf(q', y)] = \alpha g(q, y) + (1-\alpha)g(q', y)$. To see that g is onto the interval $[g(q_2, y_2), g(q_1, y_1)]$ whenever f is onto the interval $[f(q_2, y_2), f(q_1, y_1)]$, pick any $\alpha \in [g(q_2, y_2), g(q_1, y_1)]$ and define $\beta := \frac{\alpha - a}{b}$. We note that:

$$\begin{array}{rcl} f(q_2, y_2) & = & \frac{g(q_2, y_2) - a}{b} \\ & \leq & \frac{\alpha - a}{b} \\ & \leq & \frac{g(q_1, y_1) - a}{b} \\ & = & f(q_1, y_1) \end{array}$$

Hence $\beta \in [f(q_2, y_2), f(q_1, y_1)]$. Since f is onto the interval $[f(q_2, y_2), f(q_1, y_1)]$, it follows that there exists $(q, y) \in A$ such that $f(q, y) = \beta$. Hence it follows from the definition of g that $g(q, y) = a + b * \beta = a + b * \frac{(\alpha - a)}{b} = \alpha$. Q.E.D.

4.6.2 Representation and affinity for mixture intervals

Consider any $y_1 \in Y$, and any $q_1, q_2 \in \Delta(\Omega)$ such that $(q_1, y_1) \succ (q_2, y_1)$. Define $I := \{(q, y) \in \Delta(\Omega) \times Y | (q_1, y_1) \succeq_{E'}^{i,t^i} (q, y) \succeq (q_2, y_1)\}$. For each $(q, y) \in I$, define $f(q, y)$ as a number such that $(q, y) \sim (f(q, y)q_1 + (1 - f(p))q_2, y_1)$. If the assumptions of Lemma 2 are satisfied, then such a $f(q, y)$ exists and is unique. We now make two claims:

Claim 3 (Representation on I). *$f(.)$ represents preferences on I, i.e., for all $(q, y), (q, y') \in I$, $f(q, y) \geq f(q', y')$ if an only if $(f(q, y)q_1 + (1 - f(q, y))q_2, y_1) \succeq (f(q', y')q_1 + (1 - f(q', y'))q_2, y_1)$.*

Proof or Claim 1: Consider that by mixture monotonicity (Lemma 1), $(q_1, y_1) \succ (q_2, y_1)$ and $f(q, y) \geq f(q', y')$ implies that $(f(q, y)q_1 + (1 - f(q, y))q_2, y_1) \succeq (f(q', y')q_1 + (1 - f(q', y'))q_2, y_1)$. But by the definition of $f(q, y)$ and $f(q', y')$, (i.e., $(q, y) \sim (f(q, y)q_1 + (1 - f(q, y))q_2, y_1)$ and $(q', y') \sim (f(q', y')q_1 + (1 - f(q', y'))q_2, y_1)$), we can note immediately by transitivity $A3$ that this implies that $(q, y) \succeq (q', y')$. The same argument works in reverse. Thus, $[f(q, y) \geq f(q', y')] \Leftrightarrow [(q, y) \succeq (q', y')]$, i.e., $f(.)$ represents the preferences on I, and we are done. Q.E.D.

Claim 4 (Affinity on I)**.** *Let $y \in Y$. Then $f(., y)$ is affine for all $q, q' \in \Delta(\Omega)$ such that $(q, y), (q', y) \in I$, i.e. $f(\alpha q + (1 - \alpha)q', y) = \alpha f(q, y) + (1 - \alpha)f(q', y)$. The function $f(.)$ is furthermore onto the interval* $[f(q_2, y_1), f(q_1, y_1)]$.

Proof of Claim 2: Consider any $y \in Y$ and any $q, q' \in \Delta(\Omega)$ such that $(q, y), (q', y) \in I$ and define $q'' = \alpha q + (1 - \alpha)q'$. We note that it follows from the definition of $f(.)$ and $A5$ that $(\alpha q + (1 - \alpha)q', y) \sim (\alpha(f(q, y)q_1 + (1 - f(q, y))q_2) + (1 - \alpha)(f(q', y)q_1 + (1 - f(q', y))q_2), y_1)$. Rearranging, we have $(\alpha q + (1 - \alpha)q', y) \sim ([\alpha f(q, y) + (1 - \alpha)f(q', y)]q_1 + [1 - [\alpha f(q, y) + (1 - \alpha)f(q', y)]]q_2, y_1)$. From mixture monotonicity (Lemma 1), we have $(q_1, y_1) \succeq ([\alpha f(q, y) + (1 - \alpha)f(q', y)]q_1 + [1 - [\alpha f(q, y) + (1 - \alpha)f(q', y)]]q_2, y_1) \succeq (q_2, y_1)$. From transitivity, $A3$, it then follows that $(q_1, y_1) \succeq (\alpha q + (1 - \alpha)q', y) \succeq (q_2, y_1)$. Hence $(q'', y) \in I$ for any $\alpha \in (0, 1)$. Note from above that $(q'', y) \sim ([\alpha f(q, y) + (1 - \alpha)f(q', y)]q_1 + [1 - [\alpha f(q, y) + (1 - \alpha)f(q', y)]]q_2, y_1)$. By the unique solvability (Lemma 2), it must be that $\alpha^* := f(q'', y) = [\alpha f(q, y) + (1 - \alpha)f(q', y)]$, or by the definition of q'', $f(\alpha q + (1 - \alpha)q', y) = \alpha f(q, y) + (1 - \alpha)f(q', y)$. This is the definition of affinity. To see that the function is onto, pick any $\alpha \in [f(q_2, y_1), f(q_1, y_1)]$. We note that α is actually in the interval $[0, 1]$, since it follows from how the function was defined that $f(q_2, y_1) = 0$ and $f(q_1, y_1) = 1$. Then $f(\alpha q_1 + (1 - \alpha)q_2, y_1) = \alpha$ by the definition of f. Q.E.D.

4.6.3 Properties of intersections and unions of intervals

Consider any $y_1, y_2, y_2', \overline{y}, \underline{y} \in Y$, and any $q_1, q_1', q_2, q_2', \overline{q}, \underline{q} \in \Delta(\Omega)$ such that:

(i) $(q_1, y_1) \succeq (\overline{q}, \overline{y}) \succ (\underline{q}, \underline{y}) \succeq (q_1', y_1)$; and

(ii) $(q_2, y_2) \succeq (\overline{q}, \overline{y}) \succ (\underline{q}, \underline{y}) \succeq (q_2', y_2')$.

Define I_1, and I_2 respectively by $I_1 := \{(q, y) \in \Delta(\Omega) \times Y | (q_1, y_1) \succeq (q, y) \succeq (q_1', y_1)\}$ and $I_2 := \{(q, y) \in \Delta(\Omega) \times Y | (q_2, y_2) \succeq (q, y) \succeq (q_2', y_2')\}$.

Let f_1 and f_2 be affine representations of the preferences on I_1 and I_2 respectively satisfying the properties that (iii) $f_1(\overline{q}, \overline{y}) = f_2(\overline{q}, \overline{y})$; and (iv) $f_1(\underline{q}, \underline{y}) = f_2(\underline{q}, \underline{y})$. We make the following claim.

Claim 5 (Coincidence on $I_1 \cap I_2$)**.** *f_1 and f_2 coincides on $I_1 \cap I_2$.*

Proof of Claim 4: Let $(q, y) \in I_1 \cap I_2$. Since $(q, y) \in I_1$, it follows from unique solvability and mixture monotonicity that there exists uniquely $\overline{\alpha}, \underline{\alpha}$ $(\overline{\alpha} > \underline{\alpha})$, and $\alpha \in [0, 1]$ such that

(a) $(\overline{\alpha}q_1+(1-\overline{\alpha})q_1', y_1) \sim (\overline{q},\overline{y})$; (b) $(\underline{\alpha}q_1+(1-\underline{\alpha})q_1', y_1) \sim (\underline{q},\underline{y})$; and (c) $(\alpha q_1+(1-\alpha)q_1', y_1) \sim (q,y)$. We now consider each of the three possible cases:

Case 1: $(\overline{\alpha}q_1 + (1-\overline{\alpha})q_1', y_1) \succeq (\alpha q_1 + (1-\alpha)q_1', y_1) \succeq (\underline{\alpha}q_1 + (1-\underline{\alpha})q_1', y_1)$. Then by unique solvability, there exists $\alpha^* \in [0,1]$ such that $(\alpha q_1 + (1-\alpha)q', y_1) \sim (\alpha^*(\overline{\alpha}q_1 + (1-\overline{\alpha})q_1') + (1-\alpha^*)(\underline{\alpha}q_1 + (1-\underline{\alpha})q_1'), y_1)$. It then follows that:

$$\begin{aligned}
f_1(q,y) &= f_1(\alpha q_1 + (1-\alpha)q_1', y_1) && \text{(by representation and (c))}\\
&= f_1(\alpha^*(\overline{\alpha}q_1 + (1-\overline{\alpha})q_1') \\
&\quad +(1-\alpha^*)(\underline{\alpha}q_1 + (1-\underline{\alpha})q_1'), y_1) && \text{(by representation and definition of } \alpha^*\text{)}\\
&= \alpha^* f_1(\overline{\alpha}q_1 + (1-\overline{\alpha})q_1', y_1) \\
&\quad +(1-\alpha^*) f_1(\underline{\alpha}q_1 + (1-\underline{\alpha})q_1', y_1) && \text{(by affinity of } f_1\text{)}\\
&= \alpha^* f_1(\overline{q},\overline{y}) + (1-\alpha^*) f_1(\underline{q},\underline{y}) && \text{(by representation and definitions of } \overline{\alpha} \text{ and } \underline{\alpha}\text{)}\\
&= \alpha^* f_2(\overline{q},\overline{y}) + (1-\alpha^*) f_2(\underline{q},\underline{y}) && \text{(by (iii) and (iv))}\\
&= \alpha^* f_2(\overline{\alpha}q_1 + (1-\overline{\alpha})q_1', y_1) && \text{(by representation and definitions}\\
&\quad +(1-\alpha^*) f_2(\underline{\alpha}q_1 + (1-\underline{\alpha})q_1', y_1) && \text{of } \overline{\alpha} \text{ and } \underline{\alpha}\text{)}\\
&= f_2(\alpha^*(\overline{\alpha}q_1 + (1-\overline{\alpha})q_1') \\
&\quad +(1-\alpha^*)(\underline{\alpha}q_1 + (1-\underline{\alpha})q_1'), y_1) && \text{(by affinity of } f_2\text{)}\\
&= f_2(\alpha q_1 + (1-\alpha)q_1', y_1) && \text{(by representation and definition of } \alpha^*\text{)}\\
&= f_2(q,y) && \text{(by representation and (c))}
\end{aligned}$$

Case 2: $(\alpha q_1 + (1-\alpha)q_1', y_1) \succ (\overline{\alpha}q_1 + (1-\overline{\alpha})q_1', y_1) \succ (\underline{\alpha}q_1 + (1-\underline{\alpha})q_1', y_1)$. Then again by unique solvability, there exists $\alpha^* \in (0,1)$ such that $(\overline{\alpha}q_1 + (1-\overline{\alpha})q', y_1) \sim (\alpha^*(\alpha q_1 + (1-\alpha)q_1') + (1-\alpha^*)(\underline{\alpha}q_1 + (1-\underline{\alpha})q_1'), y_1)$. It then follows that:

$$\begin{aligned}
f_1(\overline{q},\overline{y}) &= f_1(\overline{\alpha}q_1 + (1-\overline{\alpha})q', y_1) && \text{(by representation and (a))}\\
&= f_1(\alpha^*(\alpha q_1 + (1-\alpha)q_1') \\
&\quad +(1-\alpha^*)(\underline{\alpha}q_1 + (1-\underline{\alpha})q_1'), y_1) && \text{(by representation and definition of } \alpha^*\text{)}\\
&= \alpha^* f_1(\alpha q_1 + (1-\alpha)q_1', y_1) \\
&\quad +(1-\alpha^*) f_1(\underline{\alpha}q_1 + (1-\underline{\alpha})q_1', y_1) && \text{(by affinity of } f_1\text{)}\\
&= \alpha^* f_1(q,y) + (1-\alpha^*) f_1(\underline{q},\underline{y}) && \text{(by representation, (b), and (c))}
\end{aligned}$$

Rearranging, we have

$$f_1(q,y) = (\frac{1}{\alpha^*})[f_1(\overline{q},\overline{y}) - (1-\alpha^*) f_1(\underline{q},\underline{y})].$$

By the same token, we have

$$\begin{aligned}
f_2(\overline{q},\overline{y}) &= f_2(\overline{\alpha}q_1 + (1-\overline{\alpha})q', y_1) && \text{(by representation and (a))}\\
&= f_2(\alpha^*(\alpha q_1 + (1-\alpha)q_1') \\
&\quad +(1-\alpha^*)(\underline{\alpha}q_1 + (1-\underline{\alpha})q_1'), y_1) && \text{(by representation and definition of } \alpha^*\text{)}\\
&= \alpha^* f_2(\alpha q_1 + (1-\alpha)q_1', y_1) \\
&\quad +(1-\alpha^*) f_2(\underline{\alpha}q_1 + (1-\underline{\alpha})q_1', y_1) && \text{(by affinity of } f_2\text{)}\\
&= \alpha^* f_2(q,y) + (1-\alpha^*) f_2(\underline{q},\underline{y}) && \text{(by representation, (b), and (c))}
\end{aligned}$$

Again, after rearranging, we have

$$f_2(q, y) = (\frac{1}{\alpha^*})[f_2(\overline{q}, \overline{y}) - (1 - \alpha^*)f_2(\underline{q}, \underline{y})]$$

.

From the two above rearranged equalities, it then follows that:

$$\begin{aligned} f_1(q, y) &= (\tfrac{1}{\alpha^*})[f_1(\overline{q}, \overline{y}) - (1 - \alpha^*)f_1(\underline{q}, \underline{y})] && \text{(from first equality)} \\ &= (\tfrac{1}{\alpha^*})[f_2(\overline{q}, \overline{y}) - (1 - \alpha^*)f_2(\underline{q}, \underline{y})] && \text{(from (iii) and (iv)} \\ &= f_2(q, y) && \text{(from second equality)} \end{aligned}$$

as required.

Case 3: $(\overline{\alpha}q_1 + (1 - \overline{\alpha})q_1', y_1) \succ (\underline{\alpha}q_1 + (1 - \underline{\alpha})q_1', y_1) \succ (\alpha q_1 + (1 - \alpha)q_1', y_1)$. Then again by unique solvability, there exists $\alpha^* \in (0, 1)$ such that $(\underline{\alpha}q_1 + (1 - \underline{\alpha})q', y_1) \sim (\alpha^*(\overline{\alpha}q_1 + (1 - \overline{\alpha})q_1') + (1 - \alpha^*)(\alpha q_1 + (1 - \alpha)q_1'), y_1)$. It then follows that:

$$\begin{aligned} f_1(\underline{q}, \underline{y}) &= f_1(\underline{\alpha}q_1 + (1 - \underline{\alpha})q', y_1) && \text{(by representation and (b))} \\ &= f_1(\alpha^*(\overline{\alpha}q_1 + (1 - \overline{\alpha})q_1') \\ &\quad +(1 - \alpha^*)(\alpha q_1 + (1 - \alpha)q_1'), y_1) && \text{(by representation and definition of } \alpha^*) \\ &= \alpha^* f_1(\overline{\alpha}q_1 + (1 - \overline{\alpha})q_1', y_1) \\ &\quad +(1 - \alpha^*)f_1(\alpha q_1 + (1 - \alpha)q_1', y_1) && \text{(by affinity of } f_1) \\ &= \alpha^* f_1(\overline{q}, \overline{y}) + (1 - \alpha^*)f_1(q, y) && \text{(by representation, (a), and (c))} \end{aligned}$$

Rearranging, we have

$$f_1(q, y) = \frac{1}{(1 - \alpha^*)}[f_1(\underline{q}, \underline{y}) - \alpha^* f_1(\overline{q}, \overline{y})].$$

By the same token, we have

$$\begin{aligned} f_2(\underline{q}, \underline{y}) &= f_2(\underline{\alpha}q_1 + (1 - \underline{\alpha})q', y_1) && \text{(by representation and (b))} \\ &= f_2(\alpha^*(\overline{\alpha}q_1 + (1 - \overline{\alpha})q_1') \\ &\quad +(1 - \alpha^*)(\alpha q_1 + (1 - \alpha)q_1'), y_1) && \text{(by representation and definition of } \alpha^*) \\ &= \alpha^* f_2(\overline{\alpha}q_1 + (1 - \overline{\alpha})q_1', y_1) \\ &\quad +(1 - \alpha^*)f_2(\alpha q_1 + (1 - \alpha)q_1', y_1) && \text{(by affinity of } f_2) \\ &= \alpha^* f_2(\overline{q}, \overline{y}) + (1 - \alpha^*)f_2(q, y) && \text{(by representation, (a), and (c))} \end{aligned}$$

Again, after rearranging, we have

$$f_2(q, y) = \frac{1}{(1 - \alpha^*)}[f_2(\underline{q}, \underline{y}) - \alpha^* f_2(\overline{q}, \overline{y})].$$

From the two above rearranged equalities, it then follows that:

$$\begin{aligned} f_1(q, y) &= \tfrac{1}{(1-\alpha^*)}[f_1(\underline{q}, \underline{y}) - \alpha^* f_1(\overline{q}, \overline{y})] && \text{(from first equality)} \\ &= \tfrac{1}{(1-\alpha^*)}[f_2(\underline{q}, \underline{y}) - \alpha^* f_2(\overline{q}, \overline{y})] && \text{(from (iii) and (iv)} \\ &= f_2(q, y) && \text{(from second equality)} \end{aligned}$$

as required. Q.E.D.

Consider any $(q_1, y_1), , (q_1', y_1'), (q_2, y_2), (q_2', y_2'), (\overline{q}, \overline{y}), (\underline{q}, \underline{y}) \in D$ such that:

(i) $(q_1, y_1) \succeq (\overline{q}, \overline{y}) \succ (\underline{q}, \underline{y}) \succeq (q_1', y_1')$; and

(ii) $(q_2, y_2) \succeq (\overline{q}, \overline{y}) \succ (\underline{q}, \underline{y}) \succeq (q_2', y_2')$.

Define I_1, and I_2 respectively by $I_1 := \{(q, y) \in \Delta(\Omega) \times Y | (q_1, y_1) \succeq (q, y) \succeq (q_1', y_1')\}$ and $I_2 := \{(q, y) \in \Delta(\Omega) \times Y | (q_2, y_2) \succeq (q, y) \succeq (q_2', y_2')\}$.

Let f_1 and f_2 be affine representations of the preferences on I_1 and I_2 respectively with the further property that f_1 coincides with f_2 on $I_1 \cap I_2$. We make the following claim.

Claim 6 (Representation on $I_1 \cup I_2$)**.** *The function f that coincides with f_1 on I_1 and f_2 on I_2 represents preferences on $I_1 \cup I_2$.*

Proof of Claim 5: Assume without loss of generality that $(q_1, y_1) \succeq (q_2, y_2)$. We consider all of three possible cases. (i) If $(q_2', y_2') \succeq (q_1', y_1')$ then $I_2 \subset I_1$. Thus $I_1 \cup I_2 = I_1$, and $f(q, y) = f_1(q, y)$ for all $(q, y) \in I_1 \cup I_2$. Hence, we are done since f_1 represents preferences on I_1. (ii) If $(q_1', y_1') \succeq (q_2', y_2')$, and $(q_2, y_2) \succeq (q_1, y_1)$, then the present assumption implies that $I_1 \subset I_2$. Thus $I_1 \cup I_2 = I_2$, and $f(q, y) = f_2(q, y)$ for all $(q, y) \in I_1 \cup I_2$. Hence, we are done since f_2 represents preferences on I_2.

(iii) $(q_1, y_1) \succ (q_2, y_2) \succeq (\overline{q}, \overline{y}) \succ (\underline{q}, \underline{y}) \succeq (q_1', y_1') \succ (q_2', y_1')$.

Step 1: Show that $[(q, y) \succeq (q', y')] \Rightarrow [f(q, y) \geq f(q', y')]$. Pick any $(q, y), (q', y') \in I_1 \cup I_2$ such that $(q, y) \succeq (q', y')$. (1a) If $(q_2, y_2) \succeq (q, y)$ then (q, y) and (q', y') are both in I_2. Hence it follows from the assumption that f_2 represents preferences on I_2 that $f_2(q, y) \geq f_2(q', y')$. But from the definition of f it then it also follows that $f(q, y) \geq f(q', y')$. (1b) If $(q', y') \succeq (q_1', y_1')$, then (q, y) and (q', y') are both in I_1. Hence from the assumption that f_1 represents preferences on I_1 it follows that $f_1(q, y) \geq f_1(q', y')$. But from the definition of f it then also follows that $f(q, y) \geq f(q', y')$. (1c) $(q, y) \succeq (q_2, y_2) \succeq (\overline{q}, \overline{y}) \succ (\underline{q}, \underline{y}) \succeq (q_1', y_1') \succeq (q', y')$. Then $(q, y) \in I_1$, and $q', y') \in I_2$. But then it follows that:

$$\begin{aligned} f(q, y) &= f_1(q, y) & \text{(from definition of } f\text{)} \\ &> f_1(\underline{q}, \underline{y}) & \text{(from properties of } f_1\text{)} \\ &= f_2(\underline{q}, \underline{y}) & \text{(by assumption)} \\ &> f_2(q', y') & \text{(from properties of } f_2\text{)} \\ &= f(q', y') & \text{(from definition of } f\text{)} \end{aligned}$$

as required.

Step 2: Show that $[f(q, y) \geq f(q', y_1')] \Rightarrow [(q, y) \in I_1]$.

Pick any $(q, y) \in I_1 \cup I_2$ such that $f(q, y) \geq f(q', y_1')$. Clearly, it suffices to show that $[(q, y) \in I_2] \Rightarrow [(q, y) \in I_1]$. We note that $[(q, y) \in I_2] \Leftrightarrow [(q_2, y_2) \succeq (q, y) \succeq (q_2', y_2')]$. Since f_2 represents preferences on I_2, and f coincides with f_2 on I_2, it follows that $[f(q_2, y_2) \geq f(q, y) \geq f(q_2', y_2')]$. Likewise, we note that since f_1 represents preferences on I_1, and f coincides with f_1 on I_1, it follows from the present assumption that $f(q_2, y_2) \geq f(q_1', y_1') \geq$

$f(q_2', y_2')$. Combining these with the present assumption that $f(q, y) \geq f(q', y_1')$, it follows that $f(q_2, y_2) \geq f(q, y) \geq f(q_1', y_1') \geq f(q_2', y_2')$. But then it follows from the facts that f_2 represents preferences on I_2 and that f coincides with f_2 on I_2 that $(q_2, y_2) \succeq (q, y) \succeq (q_1', y_1')$. Combining this with the fact that $(q_1, y_1) \succeq (q_2, y_2)$ then implies that $(q_1, y_1) \succeq (q, y) \succeq (q_1', y_1')$. Hence $(q, y) \in I_1$.

Step 3: Show that $[f(q_2, y_2) \geq f(q, y)] \Rightarrow [(q, y) \in I_2]$.

Pick any $(q, y) \in I_1 \cup I_2$ such that $f(q_2, y_2) \geq f(q, y)$. Clearly, it suffices to show that $[(q, y) \in I_1] \Rightarrow [(q, y) \in I_2]$. We note that $[(q, y) \in I_1] \Leftrightarrow [(q_1, y_1) \succeq (q, y) \succeq (q_1', y_1')]$. Since f_1 represents preferences on I_1, and f coincides with f_1 on I_1, it follows that $[f(q_1, y_1) \geq f(q, y) \geq f(q_1', y_1')]$. Likewise, we note that since f_2 represents preferences on I_2, and f coincides with f_2 on I_2, it follows from the present assumption that $f(q_1, y_1) \geq f(q_2, y_2) \geq f(q_1', y_1')$. Combining these with the present assumption that $f(q_2, y_2) \geq f(q, y)$, it follows that $f(q_1, y_1) \geq f(q_2, y_2 \geq f(q, y) \geq f(q_1', y_1')$. But then it follows from the facts that f_1 represents preferences on I_1 and that f coincides with f_1 on I_1 that $(q_2, y_2) \succeq (q, y) \succeq (q_1', y_1')$. Combining this with the fact that $(q_1', y_1') \succeq (q_2', y_2')$ then implies that $(q_2, y_2) \succeq (q, y) \succeq (q_2', y_2')$. Hence $(q, y) \in I_2$.

Step 4: Show that $[f(q, y) \geq f(q', y')] \Rightarrow [(q, y) \succeq (q', y')]$.

Pick any $(q, y), (q', y') \in I_1 \cup I_2$ such that $f(q, y) \geq f(q', y')$. (4a) If $f(q_2, y_2) \geq f(q, y)$ then we know from step 3 that both (q, y), and (q', y') are in I_2. But then we are done since f_2 represents preferences on I_2 and f coincides with f_2 on I_2. (4b) If $f(q', y') \geq f(q_1', y_1')$ then we know from step 2 that both (q, y), and (q', y') are in I_1. But then we are done since f_1 represents preferences on I_1 and f coincides with f_1 on I_1. (4c) $f(q, y) > f(q_2, y_2) > f(q_1', y_1') > f(q', y')$. Then we know from step 2 that $(q, y) \in I_1$, and from step 3 that $(q', y') \in I_2$. Since f_1 represents preferences on I_1 and f coincides with f_1 on I_1, we have $(q, y) \succ (q_2, y_2)$. Likewise, since f_2 represents preferences on I_2 and f coincides with f_2 on I_2, we have $(q_1', y_1') \succ (q', y')$. But under the present assumptions it then follows that $(q, y) \succ (q_2, y_2) \succ (q_1', y_1') \succ (q', y')$ as required. Q.E.D.

Claim 7 (Affinity on $I_1 \cup I_2$)**.** *Let $y \in Y$. Then the function f that coincides with f_1 on I_1 and f_2 on I_2 is affine on $I_1 \cup I_2$ for any $q, q' \in \Delta(\Omega)$ such that $(q, y), (q', y) \in I_1 \cup I_2$.*

Proof of Claim 6: Pick any $y \in Y$, and any $q, q' \in \Delta(\Omega)$ such that $(q, y), (q', y) \in I_1 \cup I_2$. Without loss of generality, assume that $(q, y) \succeq (q', y)$. (i) If (q, y) and (q', y) are both in I_1, then we are done since f_1 is affine and f coincides with f_1 on I_1. (ii) If (q, y) and (q', y) are both in I_2, then we are done since f_2 is affine and f coincides with f_2 on I_2. (iii) $(q, y) \notin I_1$, $(q', y) \notin I_2$. It follows from completeness, transitivity, and the present assumptions that $(q_2, y_2) \succeq (q, y) \succ (q_1, y_1) \succeq (\overline{q}, \overline{y}) \succ (\underline{q}, \underline{y}) \succeq (q_2', y_2) \succ (q', y) \succeq (q_1', y_1)$. Define $I_3 := \{(q'', y'') \in D | (q, y) \succeq (q'', y'') \succeq (q', y)\}$ and using unique solvability , define a function $f_3 : I_3 \to \mathbf{R}$ by $f_3(q'', y'') = \alpha^*_{q'', y''}$ where $\alpha^*_{q'', y''}$ is the unique element of $[0, 1]$ for which $(\alpha^*_{q'', y''} q + (1 - \alpha^*_{q'', y''}) q', y) \sim (q'', y'')$. From Claim 1, we know that f_3 represents preferences on I_3, from Claim 2, we know that f_3 is affine, and from Claim 3, we know that for any a, b where $(b > 0)$, the function $g := a + bf$ both represents preferences on I_3 and is affine. Let $a = f(\overline{q}, \overline{y}) - \frac{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]}{[f_3(\overline{q}, \overline{y}) - f_3(\underline{q}, \underline{y})]} * f_3(\overline{q}, \overline{y})$, and let $b = \frac{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]}{[f_3(\overline{q}, \overline{y}) - f_3(\underline{q}, \underline{y})]} > 0$. Then we have, on the one hand that $g(\overline{q},\overline{y}) = f(\overline{q},\overline{y}) - \frac{[f(\overline{q},\overline{y})-f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y})-f_3(\underline{q},\underline{y})]} * f_3(\overline{q},\overline{y}) + \frac{[f(\overline{q},\overline{y})-f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y})-f_3(\underline{q},\underline{y})]} * f_3(\overline{q},\overline{y}) = f(\overline{q},\overline{y}) = f_1(\overline{q},\overline{y}) = f_2(\overline{q},\overline{y})$. Likewise, we have that $g(\underline{q},\underline{y}) = f(\overline{q},\overline{y}) - \frac{[f(\overline{q},\overline{y})-f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y})-f_3(\underline{q},\underline{y})]} * f_3(\overline{q},\overline{y}) + \frac{[f(\overline{q},\overline{y})-f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y})-f_3(\underline{q},\underline{y})]} * f_3(\underline{q},\underline{y}) = f(\underline{q},\underline{y}) = f_1(\underline{q},\underline{y}) = f_2(\underline{q},\underline{y})$. But then it follows from claim 4 that g (a) coincides with f_1, and hence with f, on $I_3 \cap I_1$; and (b) coincides with f_2, and hence with f, on $I_3 \cap I_2$. But $I_3 \subset I_1 \cup I_2$ implies that $I_3 = I_3 \cap (I_1 \cup I_2) = (I_3 \cap I_1) \cup (I_3 \cap I_2)$. Hence we can conclude that f coincides with g on I_3. But then we are done since g is affine on I_3.

(iv) $(q,y) \notin I_2$, $(q',y) \notin I_1$. It follows from completeness, transitivity, and the present assumptions that $(q_1,y_1) \succeq (q,y) \succ (q_2,y_2) \succeq (\overline{q},\overline{y}) \succ (\underline{q},\underline{y}) \succeq (q'_1,y_1) \succ (q',y) \succeq (q'_2,y_2)$. Define $I_3 := \{(q'',y'') \in D | (q,y) \succeq (q'',y'') \succeq (q',y)\}$ and using unique solvability , define a function $f_3 : I_3 \to \mathbf{R}$ by $f_3(q'',y'') = \alpha^*_{q'',y''}$ where $\alpha^*_{q'',y''}$ is the unique element of $[0,1]$ for which $(\alpha^*_{q'',y''}q + (1-\alpha^*_{q'',y''})q', y) \sim (q'',y'')$. From Claim 1, we know that f_3 represents preferences on I_3, from Claim 2 we know that f_3 is affine, and from Claim 3 we know that for any a, b where $(b > 0)$, the function $g := a + bf$ both represents preferences on I_3 and is affine. Let $a = f(\overline{q},\overline{y}) - \frac{[f(\overline{q},\overline{y})-f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y})-f_3(\underline{q},\underline{y})]} * f_3(\overline{q},\overline{y})$, and let $b = \frac{[f(\overline{q},\overline{y})-f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y})-f_3(\underline{q},\underline{y})]} > 0$. Then we have, on the one hand that $g(\overline{q},\overline{y}) = f(\overline{q},\overline{y}) - \frac{[f(\overline{q},\overline{y})-f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y})-f_3(\underline{q},\underline{y})]} * f_3(\overline{q},\overline{y}) + \frac{[f(\overline{q},\overline{y})-f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y})-f_3(\underline{q},\underline{y})]} * f_3(\overline{q},\overline{y}) = f(\overline{q},\overline{y}) = f_1(\overline{q},\overline{y}) = f_2(\overline{q},\overline{y})$. Likewise, we have that $g(\underline{q},\underline{y}) = f(\overline{q},\overline{y}) - \frac{[f(\overline{q},\overline{y})-f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y})-f_3(\underline{q},\underline{y})]} * f_3(\overline{q},\overline{y}) + \frac{[f(\overline{q},\overline{y})-f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y})-f_3(\underline{q},\underline{y})]} * f_3(\underline{q},\underline{y}) = f(\underline{q},\underline{y}) = f_1(\underline{q},\underline{y}) = f_2(\underline{q},\underline{y})$. But then it follows from claim 4 that g (a) coincides with f_1, and hence with f, on $I_3 \cap I_1$; and (b) coincides with f_2, and hence with f, on $I_3 \cap I_2$. But $I_3 \subset I_1 \cup I_2$ implies that $I_3 = I_3 \cap (I_1 \cup I_2) = (I_3 \cap I_1) \cup (I_3 \cap I_2)$. Hence we can conclude that f coincides with g on I_3. But then we are done since g is affine on I_3. Q.E.D.

4.6.4 Definition and properties of a basis interval

Let A be a subset of $\Delta(\Omega) \times Y$ with the property that $[(q,y),(q',y) \in A] \Rightarrow [\forall \alpha \in (0,1), (\alpha q + (1-\alpha)q', y) \in A]$. Let $(\overline{q},\overline{y}), (\underline{q},\underline{y}) \in A$ satisfy the property that $(\overline{q},\overline{y}) \succeq (\underline{q},\underline{y})$. Define $I^* := \{(q,y) \in A | (\overline{q},\overline{y}) \succeq (q,y) \succeq (\underline{q},\underline{y})\}$.

Definition 4. I^* is a basis interval for A if it satisfies the following three criteria:

(i) there exists an affine function $f^* : I^* \to \mathbf{R}$ that represents the preferences on I^* and is onto the interval $[f^*(\underline{q},\underline{y}), f^*(\overline{q},\overline{y})]$;
(ii) for every $(q_1,y_1) \in A$ such that $(q,y) \succ (\overline{q},\overline{y})$, there exists $(q_1^*,y_1^*), (q_1^{**},y_1^*) \in A$ such that $(q_1^*,y_1^*) \sim (q_1,y_1) \succ (\overline{q},\overline{y}) \succ (q_1^{**},y_1^*) \succeq (\underline{q},\underline{y})$; and
(iii) for every $(q_2,y_2) \in A$ such that $(\underline{q},\underline{y}) \succ (q_2,y_2)$, there exists $(q_2^*,y_2^*), (q_2^{**},y_2^*) \in A$ such that $(\overline{q},\overline{y}) \succeq (q_2^*,y_2^*) \succ (\underline{q},\underline{y}) \succ (q_2,y_2) \sim (q_2^{**},y_2^*)$.

We now turn to a couple of additional claims. Let $I^* := \{(q,y) \in A | (\overline{q},\overline{y}) \succeq (q,y) \succeq (\underline{q},\underline{y})\}$ be a basis interval for A and let f^* be an affine representation of the preferences on I^* such that f^* is onto the interval $[f^*(\underline{q},\underline{y}), f^*(\overline{q},\overline{y})]$. Let $(q_1,y_1), (q'_1,y'_1) \in A$ satisfy the property that $(q_1, y_1) \succeq (\overline{q}, \overline{y}) \succ (\underline{q}, \underline{y}) \succeq (q_1', y_1')$ and let $I := \{(q, y) \in A | (q_1, y_1) \succeq (q, y) \succeq (q_1', y_1')\}$.

Claim 8 (Representation of I^*)**.** *There exists uniquely an affine representation f of the preferences on I such that a) f coincides with f^* on I^*, and b) f is onto the interval $[f(q_1', y_1'), f(q_1, y_1)]$.*

Proof of Claim 7: Define $I_1' := \{(q, y) \in A | (q_1, y_1) \succeq (q, y) \succeq (\underline{q}, \underline{y})\}$. We divide the rest of our proof into two steps.

Step 1: Show that there exists uniquely an affine representation f_1 of the preferences on I_1' such that a) f_1 coincides with f^* on I^*; and b) f_1 is onto the interval $[f_1(\underline{q}, \underline{y}), f_1(q_1, y_1)]$.

(i) If $(q_1, y_1) \sim (\overline{q}, \overline{y})$, then it follows from completeness and transitivity that $I_1' = I^*$. Since f_1 must coincide with f^* on I^*, it follows that the only function satisfying the desired properties is the function for which $f_1(q, y) = f^*(q, y)$ for every $(q, y) \in I^*$.

(ii) If $(q_1, y_1) \succ (\overline{q}, \overline{y})$ we know from the definition of a basis that there exists $(q_1^*, y_1^*), (q_1^{**}, y_1^*) \in A$ such that $(q_1^*, y_1^*) \sim (q_1, y_1) \succ (\overline{q}, \overline{y}) \succ (q_1^{**}, y_1^*) \succeq (\underline{q}, \underline{y})$. Let $I_2' := \{(q, y) \in A | (q_1^*, y_1^*) \succeq (q, y) \succeq (q_1^{**}, y_1^*)\} = \{(q, y) \in A | (q_1, y_1) \succeq (q, y) \succeq (q_1^{**}, y_1^*)\}$. By mixture monotonicity, we know that for every $(q, y) \in I_2'$, there exists uniquely $\alpha_{q,y}^* \in [0, 1]$ such that $(\alpha_{q,y}^* q_1^* + (1 - \alpha_{q,y}^*) q_1^{**}, y_1^*) \sim (q, y)$. Hence we can define a function $g : I_2' \rightarrow \mathbf{R}$ by $g(q, y) = \alpha_{q,y}^*$. From Claims 1 and 2, we know that g is an affine representation of the preferences on I_2' with the further property that it is onto the interval $[g(q_1^{**}, y_1^*), g(q_1^*, y_1^*)] = [g(q_1^{**}, y_1^*), g(q_1, y_1)]$. Let $a = f^*(q_1^{**} y_1^*) - \frac{[f^*(\overline{q},\overline{y}) - f^*(q_1^{**},y_1^*)]}{[g(\overline{q},\overline{y}) - g(q_1^{**},y_1^*)]} * g(q_1^{**}, y_1^*)$, $b = \frac{[f^*(\overline{q},\overline{y}) - f^*(q_1^{**},y_1^*)]}{[g(\overline{q},\overline{y}) - g(q_1^{**},y_1^*)]}$, and define $h := a + bg$. From Claim 3, we know that h is an affine representation of the preferences on I_2' that is onto the interval $[h(q_1^{**}, y_1^*), h(q_1, y_1)]$. We note that $h(\overline{q}, \overline{y}) = f^*(q_1^{**}, y_1^*) - \frac{[f^*(\overline{q},\overline{y}) - f^*(q_1^{**},y_1^*)]}{[g(\overline{q},\overline{y}) - g(q_1^{**},y_1^*)]} * g(q_1^{**}, y_1^*) + \frac{[f^*(\overline{q},\overline{y}) - f^*(q_1^{**},y_1^*)]}{[g(\overline{q},\overline{y}) - g(q_1^{**},y_1^*)]} * g(\overline{q}, \overline{y}) = f^*(\overline{q}, \overline{y})$. Likewise, we have $h(q_1^{**}, y_1^*) = f^*(q_1^{**}, y_1^*) - \frac{[f^*(\overline{q},\overline{y}) - f^*(q_1^{**},y_1^*)]}{[g(\overline{q},\overline{y}) - g(q_1^{**},y_1^*)]} * g(q_1^{**}, y_1^*) + \frac{[f^*(\overline{q},\overline{y}) - f^*(q_1^{**},y_1^*)]}{[g(\overline{q},\overline{y}) - g(q_1^{**},y_1^*)]} * g(q_1^{**}, y_1^*) = f^*(q_1^{**}, y_1^*)$. From Claim 4, it then follows that f^* and h coincide on $I^* \cap I_2'$.

We claim that h is the only affine representation off preferences on I_2' that also coincides with f^* on $I^* \cap I_2'$. Indeed, let h' be another such affine representation. Since $(\overline{q}, \overline{y})$, and $(\underline{q}, \underline{y})$ are both elements of I^*, we must have $h'(\overline{q}, \overline{y}) = h(\overline{q}, \overline{y})$ and $h'(\underline{q}, \underline{y}) = h(\underline{q}, \underline{y})$. But then all the assumptions of claim 4 are satisfied, so h' and h coincide on $I_2' \cap I_2' = I_2'$.Hence $h' = h$.

Let $f_1 : I^* \cup I_2' \rightarrow \mathbf{R}$ be the function that coincides with f^* on I^* and with h on I_2'. From claims 5 and 6, it follows that f_1 is an affine representation of the preferences on $I^* \cup I_2' = I_1'$. We claim that f_1 is the only affine representation of the preferences on I_1' that coincides with f^* on I^*. Indeed, let f_1' be another such representation, Then it must coincide with f_1 on I^*. Since it is affine, the restriction of f_1' to I_2' must then also be an affine representation of the preferences on I_2'. But then we know from above that the restriction must coincide with h (and hence with f_1) on I_2'. But then it coincides with f_1 on $I^* \cup I_2' = I_1'$.

To see that f_1 is also onto the required interval, note that $[f_1(\underline{q}, \underline{y}), f_1(q, y)] = [f^*(\underline{q}, \underline{y}), f^*(\overline{q}, \overline{y})] \cup [h(q_1^{**}, y_1^*), h(q_1, y_1)]$. We know that f^* is onto the interval $[f^*(\underline{q}, \underline{y}), f^*(\overline{q}, \overline{y})]$ and that h is onto the interval $[h(q_1^{**}, y_1^*), h(q_1, y_1)]$. But since f_1 coincides with f^* on I^* and with h on I_2' it must be that f_1 is onto both intervals and hence onto the interval $[f_1(\underline{q}, \underline{y}), f_1(q, y)]$.

Step 2: Show that there exists uniquely an affine representation f of the preferences on I such that a) f coincides with f^* on I^*; and b) f is onto the interval $[f(q_1', y_1'), f(q_1, y_1)]$.

(i) If $(q_1', y_1') \sim (\underline{q}, \underline{y})$, then it follows from completeness and transitivity that $I = I_1'$. Since f must coincide with f_1 on I_1', it hence follows that the only function satisfying the desired properties is the function f for which $f(q, y) = f_1(q, y)$ for every $(q, y) \in I = I_1'$.

(ii) If $(\underline{q}, \underline{y}) \succ (q_1', y_1')$, we know from the definition of a basis that there exists $(q_2^{**}, y_2^*), (q_2^{**}, y_2^*) \in A$ such that $(\overline{q}, \overline{y}) \succeq (q_2^*, y_2^*) \succ (\underline{q}, \underline{y}) \succ (q_1', y_1') \sim (q_2^{**}, y_2^*)$. Let $I_3' := \{(q, y) \in A | (q_2^*, y_2^*) \succeq (q, y) \succeq (q_2^{**}, y_2^*)\} = \{(q, y) \in A | (q_2^*, y_2^*) \succeq (q, y) \succeq (q_1', y_1')\}$. By mixture monotonicity, we know that for every $(q, y) \in I_3'$, there exists uniquely $\alpha_{q,y}^* \in [0, 1]$ such that $(\alpha_{q,y}^* q_2^* + (1 - \alpha_{q,y}^*) q_2^{**}, y_2^*) \sim (q, y)$. Hence we can define a function $g_2 : I_3' \to \mathbf{R}$ by $g_2(q, y) = \alpha_{q,y}^*$. From Claims 1 and 2, we know that g_2 is an affine representation of the preferences on I_3' with the further property that it is onto the interval $[g_2(q_2^{**}, y_2^*), g_2(q_2^*, y_2^*)] = [g_2(q_1', y_2^*), g_2(q_2^*, y_2^*)]$. Let $a = f^*(\underline{q}, \underline{y}) - \frac{[f^*(q_2^*, y_2^*) - f^*(\underline{q}, \underline{y})]}{[g_2(q_2^*, y_2^*) - g_2(\underline{q}, \underline{y})]} * g_2(\underline{q}, \underline{y})$, $b = \frac{[f^*(q_2^*, y_2^*) - f^*(\underline{q}, \underline{y})]}{[g_2(q_2^*, y_2^*) - g_2(\underline{q}, \underline{y})]}$, and define $h_2 := a + b g_2$. From Claim 3, we know that h_2 is an affine representation of the preferences on I_3' that is onto the interval $[h(q_1', y_1'), q_1^*, y_1^*)]$. We note that $h_2(q_2^*, y_2^*) = f^*(\underline{q}, \underline{y}) - \frac{[f^*(q_2^*, y_2^*) - f^*(\underline{q}, \underline{y})]}{[g_2(q_2^*, y_2^*) - g_2(\underline{q}, \underline{y})]} * g_2(\underline{q}, \underline{y}) + \frac{[f^*(q_2^*, y_2^*) - f^*(\underline{q}, \underline{y})]}{[g_2(q_2^*, y_2^*) - g_2(\underline{q}, \underline{y})]} * g_2(q_2^*, y_2^*) == f^*(q_2^*, y_2^*)$. Likewise, we have $h_2(\underline{q}, \underline{y}) = f^*(\underline{q}, \underline{y}) - \frac{[f^*(q_2^*, y_2^*) - f^*(\underline{q}, \underline{y})]}{[g_2(q_2^*, y_2^*) - g_2(\underline{q}, \underline{y})]} * g_2(\underline{q}, \underline{y}) + \frac{[f^*(q_2^*, y_2^*) - f^*(\underline{q}, \underline{y})]}{[g_2(q_2^*, y_2^*) - g_2(\underline{q}, \underline{y})]} * g_2(\underline{q}, \underline{y}) == f^*(\underline{q}, \underline{y})$. From Claim 4, it then follows that f_1 and h_2 coincide on $I_2' \cap I_3'$. We claim that h_2 is the only affine representation of the preferences on I_3' that coincides with f_1 on I_1'. Indeed, let h_2' be another such affine representation. Since h_2' and h_2 both coincide with f_1 on I_1', it follows that $h_2'(\overline{q}, \overline{y}) = h_2(\overline{q}, \overline{y})$ and $h_2'(\underline{q}, \underline{y}) = h_2(\underline{q}, \underline{y})$. But then all the assumptions of claim 4 are satisfied, so h_2' and h_2 coincide on $I_3' \cap I_3' = I_3'$. Hence $h_2' = h_2$.

Let $f : I_2' \cup I_3' \to \mathbf{R}$ be the function that coincides with f_1 on I_2' and with h_2 on I_3'. From claims 5 and 6, it follows that f is an affine representation of the preferences on $I_2' \cup I_3' = I$. Note also that it follows from the facts that f coincides with f_1 on I_2' and f_1 coincides with f^* on I^* that f coincides with f^* on I^*. We claim that f is the only affine representation of preferences on I that also coincides with f^* on I^*. Indeed, let f' be another such representation. Then it must coincide with f^* on I^*. Since f' is affine, the restriction of f' to I_1' must then also be affine and represent the preferences on I_1'. But we know from step 1 that f_1 is the only such representation of the preferences on I_1'. Hence f' must coincide with f_1 on I_1'. Again, by affinity, we have that the restriction of f' to I_3' must be affine on I_3'. But we know from above that h_2 is the only such affine representation of the preferences on I_3' that also coincides with f_1 on I_1'. Hence f' must coincide with h_2 on I_3'. But then f' coincides with f on I.

To see that f is also onto the required interval, note that $[f(q', y'), f(q, y)] = [f_1(\underline{q}, \underline{y}), f_1(q, y)] \cup [h_2(q_1', y_1'), h_2(q_2^*, y_2^*)]$. We know that f_1 is onto the interval $[f_1(\underline{q}, \underline{y}), f_1(q, y)]$ and that h_2 is onto the interval $[h_2(q_1', y_1'), h_2(q_2^*, y_2^*)]$. But since f coincides with f_1 on I_2' and with h_2 on I_3' it must be that f is onto both intervals and hence onto the interval $[f(q', y'), f_1(q, y)]$. Q.E.D.

Let $I^* := \{(q, y) \in A | (\overline{q}, \overline{y}) \succeq (q, y) \succeq (\underline{q}, \underline{y})\}$ be a basis interval for A. Let (q_1, y_1), (q_1', y_1'), (q_2, y_2), $(q_2', y_2') \in A$ satisfy the properties that
(i) $(q_1, y_1) \succeq (\overline{q}, \overline{y}) \succeq (\underline{q}, \underline{y}) \succeq (q_1', y_1')$; and
(ii) $(q_2, y_2) \succeq (\overline{q}, \overline{y}) \succeq (\underline{q}, \underline{y}) \succeq (q_2', y_2')$.

Define $I_1 := \{(q, y) \in A | (q_1, y_1) \succeq (q, y) \succeq (q_1', y_1')\}$ and $I_2 := \{(q, y) \in A | (q_2, y_2) \succeq$

$(q, y) \succeq (q_2', y_2')\}$. Finally, let f_1 and f_2 be affine representations of I_1 and I_2 respectively such that a) f_1 and f_2 coincide with f^* on I^*; b) f_1 is onto the interval $[f_1(q_1', y_1'), f_1(q_1, y_1)]$; and c) f_2 is onto the interval $[f_2(q_2', y_2'), f_2(q_2, y_2)]$.

Claim 9 (Coincidence on $I_1 \cap I_2$)**.** *The functions f_1 and f_2 coincide on $I_1 \cap I_2$*

Proof of Claim 8: Pick any $(q', y') \in I_1 \cap I_2$. (i) If $(q', y') \in I^*$ then we are done since f_1 and f_2 both coincide with f^* on I^*.

(ii) $(q', y') \succ (\overline{q}, \overline{y})$. Then we must have $(q_1, y_1) \succeq (q', y') \succ (\overline{q}, \overline{y})$ and $(q_2, y_2) \succeq (q', y') \succ (\overline{q}, \overline{y})$. From the definition of a basis, it then follows that there exists $(q_1^*, y_1^*), (q_1^{**}, y_1^*), (q_2^*, y_2^*), (q_2^{**}, y_2^*) \in A$ such that $(q_1^*, y_1^*) \sim (q_1, y_1) \succ (\overline{q}, \overline{y}) \succ (q_1^{**}, q_1^*) \succeq (\underline{q}, \underline{y})$ and $(q_2^*, y_2^*) \sim (q_2, y_2) \succ (\overline{q}, \overline{y}) \succ (q_2^{**}, q_2^*) \succeq (\underline{q}, \underline{y})$. Define $I_1' := \{(q, y) \in A | (q_1^*, y_1^*) \succeq (q, y) \succeq (q_1^{**}, y_1^*)\}$ and $I_2' := \{(q, y) \in A | (q_2^*, y_2^*) \succeq (q, y) \succeq (q_2^{**}, y_2^*)\}$. We note that it follows from above that $(q', y') \in I_1' \cap I_2'$. Define a function $f_1' : I_1' \to \mathbf{R}$ by $f_1'(q, y) = f_1(q, y)$ and a function $f_2' : I_2' \to \mathbf{R}$ by $f_2'(q, y) = f_2(q, y)$. Suppose without loss of generality that $(q_1^{**}, y_1^*) \succeq (q_2^{**}, y_2^*)$. We note the following four properties:

a) $(q_1^*, y_1^*) \succ (\overline{q}, \overline{y}) \succ (q_1^{**}, q_1^*) \succeq (q_1^{**}, q_1^*)$;
b) $(q_2^*, y_2^*) \succ (\overline{q}, \overline{y}) \succ (q_1^{**}, q_1^*) \succeq (q_2^{**}, q_2^*)$;
c) $f_1'(\overline{q}, \overline{y}) = f_1(\overline{q}, \overline{y}) = f(\overline{q}, \overline{y}) = f_2(\overline{q}, \overline{y}) = f_2'(\overline{q}, \overline{y})$; and
d) $f_1'(q_1^{**}, y_1^*) = f_1(q_1^{**}, y_1^*) = f(q_1^{**}, y_1^*) = f_2(q_1^{**}, y_1^*) = f_2'(q_1^{**}, y_1^*)$.

It follows from Claim 4 that f_1' and f_2' coincide on $I_1' \cap I_2'$. But then we are done since this implies $f_1(q', y') = f_1'(q', y') = f_2'(q', y') = f_2(q', y')$.

(iii) $(\underline{q}, \underline{y}) \succ (q', y')$. Then we must have $(\underline{q}, \underline{y}) \succ (q', y') \succeq (q_1', y_1')$ and $(\underline{q}, \underline{y}) \succ (q', y') \succeq (q_2', y_2')$. From the definition of a basis, it then follows that there exists $(q_1^*, y_1^*), (q_1^{**}, y_1^*), (q_2^*, y_2^*), (q_2^{**}, y_2^*) \in A$ such that $(\overline{q}, \overline{y}) \succeq (q_1^*, q_1^*) \succ (\underline{q}, \underline{y}) \succ (q_1', y_1') \sim (q_1^{**}, y_1^*)$ and $(\overline{q}, \overline{y}) \succeq (q_2^*, q_2^*) \succ (\underline{q}, \underline{y}) \succ (q_2', y_2') \sim (q_2^{**}, y_2^*)$. Define $I_1' := \{(q, y) \in A | (q_1^*, y_1^*) \succeq (q, y) \succeq (q_1^{**}, y_1^*)\}$ and $I_2' := \{(q, y) \in A | (q_2^*, y_2^*) \succeq (q, y) \succeq (q_2^{**}, y_2^*)\}$. We note that it follows from above that $(q', y') \in I_1' \cap I_2'$. Define a function $f_1' : I_1' \to \mathbf{R}$ by $f_1'(q, y) = f_1(q, y)$ and a function $f_2' : I_2' \to \mathbf{R}$ by $f_2'(q, y) = f_2(q, y)$. Suppose without loss of generality that $(q_1^*, y_1^*) \succeq (q_2^*, y_2^*)$. We note the following four properties:

a) $(q_1^*, y_1^*) \succeq (q_2^*, y_2^*) \succ (\underline{q}, \underline{y}) \succ (q_1^{**}, q_1^*)$;
b) $(q_2^*, y_2^*) \succeq (q_2^*, y_2^*) \succ (\underline{q}, \underline{y}) \succ (q_2^{**}, q_2^*)$;
c) $f_1'(\underline{q}, \underline{y}) = f_1(\underline{q}, \underline{y}) = f(\underline{q}, \underline{y}) = f_2(\underline{q}, \underline{y}) = f_2'(\underline{q}, \underline{y})$; and
d) $f_1'(q_2^*, y_2^*) = f_1(q_2^*, y_2^*) = f(q_2^*, y_2^*) = f_2(q_2^*, y_2^*) = f_2'(q_2^*, y_2^*)$.

It follows from Claim 4 that f_1' and f_2' coincide on $I_1' \cap I_2'$. But then we are done since this implies $f_1(q', y') = f_1'(q', y') = f_2'(q', y') = f_2(q', y')$. Q.E.D.

We are now ready to state a key Lemma that will greatly assist us in the rest of our proof.

Lemma 7. *If a basis interval I^* exists for A and f^* is a representation of the preferences on I^*, then there exists an unique affine extension f, of f^* to A such that f represents the preferences on A. Furthermore, for any two elements $(q, y), (q', y') \in A$ such that $(q, y) \succeq (q', y')$, the representation f is onto the interval $[f(q', y'), f(q, y)]$.*

Proof of Lemma 3: Suppose there exists a basis $I^* := \{(q,y) \in A | (\overline{q},\overline{y}) \succeq (q,y) \succeq (\underline{q},\underline{y})\}$. Define $I_1 := I^*$, and consider an increasing sequence of intervals $I_1 \subset I_2 \subset I_3 \ldots \subset D$ where $I_k = \{(q,y) \in A | (q_k, y_k) \succeq (q,y) \succeq (q'_k, y'_k)\}$. Let $f_1 := f^*$ be the affine representation of the preferences on I_1. From Claim 7, it follows that at each step, we can find a unique representation of the preference on I_k such that f_k is onto the interval $[f(q'_k, y'_k), f(q_k, y_k)]$, and for every $k' < k$, $[(q,y) \in I_{k'}] \Rightarrow [f_k(q,y) = f_{k-1}(q,y) = f_{k-2}(q,y) = \ldots = f_{k'}(q,y)]$. Thus, let us define this common value $f_k(q,y) = f_{k-1}(q,y) := f(q,y)$. We can thereby construct a function f that represents preferences on the entire set A. Since at each step, f_k is unique, it follows that the resulting function f is the only such extension of f^* to A. To see that f must be onto any interval $[f(q',y'), f(q,y)]$, consider any sufficiently large k so that $(q_k, y_k) \succeq (q,y) \succeq (q',y') \succeq (q'_k, y'_k)$. Then the onto property follows from the fact that f_k is onto the interval $[f(q'_k, y'_k), f(q_k, y_k)]$. Q.E.D.

4.6.5 Some additional preliminary results

Claim 10. *Let* $(q,y) \in \Delta(\Omega) \times Y$. *Then there exists* $\overline{\omega}, \underline{\omega} \in C(q)$ *such that* $(i_{\overline{\omega}}, y) \succeq (q,y) \succeq (i_{\underline{\omega}}, y)$.

Proof of Claim 9: We note that $C(q)$ has only a finite number of elements. An implication of completeness and transitivity is that there exists $\overline{\omega}, \underline{\omega} \in C(q)$ such that for all $\omega \in C(q)$, $(i_{\overline{\omega}}, y) \succeq (i_E, y) \succeq (i_{\underline{\omega}}, y)$.

(i) if $(i_{\overline{\omega}}, y) \succ (i_{\underline{\omega}}, y)$, then it follows from unique solvability that for every $\omega \in C(q)$, there exists uniquely $\alpha^*_\omega \in [0,1]$ such that $(\alpha^*_\omega i_{\overline{\omega}} + (1-\alpha^*_\omega) i_{\underline{\omega}}, y) \sim (i_\omega, y)$. Now, order the elements of $C(q)$ as $\omega_1, \omega_2, \ldots, \omega_K$ where $K := \#C(q)$. Define for each $k = 1, 2, \ldots, K$ an element $q^*_{\omega_k}$ by:

$$q^*_{\omega_k} = \frac{\sum_{k' \leq k} q(\omega_{k'}) i_{E_{k'}}}{\sum_{k'' \leq k} q(\omega_{k''})}$$

We claim that for every k, $(i_{\overline{\omega}}, y) \succeq (q^*_{\omega_k}, y) \succeq (i_{\underline{\omega}}, y)$. To see this, first note that $(q^*_{\omega_1}, y) = (i_{\omega_1}, y)$ which in turn implies that $(i_{\overline{\omega}}, y) \succeq (q^*_{\omega_1}, y) \succeq (i_{\underline{\omega}}, y)$. Hence the desired property holds for $k = 1$. Consider now any k such that the desired property holds for $k' = 1,,2,\ldots,k-1$. For each such k' denote by $\beta_{k'}$ the unique element of $[0,1]$ (implied by unique solvability) for which $(\beta_{k'} i_{\overline{\omega}} + (1-\beta_{k'}) i_{\underline{\omega}}, y) \sim^{i,t^i} (q^*_{\omega_{k'}}, y)$. Likewise, for each $k' = 1, 2, \ldots, K$ denote by $\alpha^*_{k'}$ the unique element of $[0,1]$ (implied by unique solvability) for which $(\alpha_{k'} i_{\overline{\omega}} + (1-\alpha_{k'}) i_{\underline{\omega}}, y) \sim^{i,t^i} (i_{\omega_{k'}}, y)$. Then we have,

$$\begin{aligned}
[q^*_{\omega_k}, y] &= [(\frac{\sum_{k'\leq k} q(\omega_{k'}) i_{E_{k'}}}{\sum_{k''\leq k} q(\omega_{k''})}, y] \\
&= [(\frac{q(\omega_k) i_{\omega_k}}{\sum_{k''\leq k} q(\omega_{k''})}) + \frac{(\sum_{k''<k} q(\omega_{k''}))}{(\sum_{k''\leq k} q(\omega_{k''}))} * \frac{(\sum_{k'<k} q(\omega_{k'}) i_{\omega_{k'}})}{(\sum_{k''<k} q(\omega_{k''}))}, y] \\
&= [(\frac{q(\omega_k) i_{\omega_k}}{\sum_{k''\leq k} q(\omega_{k''})}) + \frac{(\sum_{k''<k} q(\omega_{k''}))}{(\sum_{k''\leq k} q(\omega_{k''}))} * q^*_{\omega_{k-1}}, y] \\
&\sim^{i,t^i} [(\frac{q(\omega_k)}{\sum_{k''\leq k} q(\omega_{k''})}) * (\alpha_{\omega_k} i_{\overline{\omega}} + (1-\alpha_{\omega_k}) i_{\underline{\omega}}) + \frac{(\sum_{k''<k} q(\omega_{k''}))}{(\sum_{k''\leq k} q(\omega_{k''}))} * q^*_{\omega_{k-1}}, y] \\
&\sim^{i,t^i} [(\frac{q(\omega_k)}{\sum_{k''\leq k} q(\omega_{k''})}) * (\alpha_{\omega_k} i_{\overline{\omega}} + (1-\alpha_{\omega_k}) i_{\underline{\omega}}) \\
&\quad + \frac{(\sum_{k''<k} q(\omega_{k''}))}{(\sum_{k''\leq k} q(\omega_{k''}))} * (\beta_{\omega_{k-1}} i_{\overline{\omega}} + (1-\beta_{\omega_{k-1}}) i_{\underline{\omega}}), y] \\
&= [[(\frac{q(\omega_k)}{\sum_{k''\leq k} q(\omega_{k''})}) * \alpha_{\omega_k} + \frac{(\sum_{k''<k} q(\omega_{k''}))}{(\sum_{k''\leq k} q(\omega_{k''}))} * \beta_{\omega_{k-1}}] i_{\overline{\omega}} \\
&\quad + [1 - [(\frac{q(\omega_k)}{\sum_{k''\leq k} q(\omega_{k''})}) * \alpha_{\omega_k} + \frac{(\sum_{k''<k} q(\omega_{k''}))}{(\sum_{k''\leq k} q(\omega_{k''}))} * \beta_{\omega_{k-1}}]] i_{\underline{\omega}}, y]
\end{aligned}$$

Letting $\beta_k := [(\frac{q(\omega_k)}{\sum_{k''\leq k} q(\omega_{k''})}) * \alpha_{\omega_k} + \frac{(\sum_{k''<k} q(\omega_{k''}))}{(\sum_{k''\leq k} q(\omega_{k''}))} * \beta_{\omega_{k-1}}]$, we note that the desired property holds for any k. Now, simply note that $q_K = q$. Hence, we can let $\beta = \beta_K$. But then it follows that $(q, y) \sim^{i,t^i} (\beta i_{\overline{\omega}} + (1-\beta) i_{\underline{\omega}}, y)$. This in turn implies that $(i_{\overline{\omega}}, y) \succeq (q, y) \succeq (i_{\underline{\omega}}, y)$ by mixture monotonicity.

(ii) if $(i_{\overline{\omega}}, y) \sim (i_{\underline{\omega}}, y)$, then it follows from completeness and transitivity that for every $E \in C(q)$, $(i_{\overline{\omega}}, y) \sim (i_E, y) \sim (i_{\underline{\omega}}, y)$.

Now, order the elements of $C(q)$ as $\omega_1, \omega_2, \dots, \omega_K$ where $K := \#C(q)$. Define for each $k = 1, 2, \dots, K$ an element $q^*_{\omega_k}$ by:

$$q^*_{\omega_k} = \frac{\sum_{k'\leq k} q(\omega_{k'}) i_{\omega_{k'}}}{\sum_{k''\leq k} q(\omega_{k''})}$$

We claim that for every k, $(i_{\overline{\omega}}, y) \sim (q^*_{\omega_k}, y) \sim (i_{\underline{\omega}}, y)$. To see this, first note that $(q^*_{\omega_1}, y) = (i_{\omega_1}, y)$ which in turn implies that $(i_{\overline{\omega}}, y) \sim (q^*_{\omega_1}, y) \sim (i_{\underline{\omega}}, y)$. Hence the desired property holds for $k = 1$. Consider now any k such that the desired property holds for $k' = 1, , 2, \dots, k-1$.

Then we have,

$$\begin{aligned}
[q^*_{\omega_k}, y] &= [(\frac{\sum_{k'\leq k} q(\omega_{k'}) i_{\omega_{k'}}}{\sum_{k''\leq k} q(\omega_{k''})}, y] \\
&= [(\frac{q(\omega_k) i_{\omega_k}}{\sum_{k''\leq k} q(\omega_{k''})}) + \frac{(\sum_{k''<k} q(\omega_{k''}))}{(\sum_{k''\leq k} q(\omega_{k''}))} * \frac{(\sum_{k'<k} q(\omega_{k'}) i_{\omega_{k'}})}{(\sum_{k''<k} q(\omega_{k''}))}, y] \\
&= [(\frac{q(\omega_k) i_{\omega_k}}{\sum_{k''\leq k} q(\omega_{k''})}) + \frac{(\sum_{k''<k} q(\omega_{k''}))}{(\sum_{k''\leq k} q(\omega_{k''}))} * q^*_{\omega_{k-1}}, y] \\
&\sim^{i,t^i} [(\frac{q(\omega_k)}{\sum_{k''\leq k} q(\omega_{k''})}) * i_{\underline{\omega}} + \frac{(\sum_{k''<k} q(\omega_{k''}))}{(\sum_{k''\leq k} q(\omega_{k''}))} * q^*_{\omega_{k-1}}, y] \\
&\sim^{i,t^i} [(\frac{q(\omega_k)}{\sum_{k''\leq k} q(\omega_{k''})}) * i_{\underline{\omega}} + \frac{(\sum_{k''<k} q(\omega_{k''}))}{(\sum_{k''\leq k} q(\omega_{k''}))} * i_{\underline{\omega}}, y] \\
&= [i_{\underline{\omega}}, y]
\end{aligned}$$

But then it follows from completeness and transitivity that $(i_{\overline{\omega}}, y) \sim (q^*_{\omega_k}, y) \sim (i_{\underline{\omega}}, y)$. Hence the desired property holds for every k. Now, simply note that $q_K = q$. Hence $(i_{\overline{\omega}}, y) \sim (q, y) \sim (i_{\underline{\omega}}, y)$. Q.E.D.

Claim 11. *Let ω_1 and ω_2 be distinct elements of Ω and let y_1 and y_2 be elements of Y. Then there exists $y^* \in Y$ such that $(i_{\omega_1}, y^*) \sim (i_{\omega_1}, y_1)$ and $(i_{\omega_2}, y^*) \sim (i_{\omega_2}, y_2)$.*

Proof of Claim 10: Let $y^*(\omega) = y_2(\omega)$ if $\omega = \omega_2$ and let $y^*(\omega) = y_1(\omega)$ otherwise. Then it follows from $A1$ that $(i_{\omega_1}, y^*) \sim (i_{\omega_1}, y_1)$ and $(i_{\omega_2}, y^*) \sim (i_{\omega_2}, y_2)$. Q.E.D.

4.6.6 Special Case 1:

Claim 12. *If there exists* $\omega_1, \omega_2, \omega_3, \omega_4 \in \Omega[\omega_1 \neq \omega_3, \omega_1 \neq \omega_4, \omega_2 \neq \omega_3, \omega_2 \neq \omega_4]$ *and* $y_1, y_2, y_3, y_4 \in Y$ *such that* $(i_{\omega_1}, y_1) \succ (i_{\omega_2}, y_2)$ *and* $(i_{\omega_3}, y_3) \succ (i_{\omega_4}, y_4)$*, then there exists* $(\overline{q}, \overline{y}), (\underline{q}, \underline{y}) \in \Delta(\Omega) \times Y$ *such that:*

(i) $(\overline{q}, \overline{y}) \succeq (i_{\omega_1}, y_1) \succ (i_{\omega_2}, y_2) \succeq (\underline{q}, \underline{y})$*;*

(ii) $(\overline{q}, \overline{y}) \succeq (i_{\omega_3}, y_3) \succ (i_{\omega_4}, y_4) \succeq (\underline{q}, \underline{y})$*;*

(iii) the preferences on the interval $I^* := \{(q, y) \in \Delta(\Omega) \times Y | (\overline{q}, \overline{y}) \succeq (q, y) \succeq (\underline{q}, \underline{y})\}$ *are representable by an affine function* f *that is onto the interval* $[f(\underline{q}, \underline{y}), f(\overline{q}, \overline{y})]$*; and*

(iv) the interval I^* *is a basis.*

Moreover, if g *is another affine representation of the interval* I^* *then there is* a, b *(*$b > 0$*) such that* $g = a + bf$*.*

Proof of Claim 11: Step 1: Establish (i)-(iii) and show that the moreover statement holds.

Assume without loss of generality that $(i_{\omega_1}, y_1) \succeq (i_{\omega_3}, y_3)$.

1a) If $(i_{\omega_2}, y_2) \succeq (i_{\omega_4}, y_4)$ set $(\overline{q}, \overline{y}) = (i_{\omega_1}, y_1)$ and $(\underline{q}, \underline{y}) = (i_{\omega_4}, y_4)$. Then we have,

$$(\overline{q}, \overline{y}) = (i_{\omega_1}, y_1) \succ^{i,t^i} (i_{\omega_2}, y_2) \succeq (i_{\omega_4}, y_4) = (\underline{q}, \underline{y}); \text{ and}$$

$$(\overline{q}, \overline{y}) = (i_{\omega_1}, y_1) \succeq (i_{\omega_3}, y_3) \succ (i_{\omega_4}, y_4) = (\underline{q}, \underline{y}).$$

Hence (i) and (ii) are both satisfied.

Now, since $\omega_1 \neq \omega_4$, it follows from claim 10 that there exists $y^* \in Y^{i,t^i}$ such that

$$(i_{\omega_1}, y^*) \sim (i_{\omega_1}, y_1) \succ (i_{\omega_2}, y_2) \succeq (i_{\omega_4}, y_4) \sim (i_{\omega_4}, y^*)$$

Due to the unique solvability property, we can then define a function $f : I^* \Rightarrow \mathbf{R}$ by $f(q, y) = \alpha^*_{q,y}$ where $\alpha^*_{q,y}$ is the unique element of $[0, 1]$ for which $(\alpha^*_{q,y} i_{\omega_1} + (1 - \alpha^*_{q,y}) i_{\omega_4}, y^*) \sim (q, y)$. From claims 1 and 2 it follows that f is an affine representation of the preferences on I^* that is furthermore onto the interval $[f(i_{\omega_4}, y^*), f(i_{\omega_1}, y^*)] = [f(\underline{q}, \underline{y}, f(\overline{q}, \overline{y}]$. Hence (iii) holds.

Consider now any other affine representation of the preferences on I^*. Define

$$a := g(\overline{q}, \overline{y}) - \frac{[g(\overline{q}, \overline{y}) - g(\underline{q}, \underline{y})]}{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]} * f(\overline{q}, \overline{y}) \text{ ; and}$$

$$b := \frac{[g(\overline{q}, \overline{y}) - g(\underline{q}, \underline{y})]}{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]} > 0$$

.

Now, define $h = a + bf$. We know from claim 3 that h is an affine representation of the preferences on I^* that is onto the interval $[h(\underline{q}, \underline{y}), h(\overline{q}, \overline{y})]$. Now, we have

$$\begin{aligned} h(\overline{q}, \overline{y}) &= g(\overline{q}, \overline{y}) - \tfrac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]} * f(\overline{q}, \overline{y}) + \tfrac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]} * f(\overline{q}, \overline{y}) \\ &= g(\overline{q}, \overline{y}) \end{aligned}$$

Likewise, we have

$$\begin{aligned} h(\underline{q}, \underline{y}) &= g(\overline{q}, \overline{y}) - \tfrac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]} * f(\overline{q}, \overline{y}) + \tfrac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]} * f(\underline{q}, \underline{y}) \\ &= g(\underline{q}, \underline{y}) \end{aligned}$$

We know that $(i_{\omega_1}, y^*) \sim (\overline{q}, \overline{y})$ and $(i_{\omega_4}, y^*) \sim (\underline{q}, \underline{y})$. Let $I_1 = I_2 := \{(q, y) \in \Delta(\Omega) \times Y | (i_{\omega_1}, y^*) \succeq (q, y) \succeq (i_{\omega_4}, y^*)\}$. Then we note that all of the assumptions of Claim 4 are satisfied and hence h coincides with g on $I_1 \cap I_2 = I^*$. Hence we have $g = h = a + bf$ as required.

1b) If $(i_{\omega_4}, y_4) \succ (i_{\omega_2}, y_2)$, set $(\overline{q}, \overline{y}) = (i_{\omega_1}, y_1)$ and $(\underline{q}, \underline{y}) = (i_{\omega_2}, y_2)$. Then we have,

$$(\overline{q}, \overline{y}) = (i_{\omega_1}, y_1) \succeq (i_{\omega_3}, y_3) \succ^{i,t^i} (i_{\omega_4}, y_4) \succeq (i_{\omega_2}, y_2) = (\underline{q}, \underline{y})$$

so (i) and (ii) both hold. Now, since $\omega_1 \neq \omega_4$ and $\omega_2 \neq \omega_3$, it follows from claim 10 that there exists y_1^* and y_2^* in Y such that

$$(i_{\omega_1}, y_1^*) \sim (i_{\omega_1}, y_1) \succ (i_{\omega_4}, y_4) \sim (i_{\omega_4}, y_1^*)$$

and

$$(i_{\omega_3}, y_2^*) \sim (i_{\omega_3}, y_3) \succ (i_{\omega_2}, y_2) \sim (i_{\omega_2}, y_2^*)$$

.

From unique solvability it follows that for every (q, y) such that $(i_{\omega_1}, y_1^*) \succeq (q, y) \succeq (i_{\omega_4}, y_1^*)$ there exist uniquely $\alpha_{q,y}^* \in [0, 1]$ such that $(\alpha_{q,y}^* i_{\omega_1} + (1 - \alpha_{q,y}^*), y_1^*) \sim (q, y)$. Likewise, for every (q, y) such that $(i_{\omega_1}, y_1^*) \succeq (q, y) \succeq (i_{\omega_4}, y_1^*)$ there exist uniquely $\alpha_{q,y}^* \in [0, 1]$ such that $(\beta_{q,y}^* i_{\omega_1} + (1 - \beta_{q,y}^*), y_1^*) \sim (q, y)$.

Hence, we can define a function $f_1 : \{(q, y) \in \Delta(\Omega) \times Y | (i_{\omega_1}, y_1^*) \succeq (q, y) \succeq (i_{\omega_4}, y_1^*)\} \to \mathbf{R}$ by $f_1(q, y) = \alpha_{q,y}^*$ and a function $f_2 : \{(q, y) \in \Delta(\Omega) \times Y | (i_{\omega_3}, y_2^*) \succeq (q, y) \succeq (i_{\omega_2}, y_2^*)\} \to \mathbf{R}$ by $f_1(q, y) = \beta_{q,y}^*$. From claims 1 and 2 it follows that f_1 is an affine representation of the preferences on $I_1 := \{(q, y) \in \Delta(\Omega) \times Y | (i_{\omega_1}, y_1^*) \succeq (q, y) \succeq (i_{\omega_4}, y_1^*)\}$ such that f_1 is onto the interval $[f_1(i_{\omega_4}, y_1^*), f_1(i_{\omega_1}, y_1^*)]$. Likewise, it follows that f_2 is an affine representation of the preferences on $I_2 := \{(q, y) \in \Delta(\Omega) \times Y | (i_{\omega_3}, y_2^*) \succeq (q, y) \succeq (i_{\omega_2}, y_2^*)\}$ such that f_2 is onto the interval $[f_2(i_{\omega_2}, y_2^*), f_2(i_{\omega_3}, y_2^*)]$.

Let $(q^*, y^*) = (i_{\omega_3}, y_3)$ and $(q^{**}, y^{**}) = (i_{\omega_4}, y_4)$. Then we have $(i_{\omega_1}, y_1^*) \succeq (i_{\omega_3}, y_3) \sim (q^*, y^*) \succ (q^{**}, y^{**}) \sim (i_{\omega_4}, y_1^*)$ and $(i_{\omega_3}, y_2^*) \sim (q^*, y^*) \succ (q^{**}, y^{**}) \sim (i_{\omega_4}, y_1^*) \succ (i_{\omega_2}, y_2^*)$. Given claim 3, we can find a and $b[b > 0]$ such that $g = a + bf_2$ is an affine representation of the preferences on I_2 which is onto the interval $[g(i_{\omega_2}, y_2^*), g(i_{\omega_3}, y_2^*)]$ and that furthermore has the properties that $g(q^*, y^*) = f_1(q^*, y^*)$ and $g(q^{**}, y^{**}) = f_1(q^{**}, y^{**})$. From Claim 4, we know that g and f_1 coincide on $I_1 \cap I_2$. Now, let f^* be the function that coincides with f_1 in I_1 and with g on I_2. Then we know from claims 5 and 6 that f^* is an affine representation of the preferences in $I_1 \cup I_2 = I^*$ as required. Hence (iii) holds.

To see that the moreover remark holds, let g be another affine representation of the preferences on $I_1 \cup I_2 = I^*$. Define

$$a := g(i_{\omega_3}, y_3) - \frac{[g(i_{\omega_3}, y_{\omega_3}) - g(i_{\omega_4}, y_{\omega_4})]}{[f(i_{\omega_3}, y_{\omega_3}) - f(i_{\omega_4}, y_{\omega_4})]} * f(i_{\omega_3}, y_{\omega_3}) \text{ ; and}$$

$$b := \frac{[g(i_{\omega_3}, y_{\omega_3}) - g(i_{\omega_4}, y_{\omega_4})]}{[f(i_{\omega_3}, y_{\omega_3}) - f(i_{\omega_4}, y_{\omega_4})]} > 0$$

.

Now, define $h = a + bf$. Then we have,

$$\begin{aligned} h(i_{\omega_3}, y_{\omega_3}) &= g(i_{\omega_3}, y_{\omega_3}) - \tfrac{[g(i_{\omega_3}, y_{\omega_3}) - g(i_{\omega_4}, y_{\omega_4})]}{[f(i_{\omega_3}, y_{\omega_3}) - f(i_{\omega_4}, y_{\omega_4})]} * f(i_{\omega_3}, y_{\omega_3}) + \tfrac{[g(i_{\omega_3}, y_{\omega_3}) - g(i_{\omega_4}, y_{\omega_4})]}{[f(i_{\omega_3}, y_{\omega_3}) - f(i_{\omega_4}, y_{\omega_4})]} * f(i_{\omega_3}, y_{\omega_3}) \\ &= g(i_{\omega_3}, y_{\omega_3})) \end{aligned}$$

Likewise, we have

$$\begin{aligned} h(i_{\omega_4}, y_{\omega_4}) &= g(i_{\omega_3}, y_{\omega_3}) - \tfrac{[g(i_{\omega_3}, y_{\omega_3}) - g(i_{\omega_4}, y_{\omega_4})]}{[f(i_{\omega_3}, y_{\omega_3}) - f(i_{\omega_4}, y_{\omega_4})]} * f(i_{\omega_3}, y_{\omega_3}) + \tfrac{[g(i_{\omega_3}, y_{\omega_3}) - g(i_{\omega_4}, y_{\omega_4})]}{[f(i_{\omega_3}, y_{\omega_3}) - f(i_{\omega_4}, y_{\omega_4})]} * f(i_{\omega_4}, y_{\omega_4}) \\ &= g(i_{\omega_4}, y_{\omega_4})) \end{aligned}$$

We note that h and g are both affine representations of I^*. An implication of claim 4 is that the restrictions of h and g to $I_1 := \{(q, y) \in \Delta(\Omega) \times Y | (i_{\omega_1}, y_{\omega_1}) \succeq (q, y) \succeq (i_{\omega_4}, y_{\omega_4})\}$ must coincide. Likewise, it is also an implication of claim 4 that the restrictions of h and g to $I_2 := \{(q, y) \in \Delta(\Omega) \times Y | (i_{\omega_3}, y_{\omega_3}) \succeq (q, y) \succeq (i_{\omega_2}, y_{\omega_2})\}$ must coincide. But then $g = h = a + bf$ as required.

Step 2: Show that (i)-(iii) implies that I^* is a basis.

2A) Show that property (ii) from the definition of a basis holds.

Pick any $(q_1', y_1') \in \Delta(\Omega) \times Y$ such that $(q_1', y_1') \succ (\overline{q}, \overline{y})$.

From Claim 9, we know that there exist $\overline{\omega} \in \Delta(\Omega)$ such that $(i_{\overline{\omega}}, y_1') \succeq (q_1', y_1')$. Since $\omega_2 \neq \omega_4$, we know that either $\overline{\omega} \neq \omega_2$ or $\overline{\omega} \neq \omega_4$. Without loss of generality, assume that $\overline{\omega} \neq \omega_2$. Then we know from Claim 10 that there exists $y^* \in Y$ such that $(i_{\overline{\omega}}, y^*) \sim (i_{\overline{\omega}}, y_1')$ and $(i_{\omega_2}, y^*) \sim (i_{\omega_2}, y_2)$. Then we have,

$$(i_{\overline{\omega}}, y^*) \sim (i_{\overline{\omega}}, y_1') \succeq (q_1', y_1') \succ (\overline{q}, \overline{y}) \succeq (i_{\omega_1}, y_1) \succ (i_{\omega_2}, y_2) \sim (i_{\omega_2}, y^*) \succeq (\underline{q}, \underline{y})$$

From unique solvability, it follows that there exist uniquely $\alpha^* \in [0, 1]$ such that $(\alpha^* i_{\overline{\omega}} + (1 - \alpha^*) i_{\omega_2}, y^*) \sim (q_1', y_1')$ Now, let $(q^*, y^*) = (\alpha^* i_{\overline{\omega}} + (1 - \alpha^*) i_{\omega_2}, y^*)$ and $(q^{**}, y^{**}) = (i_{\omega_2}, y^*)$. Then we have

$$(q^*, y^*) \sim (q_1', y_1') \succ (\overline{q}, \overline{y}) \succeq (i_{\omega_1}, y_1) \succ (i_{\omega_2}, y_2) \sim (q^{**}, y^*) \succeq (\underline{q}, \underline{y})$$

as required.

2B) Show that property (iii) from the definition of a basis holds.

Pick any $(q_2', y_2') \in \Delta(\Omega) \times Y$ such that $(\underline{q}, \underline{y}) \succ (q_2', y_2')$.

From Claim 9, we know that there exist $\underline{\omega} \in \Delta(\Omega)$ such that $(q_2', y_2') \succeq (i_{\underline{\omega}}, y_2')$. Since $\omega_1 \neq \omega_3$, we know that either $\underline{\omega} \neq \omega_1$ or $\underline{\omega} \neq \omega_3$. Without loss of generality, assume that $\underline{\omega} \neq \omega_1$. Then we know from Claim 10 that there exists $y^* \in Y$ such that $(i_{\omega_1}, y^*) \sim (i_{\omega_1}, y_1)$ and $(i_{\underline{\omega}}, y^*) \sim (i_{\underline{\omega}}, y_2')$. Then we have,

$$(\overline{q}, \overline{y}) \succeq (i_{\omega_1}, y^*) \sim (i_{\omega_1}, y_1) \succ (\underline{q}, \underline{y}) \succ (q_2', y_2') \sim (i_{\underline{\omega}}, y_2') \sim (i_{\underline{\omega}}, y^*)$$

From unique solvability, it follows that there exist uniquely $\beta^* \in [0, 1]$ such that $(\beta^* i_{\omega_1} + (1 - \beta^*) i_{\underline{\omega}}, y^*) \sim (q_2', y_2')$. Now, let $(q^*, y^*) = (i_{\omega_1}, y^*)$ and $(q^{**}, y^{**}) = (\beta^* i_{\omega_1} + (1 - \beta^*) i_{\underline{\omega}}, y^*)$. Then we have

$$(\overline{q}, \overline{y}) \succeq (q^*, y^*) \succ (\underline{q}, \underline{y}) \succ (q_2', y_2') \sim (q^{**}, y^*)$$

as required.

Q.E.D.

Lemma 8. *If there exists $\omega_1, \omega_2, \omega_3, \omega_4 \in \Omega[\omega_1 \neq \omega_3, \omega_1 \neq \omega_4, \omega_2 \neq \omega_3, \omega_2 \neq \omega_4]$ and $y_1, y_2, y_3, y_4 \in Y$ such that $(i_{\omega_1}, y_1) \succ (i_{\omega_2}, y_2)$ and $(i_{\omega_3}, y_3) \succ (i_{\omega_4}, y_4)$, then there exists an affine representation $U : \Delta(\Omega) \times Y \to \mathbf{R}$ of the preferences on $\Delta(\Omega) \times Y$ with the further property that for any $(q, y), (q', y') \in \Delta(\Omega) \times Y$ for which $(q, y) \succ (q', y')$ the function is onto the interval $[U(q', y'), U(q, y)]$. Moreover, if $V : \Delta(\Omega) \times Y \to \mathbf{R}$ is another affine representation, then there is a, b $(b > 0)$ such that $V = a + bU$.*

Proof of Lemma 4: From claim 11, we know that there exists a basis interval I^* with an affine representation f^*. The basis interval furthermore has the property that if g^* is another affine representation of I^*, then there exists a, b, $[b > 0]$ such that $g^* = a + bf^*$. From Lemma 3, we know that there exists uniquely an affine extension f of f^* to $\Delta(\Omega) \times Y$ with the properties that f represents preferences on $\Delta(\Omega) \times Y$ and b) that f is onto the interval $[f(q, y), f(q', y')]$ for any arbitrary elements $(q, y), (q', y')$ of $\Delta(\Omega) \times Y$. Hence existence is done.

Consider now any other affine representation g of the preferences on $\Delta(\Omega) \times Y$. Denote by g^* the restriction of g to I^*. Then we know that there exists a, b $(b > 0)$ such that $g^* = a + bf^*$. Define $g' := a + bf$. Then g' has the property that it coincides with g^* on I^*. We know from claim 3 that g' also represents preferences on $\Delta(\Omega) \times Y$. Hence g' is an affine extension of g^* to $\Delta(\Omega) \times Y$. But we know from Lemma 3 that only one such extension exists. But then g' must coincide with g since g is the unique extension of g^*. Hence uniqueness has been established. Q.E.D.

4.6.7 Special Case 2:

Lemma 9. *Let ω_1^*, ω_2^* be distinct elements of Ω and let y^* be an element of Y such that $[(q, y) \not\sim (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$. Then there exists $(\overline{q}, \overline{y}), (\underline{q}, \underline{y}) \in \Delta(\Omega) \times Y$ such that:*

(i) $(\overline{q}, \overline{y}) \succeq (\underline{q}, \underline{y})$;

(ii) The preferences on the interval $I^* := \{(q, y) \in \Delta(\Omega) \times Y | (\overline{q}, \overline{y}) \succeq (q, y) \succeq (\underline{q}, \underline{y})\}$ are representable by an affine function f that is onto the interval $[f(\underline{q}, \underline{y}), f(\overline{q}, \overline{y})]$; and

(iii) the interval I^* is a basis for $I_1 := \{(q,y) \in \Delta(\Omega) \times Y | (q,y) \succeq (i_{\omega_2^*}, y^*)\}$.

Moreover, if g is another affine representation of the preferences on I^* then there exists a, b $(b > 0)$ such that $g = a + bf$.

Proof of Lemma 5: Step 1: Establish the result when there does not exist $(q,y) \in \Delta(\Omega) \times Y$ such that $(q,y) \succ (i_{\omega_2^*}, y^*)$.

Let $(\overline{q}, \overline{y}) = (\underline{q}, \underline{y}) := (i_{\omega_2^*}, y^*)$. Clearly, (i) then holds. We also have $I^* = \{(q,y) \in \Delta(\Omega) \times Y | (q,y) \sim (i_{\omega_2^*}, y^*)\} = I_1$ where I^* is nonempty since it has $(i_{\omega_2^*}, y^*)$ as an element. Now, define $f : I^* \rightarrow \mathbf{R}$ by $f(q,y) = 0$ for every $(q,y) \in I^*$. Since I^* is an indifference set, it follows that f represents the preferences on I^*. f is also clearly affine and onto the interval $[f(\underline{q}, \underline{y}), f(\overline{q}, \overline{y})]$. Hence (ii) holds. Let g be another affine representation of I^*. We note that for every $(q,y) \in I^*$, $g(q,y) = g(\omega_2^*, y^*)$. Let $a := g(\omega_2^*, y^*)$ and $b = 1$. Define $h = a + bf$. Then for every $(q,y) \in I^*$, $h(q,y) = g(\omega_2^*, y^*) = g(q,y)$. Hence $g = a + bf$ and the moreover remark holds. We now simply note that under the present assumptions $I^* = I_1$. Hence there neither exists $(q,y) \in I_1$ such that $(q,y) \succ (\overline{q}, \overline{y})$ nor $(q,y) \in I_1$ such that $(\underline{q}, \underline{y}) \succ (q,y)$. Hence it follows that I^* satisfies all the properties for being a a basis for I_1.

Step 2: Show that when there exists $(q,y) \in \Delta(\Omega) \times Y$ such that $(q,y) \succ (i_{\omega_2^*}, y^*)$, then there exists $(\overline{q}, \overline{y}), (\underline{q}, \underline{y}), [(\overline{q}, \overline{y}) \succ (\underline{q}, \underline{y})] \in \Delta(\Omega) \times Y$ such that properties (ii) and (iii) holds.

Let $(q,y) \succ (i_{\omega_2^*}, y^*)$. Then we know from claim 9 that there exists $\omega \in C(q)$ such that $(i_\omega, y) \succeq (q,y)$. Under the present assumptions, it must be that $\omega = \omega_1^* \neq \omega_2^*$. Hence we have, $(i_{\omega_1^*}, y) \succeq (q,y) \succ (i_{\omega_2^*}, y^*)$. But then we know from claim 10 that there exists $y^{**} \in Y$ such that $(i_{\omega_1^*}, y^{**}) \sim (i_{\omega_1^*}, y)$ and $(i_{\omega_2^*}, y^{**}) \sim (i_{\omega_2^*}, y^*)$. Let $(\overline{q}, \overline{y}) = (i_{\omega_1^*}, y^{**})$, $(\underline{q}, \underline{y}) = (i_{\omega_2^*}, y^{**})$, and $I^* := \{(q', y') \in \Delta(\Omega) \times Y | (\overline{q}, \overline{y}) \succeq (q', y') \succeq (\underline{q}, \underline{y})\}$. Clearly (i) is satisfied. By unique solvability, we know that for every $(q', y') \in I^*$, there exists uniquely $\alpha^*_{q',y'} \in [0,1]$ such that $(\alpha^*_{q',y'} i_{\omega_1^*} + (1 - \alpha^*_{q',y'}) i_{\omega_2^*}, y^{**}) \sim (q', y')$. Let $f : I^* \rightarrow \mathbf{R}$ be the function for which $f(q', y') = \alpha^*_{q',y'}$. From Claim 1, we know that this function represents preferences on I^*. From claim 2, we know that it is affine and onto the interval $[f(\underline{q}, \underline{y}), f(\overline{q}, \overline{y})]$. Hence (ii) holds. To see that the moreover remark holds, consider another affine representation g of the preferences on I^*. Let $a = g(\underline{q}, \underline{y}) - \frac{[g(\overline{q}, \overline{y}) - g(\underline{q}, \underline{y})]}{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]} f(\underline{q}, \underline{y})$, $b = \frac{[g(\overline{q}, \overline{y}) - g(\underline{q}, \underline{y})]}{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]}$ and $h = a + bf$ We note that it is an implication of Claim 3 that h also is an affine representation of the preferences on I^*.

Then we have:

$$\begin{aligned} h(\overline{q}, \overline{y}) &= g(\underline{q}, \underline{y}) - \frac{[g(\overline{q}, \overline{y}) - g(\underline{q}, \underline{y})]}{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]} f(\underline{q}, \underline{y}) + \frac{[g(\overline{q}, \overline{y}) - g(\underline{q}, \underline{y})]}{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]} f(\overline{q}, \overline{y}) \\ &= g(\overline{q}, \overline{y}) \end{aligned}$$

Likewise, we have

$$\begin{aligned} h(\underline{q}, \underline{y}) &= g(\underline{q}, \underline{y}) - \frac{[g(\overline{q}, \overline{y}) - g(\underline{q}, \underline{y})]}{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]} f(\underline{q}, \underline{y}) + \frac{[g(\overline{q}, \overline{y}) - g(\underline{q}, \underline{y})]}{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]} f(\underline{q}, \underline{y}) \\ &= g(\underline{q}, \underline{y}) \end{aligned}$$

Hence h and g coincides at both $(\overline{q}, \overline{y})$, and $(\underline{q}, \underline{y})$. But then it follows from claim 4 that h and g coincide on $I^* \cap I^* = I^*$. Hence $g = a + bf$ and the moreover remark holds.

All that remains in order to show that I^* is a basis is to show that for every $(q', y') \in \Delta(\Omega) \times Y$ for which $(q', y') \succ (i_{\omega_1^*}, y^{**})$, there exists $q_1^*, q_1^{**} \in \Delta(\Omega)$ and $y_1^* \in Y$ such that

$$(q_1^*, y_1^*) \sim (q', y') \succ (i_{\omega_1^*}, y^{**}) \succ (q_1^{**}, y_1^*) \succeq (i_{\omega_2^*}, y^{**}).$$

Let $(q', y') \succ (i_{\omega_1^*}, y^{**})$. We know from Claim 9 that there exists $\omega \in C(q)$ such that $(i_\omega, y') \succeq (q', y')$. Due to the present assumptions, we know that $\omega = \omega_1^*$. Now, since $\omega_1^* \neq \omega_2^*$, we know from claim 10 that there exist $y^{***} \in Y$ such that $(i_{\omega_1^*}, y^{***}) \sim (i_{\omega_1^*}, y)$ and $(i_{\omega_2^*}, y^{***}) \sim (i_{\omega_2^*}, y^{**}) \sim (i_{\omega_2^*}, y^*)$. Hence, we have

$$(i_{\omega_1^*}, y^{***}) \succeq (q', y') \succ (i_{\omega_1^*}, y^{**}) \succ (i_{\omega_2^*}, y^*) \sim (i_{\omega_2^*}, y^{**}) \sim (i_{\omega_2^*}, y^{***})$$

But then it follows from unique solvability that there exists uniquely $\beta_{q,y}^* \in [0,1]$ such that $(\beta_{q,y}^* i_{\omega_1^*} + (1 - \beta_{q,y}^*) i_{\omega_2^*}, y^{***}) \sim (q, y)$.

Hence, if we set $(q_1^*, y_1^*) = (\beta_{q,y}^* i_{\omega_1^*} + (1 - \beta_{q,y}^*) i_{\omega_2^*}, y^{***})$ and $(q_1^{**}, y_1^*) = (i_{\omega_2^*}, y^{***})$, we have $(q_1^*, y_1^*) \sim (q', y') \succ (\overline{q}, \overline{y}) \succ (q_1^{**}, y_1^*) \sim (\underline{q}, \underline{y})$ as required. Q.E.D.

Lemma 10. *Let* ω_1^*, ω_2^* *be distinct elements of* Ω *and let* y^* *be an element of* Y *such that* $[(q, y) \not\sim (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$. *Then there exists* $(\overline{q}, \overline{y}), (\underline{q}, \underline{y}) \in \Delta(\Omega) \times Y$ *such that:*

(i) $(\overline{q}, \overline{y}) \succeq (\underline{q}, \underline{y})$*;*

(ii) The preferences on the interval $I^* := \{(q, y) \in \Delta(\Omega) \times Y | (\overline{q}, \overline{y}) \succeq (q, y) \succeq (\underline{q}, \underline{y})\}$ *are representable by an affine function* f *that is onto the interval* $[f(\underline{q}, \underline{y}), f(\overline{q}, \overline{y})]$*; and*

(iii) the interval I^* *is a basis for* $I_2 := \{(q, y) \in \Delta(\Omega) \times Y | (i_{\omega_2^*}, y^*) \succeq (q, y)\}$.

Moreover, if g *is another affine representation of the preferences on* I^* *then there exists* a, b $(b > 0)$ *such that* $g = a + bf$.

Proof of Lemma 6: Step 1: Establish the result when there does not exist $(q, y) \in \Delta(\Omega) \times Y$ such that $(i_{\omega_2^*}, y^*) \succ (q, y)$.

Let $(\overline{q}, \overline{y}) = (\underline{q}, \underline{y}) := (i_{\omega_2^*}, y^*)$. Clearly, (i) then holds. We also have $I^* = \{(q, y) \in \Delta(\Omega) \times Y | (q, y) \sim (i_{\omega_2^*}, y^*)\} = I_2$ where I^* is nonempty since it has $(i_{\omega_2^*}, y^*)$ as an element. Now, define $f : I^* \to \mathbf{R}$ by $f(q, y) = 0$ for every $(q, y) \in I^*$. Since I^* is an indifference set, it follows that f represents the preferences on I^*. f is also clearly affine and onto the interval $[f(\underline{q}, \underline{y}), f(\overline{q}, \overline{y})]$. Hence (ii) holds. Let g be another affine representation of I^*. We note that for every $(q, y) \in I^*$, $g(q, y) = g(\omega_2^*, y^*)$. Let $a := g(\omega_2^*, y^*)$ and $b = 1$. Define $h = a + bf$. Then for every $(q, y) \in I^*$, $h(q, y) = g(\omega_2^*, y^*) = g(q, y)$. Hence $g = a + bf$ and the moreover remark holds. We now simply note that under the present assumptions $I^* = I_2$. Hence there neither exists $(q, y) \in I_2$ such that $(q, y) \succ (\overline{q}, \overline{y})$ nor $(q, y) \in I_2$ such that $(\underline{q}, \underline{y}) \succ (q, y)$. Hence it follows that I^* satisfies all the properties for being a a basis for I_2.

Step 2: Show that when there exists $(q, y) \in \Delta(\Omega) \times Y$ such that $(i_{\omega_2^*}, y^*) \succ (q, y)$, then there exists $(\overline{q}, \overline{y}), (\underline{q}, \underline{y}), [(\overline{q}, \overline{y}) \succ (\underline{q}, \underline{y})] \in \Delta(\Omega) \times Y$ such that properties (ii) and (iii) holds.

Let $(i_{\omega_2^*}, y^*) \succ (q, y)$. Then we know from claim 9 that there exists $\omega \in C(q)$ such that $(q, y) \succeq (i_\omega, y)$. Under the present assumptions, it must be that $\omega = \omega_1^* \neq \omega_2^*$. Hence we have, $(i_{\omega_2^*}, y^*) \succ (q, y) \succeq (i_{\omega_1^*}, y)$. But then we know from claim 10 that there exists $y^{**} \in Y$ such that $(i_{\omega_1^*}, y^{**}) \sim (i_{\omega_1^*}, y)$ and $(i_{\omega_2^*}, y^{**}) \sim (i_{\omega_2^*}, y^*)$. Let $(\overline{q}, \overline{y}) = (i_{\omega_2^*}, y^{**})$, $(\underline{q}, \underline{y}) = (i_{\omega_1^*}, y^{**})$, and $I^* := \{(q', y') \in \Delta(\Omega) \times Y | (\overline{q}, \overline{y}) \succeq (q', y') \succeq (\underline{q}, \underline{y})\}$. Clearly (i) is satisfied. By unique solvability, we know that for every $(q', y') \in I^*$, there exists uniquely $\alpha^*_{q',y'} \in [0, 1]$ such that $(\alpha^*_{q',y'} i_{\omega_2^*} + (1 - \alpha^*_{q',y'}) i_{\omega_1^*}, y^{**}) \sim (q', y')$. Let $f : I^* \to \mathbf{R}$ be the function for which $f(q', y') = \alpha^*_{q',y'}$. From Claim 1, we know that this function represents preferences on I^*. From claim 2, we know that it is affine and onto the interval $[f(\underline{q}, \underline{y}), f(\overline{q}, \overline{y})]$. Hence (ii) holds. To see that the moreover remark holds, consider another affine representation g of the preferences on I^*. Let $a = g(\underline{q}, \underline{y}) - \frac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]} f(\underline{q}, \underline{y})$, $b = \frac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]}$ and $h = a + bf$ We note that it is an implication of Claim 3 that h also is an affine representation of the preferences on I^*.

Then we have:

$$\begin{array}{rcl} h(\overline{q}, \overline{y}) & = & g(\underline{q}, \underline{y}) - \frac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]} f(\underline{q}, \underline{y}) + \frac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]} f(\overline{q}, \overline{y}) \\ & = & g(\overline{q}, \overline{y}) \end{array}$$

Likewise, we have

$$\begin{array}{rcl} h(\underline{q}, \underline{y}) & = & g(\underline{q}, \underline{y}) - \frac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]} f(\underline{q}, \underline{y}) + \frac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]} f(\underline{q}, \underline{y}) \\ & = & g(\underline{q}, \underline{y}) \end{array}$$

Hence h and g coincides at both $(\overline{q}, \overline{y})$, and $(\underline{q}, \underline{y})$. But then it follows from claim 4 that h and g coincide on $I^* \cap I^* = I^*$. Hence $g = a + bf$ and the moreover remark holds.

All that remains in order to show that I^* is a basis is to show that for every $(q', y') \in \Delta(\Omega) \times Y$ for which $(i_{\omega_1^*}, y^{**}) \succ (q', y')$, there exists $q_2^*, q_2^{**} \in \Delta(\Omega)$ and $y_2^* \in Y$ such that

$$(i_{\omega_2^*}, y^{**}) \succeq (q_2^*, y_2^*) \succ (i_{\omega_1^*}, y^{**}) \succ (q', y') \sim (q_2^{**}, y_2^*)$$

Let $(i_{\omega_1^*}, y^{**}) \succ (q', y')$. We know from Claim 9 that there exists $\omega \in C(q)$ such that $(q', y') \succeq (i_\omega, y')$. Due to the present assumptions, we know that $\omega = \omega_1^*$. Now, since $\omega_1^* \neq \omega_2^*$, we know from claim 10 that there exist $y^{***} \in Y$ such that $(i_{\omega_1^*}, y^{***}) \sim (i_{\omega_1^*}, y)$ and $(i_{\omega_2^*}, y^{***}) \sim (i_{\omega_2^*}, y^{**}) \sim (i_{\omega_2^*}, y^*)$. Hence, we have

$$(i_{\omega_2^*}, y^{***}) \sim (i_{\omega_2^*}, y^{**}) \sim (i_{\omega_2^*}, y^* \succ (i_{\omega_1^*}, y^{**})(q', y') \succeq (i_{\omega_1^*}, y^{***}))$$

But then it follows from unique solvability that there exists uniquely $\beta^*_{q,y} \in [0, 1]$ such that $(\beta^*_{q,y} i_{\omega_2^*} + (1 - \beta^*_{q,y}) i_{\omega_1^*}, y^{***}) \sim (q, y)$.

Hence, if we set $(q_1^*, y_2^*) = (i_{\omega_2^*}, y^{***})$ and $(q_1^{**}, y_1^*) = (\beta^*_{q,y} i_{\omega_2^*} + (1 - \beta^*_{q,y}) i_{\omega_1^*}, y^{***})$, we have $(\overline{q}, \overline{y}) \sim (q_2^*, y_2^*) \succ (\underline{q}, \underline{y}) \succ (q', y') \sim (q_2^{**}, y_2^*)$ as required. Q.E.D.

Lemma 11. *Let ω_1^*, ω_2^* be distinct elements of Ω and let y^* be an element of Y such that $[(q, y) \not\sim (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$. Then there exists an affine representation U of the preferences on $I_1 := \{(q, y) \in \Delta(\Omega) \times Y | (q, y) \succeq (i_{\omega_2^*}, y^*)\}$ with the further property that for any $(q, y), (q', y') \in I_1$ for which $(q, y) \succ (q', y')$ the function is onto the interval $[U(q', y'), U(q, y)]$. Moreover, if $V : I_1 \to \mathbf{R}$ is another affine representation, then there is a, b $(b > 0)$ such that $V = a + bU$.*

Proof of Lemma 7: From Lemma 5, we know that there exists a basis interval I^* for I_1 with an affine representation f^*. The basis interval furthermore has the property that if g^* is another affine representation of I^*, then there exists a, b, $[b > 0]$ such that $g^* = a + bf^*$. From Lemma 3, we know that there exists uniquely an affine extension f of f^* to I_1 with the properties that f represents preferences on I_1 and b) that f is onto the interval $[f(q, y), f(q', y')]$ for any arbitrary elements $(q, y), (q', y')$ of I_1. Hence existence is done.

Consider now any other affine representation g of the preferences on I_1. Denote by g^* the restriction of g to I^*. Then we know that there exists a, b $(b > 0)$ such that $g^* = a + bf^*$. Define $g' := a + bf$. Then g' has the property that it coincides with g^* on I^*. We know from claim 3 that g' also represents preferences on I_1. Hence g' is an affine extension of g^* to I_1. But we know from Lemma 3 that only one such extension exists. But then g' must coincide with g since g is the unique extension of g^*. Hence uniqueness has been established. Q.E.D.

Lemma 12. *Let ω_1^*, ω_2^* be distinct elements of Ω and let y^* be an element of Y such that $[(q, y) \not\succ (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$. Then there exists an affine representation U of the preferences on $I_2 := \{(q, y) \in \Delta(\Omega) \times Y | (i_{\omega_2^*}, y^*) \succeq (q, y)\}$ with the further property that for any $(q, y), (q', y') \in I_2$ for which $(q, y) \succ (q', y')$ the function is onto the interval $[U(q', y'), U(q, y)]$. Moreover, if $V : I_1 \to \mathbf{R}$ is another affine representation, then there is a, b $(b > 0)$ such that $V = a + bU$.*

Proof of Lemma 8: From Lemma 6, we know that there exists a basis interval I^* for I_2 with an affine representation f^*. The basis interval furthermore has the property that if g^* is another affine representation of I^*, then there exists a, b, $[b > 0]$ such that $g^* = a + bf^*$. From Lemma 3, we know that there exists uniquely an affine extension f of f^* to I_2 with the properties that f represents preferences on I_2 and b) that f is onto the interval $[f(q, y), f(q', y')]$ for any arbitrary elements $(q, y), (q', y')$ of I_2. Hence existence is done.

Consider now any other affine representation g of the preferences on I_2. Denote by g^* the restriction of g to I^*. Then we know that there exists a, b $(b > 0)$ such that $g^* = a + bf^*$. Define $g' := a + bf$. Then g' has the property that it coincides with g^* on I^*. We know from claim 3 that g' also represents preferences on I_2. Hence g' is an affine extension of g^* to I_2. But we know from Lemma 3 that only one such extension exists. But then g' must coincide with g since g is the unique extension of g^*. Hence uniqueness has been established. Q.E.D.

Lemma 13. *Let ω_1^*, ω_2^* be distinct elements of Ω and let y^* be an element of Y such that $[(q, y) \not\succ (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$. If an affine function $f : \Delta(\Omega) \times Y \to \mathbf{R}$ represents preferences on $\Delta(\Omega) \times Y$, then the function g defined by*

$$g(q, y) = \begin{cases} a + b_1[f(q, y) - f(i_{\omega_2^*}, y^*)] \text{ if } (q, y) \succeq (i_{\omega_2^*}, y^*) \\ a + b_2[f(q, y) - f(i_{\omega_2^*}, y^*)] \text{ otherwise} \end{cases}$$

(where $b_1 > 0$ and $b_2 > 0$) also (i) represents preferences on $\Delta(\Omega) \times Y$; and (ii) is affine. Furthermore, if the function f is onto the interval $[f(q_2, y_2), f(q_1, y_1)]$ then (iii) the function g is onto the interval $[g(q_2, y_2), g(q_1, y_1)]$.

Proof of Lemma 9: Let $I_1 := \{(q, y) \in \Delta(\Omega) \times Y | (q, y) \succeq (i_{\omega_2^*}, y^*)\}$ and $I_2 := \{(q, y) \in \Delta(\Omega) \times Y | (i_{\omega_2^*}, y^*) \succeq (q, y)\}$. Define $g_1 : I_1 \to \mathbf{R}$ by $g_1(q, y) = a + b_1[f(q, y) - f(i_{\omega_2^*}, y^*)]$ and

$g_2 : I_2 \to \mathbf{R}$ by $g_2(q, y) = a + b_2[f(q, y) - f(i_{\omega_2^*}, y^*)]$. We note that g_1 is simply the restriction of g to I_1 and that g_2 is simply the restriction of g to I_2.

Step 1: Show that g_1 is an affine representation of the preferences on I_1.

Denote by f_1 the restriction of f to I_1. Since f is an affine representation of the preferences on $\Delta(\Omega) \times Y$, it must be that f_1 is an affine representation of the preferences on I_1. Let $a_1^* = a - b_1 f(i_{\omega_2^*}, y^*)$. Then $g_1 = a_1^* + b_1 f_1$. Hence it follows from Claim 3 that g_1 is an affine representation of the preferences on I_1.

Step 2: Show that g_2 is an affine representation of the preferences on I_2.

Denote by f_2 the restriction of f to I_2. Since f is an affine representation of the preferences on $\Delta(\Omega) \times Y$, it must be that f_2 is an affine representation of the preferences on I_2. Let $a_2^* = a - b_2 f(i_{\omega_2^*}, y^*)$. Then $g_2 = a_2^* + b_2 f_2$. Hence it follows from Claim 3 that g_2 is an affine representation of the preferences on I_2.

Step 3: Show that $[(q, y) \succeq (q', y')] \Rightarrow [g(q, y) \geq g(q', y')]$.

(i) if $(q', y') \succeq (i_{\omega_2^*}, y^*)$ then (q, y) and (q', y') are both in I_1. We know that g_1 represents preferences on I_1. Hence we have $g_1(q, y) \geq g_1(q', y')$. But g_1 is simply the restriction of g to I_1. Hence $g(q, y) \geq g(q', y')$. (ii) if $(i_{\omega_2^*}, y^*) \succeq (q, y)$ then (q, y) and (q', y') are both in I_2. We know that g_2 represents preferences on I_2. Hence we have $g_2(q, y) \geq g_2(q', y')$. But g_2 is simply the restriction of g to I_2. Hence $g(q, y) \geq g(q', y')$. (iii) If $(q, y) \succ (i_{\omega_2^*}, y^*) \succ (q', y')$ then we have $(q, y) \in I_1 \setminus I_2$ and $(q', y') \in I_2 \setminus I_1$. Since g_1 represents the preferences on I_1, we have $g_1(q, y) > g_1(i_{\omega_2^*}, y^*)$. Likewise, since g_2 represents the preferences on I_2, we have $g_1(i_{\omega_2^*}, y^*) > g_1(q', y')$. But g_1 and g_2 are simply the restrictions of g to I_1 and I_2 respectively. Hence we must have $g(q, y) > g(i_{\omega_2^*}, y^*) > g(q', y')$.

Step 4: Show that $[g(q, y) \geq g(i_{\omega_2^*}, y^*)] \Rightarrow [(q, y) \in I_1]$.

Suppose $(q, y) \notin I_1$. Then $(i_{\omega_2^*}, y^*) \succ (q, y)$. Since f represents the preferences on $\Delta(\Omega) \times Y$, we must then have $f(i_{\omega_2^*}, y^*) > f(q, y)$. But then we have

$$\begin{array}{rcl} g(q, y) & = & a + b_2[f(q, y) - f(i_{\omega_2^*}, y^*)] \\ & < & a \\ & = & a + b_2[f(i_{\omega_2^*}, y^*) - f(i_{\omega_2^*}, y^*)] \\ & = & g(i_{\omega_2^*}, y^*) \end{array}$$

a contradiction. Hence $(q, y) \in I_1$.

Step 5: Show that $[g(i_{\omega_2^*}, y^*) \geq g(q, y)] \Rightarrow [(q, y) \in I_2]$.

Suppose $(q, y) \notin I_2$. Then $(q, y) \succ (i_{\omega_2^*}, y^*)$. Since f represents the preferences on $\Delta(\Omega) \times Y$, we must then have $f(q, y) > f(i_{\omega_2^*}, y^*)$. But then we have

$$\begin{array}{rcl} g(q, y) & = & a + b_1[f(q, y) - f(i_{\omega_2^*}, y^*)] \\ & > & a \\ & = & a + b_1[f(i_{\omega_2^*}, y^*) - f(i_{\omega_2^*}, y^*)] \\ & = & g(i_{\omega_2^*}, y^*) \end{array}$$

a contradiction. Hence $(q, y) \in I_2$.

Step 6: Show that $[g(q,y) \geq g(q',y')] \Rightarrow [(q,y) \succeq (q',y')]$.

(i) If $g(q',y') \geq g(i_{\omega_2^*}, y^*)$ then step 4 implies that both (q,y) and (q',y') are in I_1. Since g_1 is the restriction of g to I_1, it follows that $g_1(q,y) \geq g_1(q',y')$. From step 1 we know that g_1 represents the preferences on I_1. Hence we must then have $(q,y) \succeq (q',y')$ as required.

(ii) If $g(i_{\omega_2^*}, y^*) \geq g(q,y)$ then step 5 implies that both (q,y) and (q',y') are in I_2. Since g_2 is the restriction of g to I_2, it follows that $g_2(q,y) \geq g_2(q',y')$. From step 2 we know that g_2 represents the preferences on I_2. Hence we must then have $(q,y) \succeq (q',y')$ as required.

(iii) If $g(q,y) > g(i_{\omega_2^*}, y^*) > g(q',y')$, then we have $(q,y) \in I_1$ and $(q',y') \in I_2$. Since g_1 is the restriction of g to I_1, it follows that $g_1(q,y) \geq g_1(i_{\omega_2^*}, y^*)$. From step 1 we know that g_1 represents the preferences on I_1. Hence we must then have $(q,y) \succeq (i_{\omega_2^*}, y^*)$. Likewise, since g_2 is the restriction of g to I_2, it follows that $g_2(i_{\omega_2^*}, y^*) \geq g_2(q',y')$. From step 2 we know that g_2 represents the preferences on I_2. Hence we must then have $(i_{\omega_2^*}, y^*) \succeq (q',y')$. Then it follows that $(q,y) \succeq (i_{\omega_2^*}, y^*) \succeq (q',y')$ as required.

From (i)-(iii) we can then conclude that $[g(q,y) \geq g(q',y')] \Rightarrow [(q,y) \succeq (q',y')]$.

Step 7: Conclude from steps 3B and 6E that g represents the preferences on $\Delta(\Omega) \times Y$.

Step 8: Show that for every $y \in Y$, the set $\{(q',y') \in \Delta(\Omega) \times Y | y' = y\}$ is either (i) a subset of I_1 or (ii) a subset of I_2.

Let $y \in Y$. If neither (i) nor (ii) held, then there exists $q_1^*, q_2^* \in \Delta(\Omega)$ such that $(q_1^*, y) \succ (i_{\omega_2^*}, y^*)$ and $(i_{\omega_2^*}, y^*) \succ (q_2^*, y)$. From Claim 9, it follows that there exists $\omega_1 \in C(q_1^*)$ and $\omega_2 \in C(q_2^*)$ such that $(i_{\omega_1}, y) \succeq (q_1^*, y) \succ (i_{\omega_2^*}, y^*) \succ (q_2^*, y) \succeq (i_{\omega_1}, y)$. Under the present assumptions, it must be that $\omega_1 = \omega_1^*$ and $\omega_2 = \omega_1^*$. Hence we have $(i_{\omega_1^*}, y) \succeq (q_1^*, y) \succ (i_{\omega_2^*}, y^*) \succ (q_2^*, y) \succeq (i_{\omega_1^*}, y)$ a contradiction.

Step 9: Show that g is affine.

Let $(q,y), (q',y) \in \Delta(\Omega) \times Y$ satisfy $(q,y) \succeq (q',y)$. From step 8 it follows that either both (q,y) and (q',y) are in I_1 or both (q,y) and (q',y) are in I_2.

(i) If both (q,y) and (q,y') are in I_1, then we know from step 8 that for every $\alpha \in [0,1]$, $(\alpha q + (1-\alpha)q', y) \in I_1$. Since g_1 is an affine representation of the preferences on I_1, it follows that $g_1(\alpha q + (1-\alpha)q', y) = \alpha g_1(q,y) + (1-\alpha) g_1(q',y)$. But g_1 is simply the restriction of g to I_1. Hence we must have $g(\alpha q + (1-\alpha)q', y) = \alpha g(q,y) + (1-\alpha) g(q',y)$ as required.

(ii) If both (q,y) and (q,y') are in I_2, then we know from step 8 that for every $\alpha \in [0,1]$, $(\alpha q + (1-\alpha)q', y) \in I_2$. Since g_2 is an affine representation of the preferences on I_2, it follows that $g_2(\alpha q + (1-\alpha)q', y) = \alpha g_2(q,y) + (1-\alpha) g_2(q',y)$. But g_2 is simply the restriction of g to I_2. Hence we must have $g(\alpha q + (1-\alpha)q', y) = \alpha g(q,y) + (1-\alpha) g(q',y)$ as required.

Step 10: Show that g is onto the interval $[g(q_2,y_2), g(q_1,y_1)]$ whenever f is onto the interval $[f(q_2,y_2), f(q_1,y_1)]$.

Suppose f is onto the interval $[f(q_2,y_2), f(q_1,y_1)]$. Pick any $\alpha \in [g(q_2,y_2), g(q_1,y_1)]$.

(i) $\alpha \leq g(i_{\omega_2^*}, y^*)$. Define $\beta = \frac{\alpha - [a - b_2 f(i_{\omega_2^*}, y^*)]}{b_2}$. Then we have:

$$\begin{array}{rcl} f(q_2, y_2) & = & \frac{g(q_2,y_2)-[a-b_2 f(i_{\omega_2^*},y^*)]}{b_2} \\ & \leq & \frac{\alpha-[a-b_2 f(i_{\omega_2^*},y^*)]}{b_2} \\ & \leq & \frac{\min\{g(q_1,y_1),g(i_{\omega_2^*},y^*)\}-[a-b_2 f(i_{\omega_2^*},y^*)]}{b_2} \\ & = & \min\{f(q_1, y_1), f(i_{\omega_2^*}, y^*)\} \\ & \leq & f(q_1, y_1) \end{array}$$

Hence $\beta \in [f(q_2, y_2, f(q_1, y_1]$. Since f is onto the interval $[f(q_2, y_2, f(q_1, y_1]$, it follows that there exists $(q, y) \in \Delta(\Omega) \times Y$ such that $f(q, y) = \beta \leq f(i_{\omega_2^*}, y^*)$. Then it follows from the definition of g that

$$\begin{array}{rcl} g(q, y) & = & a + b_2[\beta - f(i_{\omega_2^*}, y^*)] \\ & = & a - b_2 f(i_{\omega_2^*}, y^*) + b_2[\frac{\alpha-[a-b_2 f(i_{\omega_2^*},y^*)]}{b_2}] \\ & = & a - b_2 f(i_{\omega_2^*}, y^*) + \alpha - [a - b_2 f(i_{\omega_2^*}, y^*)] \\ & = & \alpha \end{array}$$

as required.

(ii) $\alpha \geq g(i_{\omega_2^*}, y^*)$. Define $\beta = \frac{\alpha-[a-b_1 f(i_{\omega_2^*},y^*)]}{b_1}$. Then we have:

$$\begin{array}{rcl} f(q_2, y_2) & \leq & \max\{f(q_2, y_2), f(i_{\omega_2^*}, y^*)\} \\ & = & \frac{\max\{g(q_2,y_2),g(i_{\omega_2^*},y^*)\}-[a-b_1 f(i_{\omega_2^*},y^*)]}{b_1} \\ & \leq & \frac{\alpha-[a-b_1 f(i_{\omega_2^*},y^*)]}{b_1} \\ & \leq & \frac{g(q_1,y_1)-[a-b_1 f(i_{\omega_2^*},y^*)]}{b_1} \\ & = & f(q_1, y_1) \end{array}$$

Hence $\beta \in [f(q_2, y_2, f(q_1, y_1]$. Since f is onto the interval $[f(q_2, y_2, f(q_1, y_1]$, it follows that there exists $(q, y) \in \Delta(\Omega) \times Y$ such that $f(q, y) = \beta \leq f(i_{\omega_2^*}, y^*)$. Then it follows from the definition of g that

$$\begin{array}{rcl} g(q, y) & = & a + b_1[\beta - f(i_{\omega_2^*}, y^*)] \\ & = & a - b_1 f(i_{\omega_2^*}, y^*) + b_1[\frac{\alpha-[a-b_1 f(i_{\omega_2^*},y^*)]}{b_1}] \\ & = & a - b_1 f(i_{\omega_2^*}, y^*) + \alpha - [a - b_1 f(i_{\omega_2^*}, y^*)] \\ & = & \alpha \end{array}$$

as required.

Q.E.D.

Lemma 14. *Let* ω_1^*, ω_2^* *be distinct elements of* Ω *and let* y^* *be an element of* Y *such that* $[(q, y) \not\succ (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$. *Then there exists an affine representation* $U : \Delta(\Omega) \times Y \to \mathbf{R}$ *of the preferences on* $\Delta(\Omega) \times Y$ *with the further property that for any* $(q, y), (q', y) \in \Delta(\Omega) \times Y$ *for which* $(q, y) \succ (q', y')$ *the function* U *is onto the interval* $[U(q', y'), U(q, y)]$. *Moreover, if* $V : \Delta(\Omega) \times Y \to \mathbf{R}$ *is another affine representation of the preferences on* $\Delta(\Omega) \times Y$, *then there is* $a, b_1, b_2 (b_1 > 0, b_2 > 0)$ *such that:*

$$V(q, y) = \left\{ \begin{array}{l} a + b_1[U(q, y) - U(i_{\omega_2^*}, y^*)] \textit{ if } (q, y) \succeq (i_{\omega_2^*}, y^*) \\ a + b_2[U(q, y) - U(i_{\omega_2^*}, y^*)] \textit{ otherwise} \end{array} \right.$$

Proof of Lemma 10: Step 1: Show that Lemma 10 holds when it is false that there exists $(q, y) \in \Delta(\Omega) \times Y$ such that $(i_{\omega_2^*}, y^*) \succ (q, y)$.

If there does not exist $(q, y) \in \Delta(\Omega) \times Y$ such that $(i_{\omega_2^*}, y^*) \succ (q, y)$, then it follows that $\Delta(\Omega) \times Y = \{(q, y) \in \Delta(\Omega) \times Y | (q, y) \succeq (i_{\omega_2^*}, y^*)\}$. But then all the required properties follow from Lemma 7.

Step 2: Show that Lemma 10 holds when it is false that there exists $(q, y) \in \Delta(\Omega) \times Y$ such that $(q, y) \succ (i_{\omega_2^*}, y^*)$.

If there does not exist $(q, y) \in \Delta(\Omega) \times Y$ such that $(q, y) \succ (i_{\omega_2^*}, y^*)$, then it follows that $\Delta(\Omega) \times Y = \{(q, y) \in \Delta(\Omega) \times Y | (i_{\omega_2^*}, y^*) \succeq (q, y)\}$. But then all the required properties follow from Lemma 8.

Step 3: Show that the Lemma holds if there exists $(q_1, y_1), (q_2, y_2) \in \Delta(\Omega) \times Y$ such that $(q_1, y_1) \succ (i_{\omega_2^*}, y^*) \succ (q_2, y_2)$.

Let $I_1 := \{(q, y) \in \Delta(\Omega) \times Y | (q, y) \succeq (i_{\omega_2^*}, y^*)\}$ and $I_2 := \{(q, y) \in \Delta(\Omega) \times Y | (i_{\omega_2^*}, y^*) \succeq (q, y)\}$. Let U_1 be the affine representation of the preferences on I_1 given in Lemma 7 and let U_2^* be the affine representation of the preferences on I_2 given in Lemma 8. Now, let $a = U_1(i_{\omega_2^*}, y^*) - U_2^*(i_{\omega_2^*}, y^*)$, $b = 1$, and define $U_2 := a + bU_2^*$.

From Claim 3, we know that U_2 is an affine representation of the preferences on I_2 with the further property that for any $(q, y), (q', y') \in I_2$ such that $(q, y) \succ (q', y')$, the function U_2 is onto the interval $[U_2(q', y'), U_2(q, y)]$.

Step 3A: Show that U_1 and U_2 coincide on $I_1 \cap I_2$.

Let $(q, y) \in I_1 \cap I_2$. We note that $(q, y) \sim (i_{\omega_2^*}, y^*)$. Hence by the definition of U_2^*, and representation on the intervals I_1 and I_2 respectively, it follows that $U_2(q, y) = U_2(i_{\omega_2^*}, y^*) = U_1(i_{\omega_2^*}, y^*) - U_2^*(i_{\omega_2^*}, y^*) + U_2^*(i_{\omega_2^*}, y^*) = U_1(i_{\omega_2^*}, y^*) = U_1(q, y)$ as required.

Now, let $U : \Delta(\Omega) \times Y \to \mathbf{R}$ be the function that coincides with U_1 on I_1 and U_2 on I_2 respectively.

Step 3B: Show that $[(q, y) \succeq (q', y')] \Rightarrow [U(q, y) \geq U(q', y')]$.

(i) if $(q', y') \succeq (i_{\omega_2^*}, y^*)$ then (q, y) and (q', y') are both in I_1. We know that U_1 represents preferences on I_1. Hence we have $U_1(q, y) \geq U_1(q', y')$. But U_1 is simply the restriction of U to I_1. Hence $U(q, y) \geq U(q', y')$. (ii) if $(i_{\omega_2^*}, y^*) \succeq (q, y)$ then (q, y) and (q', y') are both in I_2. We know that U_2 represents preferences on I_2. Hence we have $U_2(q, y) \geq U_2(q', y')$. But U_2 is simply the restriction of U to I_2. Hence $U(q, y) \geq U(q', y')$. (iii) If $(q, y) \succ (i_{\omega_2^*}, y^*) \succ (q', y')$ then we have $(q, y) \in I_1 \setminus I_2$ and $(q', y') \in I_2 \setminus I_1$. Since U_1 represents the preferences on I_1, we have $U_1(q, y) > U_1(i_{\omega_2^*}, y^*)$. Likewise, since U_2 represents the preferences on I_2, we have $U_1(i_{\omega_2^*}, y^*) > U_1(q', y')$. But U_1 and U_2 are simply the restrictions of U to I_1 and I_2 respectively. Hence we must have $U(q, y) > U(i_{\omega_2^*}, y^*) > U(q', y')$ as required.

Step 3C: Show that $[U(q, y) \geq U(i_{\omega_2^*}, y^*)] \Rightarrow [(q, y) \in I_1]$.

Suppose $(q, y) \notin I_1$. Then $(i_{\omega_2^*}, y^*) \succ (q, y)$. Since U_2 represents the preferences on I_2 and U coincides with U_2 on I_2, it follows that $U(i_{\omega_2^*}, y^*) = U_2(i_{\omega_2^*}, y^*) > U_2(q, y) = U(q, y)$ a contradiction. Hence $(q, y) \in I_1$.

Step 3D: Show that $[U(i_{\omega_2^*}, y^*) \geq U(q, y)] \Rightarrow [(q, y) \in I_2]$.

Suppose $(q, y) \notin I_2$. Then $(q, y) \succ (i_{\omega_2^*}, y^*)$. Since U_1 represents the preferences on I_1 and U coincides with U_1 on I_1, it follows that $U(i_{\omega_2^*}, y^*) = U_1(i_{\omega_2^*}, y^*) < U_1(q, y) = U(q, y)$ a contradiction. Hence $(q, y) \in I_2$.

Step 3E: Show that $[U(q, y) \geq U(q', y')] \Rightarrow [(q, y) \succeq (q', y')]$.

(i) If $U(q', y') \geq U(i_{\omega_2^*}, y^*)$ then step 3C implies that both (q, y) and (q', y') are in I_1. Since U_1 is the restriction of U to I_1, it follows that $U_1(q, y) \geq U_1(q', y')$. From its definition, we know that U_1 represents the preferences on I_1. Hence we must then have $(q, y) \succeq (q', y')$ as required.

(ii) If $U(i_{\omega_2^*}, y^*) \geq U(q, y)$ then step 3D implies that both (q, y) and (q', y') are in I_2. Since U_2 is the restriction of U to I_2, it follows that $U_2(q, y) \geq U_2(q', y')$. From its definition, we know that U_2 represents the preferences on I_2. Hence we must then have $(q, y) \succeq (q', y')$ as required.

(iii) If $U(q, y) > U(i_{\omega_2^*}, y^*) > U(q', y')$, then it follows from steps 3C and 3D that $(q, y) \in I_1$ and $(q', y') \in I_2$. Since U_1 is the restriction of U to I_1, it follows that $U_1(q, y) \geq U_1(i_{\omega_2^*}, y^*)$. From its definition, we know that U_1 represents the preferences on I_1. Hence we must then have $(q, y) \succeq (i_{\omega_2^*}, y^*)$. Likewise, since U_2 is the restriction of U to I_2, it follows that $U_2(i_{\omega_2^*}, y^*) \geq U_2(q', y')$. From its definition, we know that U_2 represents the preferences on I_2. Hence we must then have $(i_{\omega_2^*}, y^*) \succeq (q', y')$. Then it follows that $(q, y) \succeq (i_{\omega_2^*}, y^*) \succeq (q', y')$ as required.

From (i)-(iii) we can then conclude that $[U(q, y) \geq U(q', y')] \Rightarrow [(q, y) \succeq (q', y')]$.

Step 3F: Conclude from steps 3B and 3E that U represents the preferences on $\Delta(\Omega) \times Y$.

Step 3G: Show that for every $y \in Y$, the set $\{(q', y') \in \Delta(\Omega) \times Y | y' = y\}$ is either (i) a subset of I_1 or (ii) a subset of I_2.

Let $y \in Y$. If neither (i) nor (ii) held, then there exists $q_1^*, q_2^* \in \Delta(\Omega)$ such that $(q_1^*, y) \succ (i_{\omega_2^*}, y^*)$ and $(i_{\omega_2^*}, y^*) \succ (q_2^*, y)$. From Claim 9, it follows that there exists $\omega_1 \in C(q_1^*)$ and $\omega_2 \in C(q_2^*)$ such that $(i_{\omega_1}, y) \succeq (q_1^*, y) \succ (i_{\omega_2^*}, y^*) \succ (q_2^*, y) \succeq (i_{\omega_1}, y)$. Under the present assumptions, it must be that $\omega_1 = \omega_1^*$ and $\omega_2 = \omega_1^*$. Hence we have $(i_{\omega_1^*}, y) \succeq (q_1^*, y) \succ (i_{\omega_2^*}, y^*) \succ (q_2^*, y) \succeq (i_{\omega_1^*}, y)$ a contradiction.

Step 3H: Show that U is affine.

Let $(q, y), (q', y) \in \Delta(\Omega) \times Y$ satisfy $(q, y) \succeq (q', y)$. From step 3G it follows that either both (q, y) and (q', y) are in I_1 or both (q, y) and (q', y) are in I_2.

(i) If both (q, y) and (q, y') are in I_1, then we know from step 3G that for every $\alpha \in [0, 1]$, $(\alpha q + (1-\alpha) q', y) \in I_1$. Since U_1 is an affine representation of the preferences on I_1, it follows that $U_1(\alpha q + (1-\alpha) q', y) = \alpha U_1(q, y) + (1-\alpha) U_1(q', y)$. But U_1 is simply the restriction of U to I_1. Hence we must have $U(\alpha q + (1-\alpha) q', y) = \alpha U(q, y) + (1-\alpha) U(q', y)$ as required.

(ii) If both (q, y) and (q, y') are in I_2, then we know from step 3G that for every $\alpha \in [0, 1]$, $(\alpha q + (1-\alpha) q', y) \in I_2$. Since U_2 is an affine representation of the preferences on I_2, it follows that $U_2(\alpha q + (1-\alpha) q', y) = \alpha U_2(q, y) + (1-\alpha) U_2(q', y)$. But U_2 is simply the restriction of U to I_2. Hence we must have $U(\alpha q + (1-\alpha) q', y) = \alpha U(q, y) + (1-\alpha) U(q', y)$ as required.

Step 3I: Show that U is onto the interval $[U(q_2, y_2), U(q_1, y_1)]$.

Let $\alpha \in [U(q_2, y_2), U(q_1, y_1)]$. (i) If $U(q_2, y_2) \geq U(i_{\omega_2^*}, y^*)$, then we know from step 3C that (q_1, y_1) and (q_2, y_2) are elements of I_1. From its definition, we know that U_1 is onto the interval $[U_1(q_2, y_2), U_1(q_1, y_1)]$. But U_1 is simply the restriction of U to I_1. Hence it follows that U is onto the interval $[U(q_2, y_2), U(q_1, y_1)]$. (ii) If $U(i_{\omega_2^*}, y^*) \geq U(q_1, y_1)$, then we know from step 3D that (q_1, y_1) and (q_2, y_2) are elements of I_2. From its definition, we know that U_2 is onto the interval $[U_2(q_2, y_2), U_2(q_1, y_1)]$. But U_2 is simply the restriction of U to I_2. Hence it follows that U is onto the interval $[U(q_2, y_2), U(q_1, y_1)]$. (iii) If $U(q_1, y_1) > U(i_{\omega_2^*}, y^*) > U(q_2, y_2)$, then we know from steps 3C and 3D that $(q_1, y_1) \in I_1$ and $(q_2, y_2) \in I_2$. From the definition of U_1, we know that it is onto the interval $[U_1(i_{\omega_2^*}, y^*), U_1(q_1, y_1)]$. Likewise, from the definition of U_2, we know that it is onto the interval $[U_2(q_2, y_2), U_2(i_{\omega_2^*}, y^*)]$. But U_1 and U_2 are simply the restrictions of U to I_1 and I_2 respectively. Hence we know that U is onto both the interval $[U(i_{\omega_2^*}, y^*), U(q_1, y_1)]$ and the interval $[U(q_2, y_2), U(i_{\omega_2^*}, y^*)]$. But then U is onto the interval $[U(q_2, y_2), U(q_1, y_1)]$ as required.

Step 3J: Show that the moreover statement holds.

Let $V : \Delta(\Omega) \times Y \to \mathbf{R}$ be another affine representation of the preferences on $\Delta(\Omega) \times Y$. Denote by V_1 and V_2 the restrictions of V to I_1 and I_2 respectively. From Lemma 7, we know that there exists a_1, b_1 $(b_1 > 0)$ such that $V_1 = a_1 + b_1 U_1$. Likewise, from Lemma 8, we know that there exists a_2, b_2 $(b_2 > 0)$ such that $V_2 = a_2 + b_2 U_2$. Let $a_1^* = a_1 + b_1 U_1(i_{\omega_2^*}, y^*)$ and $a_2^* = a_2 + b_2 U_2(i_{\omega_2^*}, y^*)$. Then we have $V_1 = a_1^* + b_1[U_1 - U_1(i_{\omega_2^*}, y^*)]$ and $V_2 = a_2^* + b_2[U_2 - U_2(i_{\omega_2^*}, y^*)]$. Since $(i_{\omega_2^*}, y^*) \in I_1 \cap I_2$, we know that

$$\begin{aligned}
a_1^* &= a_1^* + b_1[U_1(i_{\omega_2^*}, y^*) - U_1(i_{\omega_2^*}, y^*)] \\
&= V_1(i_{\omega_2^*}, y^*) \\
&= V(i_{\omega_2^*}, y^*) \\
&= V_2(i_{\omega_2^*}, y^*) \\
&= a_2^* + b_2[U_2(i_{\omega_2^*}, y^*) - U_2(i_{\omega_2^*}, y^*)] \\
&= a_2^*.
\end{aligned}$$

Let $a := a_1^* = a_2^*$. Then we have $V_1 = a + b_1[U_1 - U_1(i_{\omega_2^*}, y^*)]$ and $V_2 = a + b_2[U_2 - U_2(i_{\omega_2^*}, y^*)]$. But V_1 and V_2 are simply the restrictions of V to I_1 and I_2 respectively. Likewise, U_1 and U_2 are simply the restrictions of U to I_1 and I_2 respectively. Hence we must have that

$$V(q, y) = \begin{cases} a + b_1[U(q, y) - U(i_{\omega_2^*}, y^*)] & \text{if } (q, y) \succeq (i_{\omega_2^*}, y^*) \\ a + b_2[U(q, y) - U(i_{\omega_2^*}, y^*)] & \text{otherwise} \end{cases}$$

as required.

Q.E.D.

4.6.8 Special Case 3:

Lemma 15. *If $\#\{\omega \in \Omega | \exists y_1, y_2 \in Y : (i_\omega, y_1) \succ (i_\omega, y_2)\} = 0$ then there exists $(\overline{q}, \overline{y}), (\underline{q}, \underline{y}) \in \Delta(\Omega) \times Y$ such that:*

(i) $(\overline{q}, \overline{y}) \succeq (\underline{q}, \underline{y})$;

(ii) The preferences on the interval $I^* := \{(q,y) \in \Delta(\Omega) \times Y | (\overline{q},\overline{y}) \succeq (q,y) \succeq (\underline{q},\underline{y})\}$ *are representable by an affine function* f *that is onto the interval* $[f(\underline{q},\underline{y}), f(\overline{q},\overline{y})]$*; and*

(iii) the interval I^* *is a basis for* $\Delta(\Omega) \times Y$.

Moreover, if g *is another affine representation of the preferences on* I^* *then there exists* a, b $(b > 0)$ *such that* $g = a + bf$.

Proof of Lemma 11: Step 1: Establish the result when there does not exist $(q,y), (q',y') \in \Delta(\Omega) \times Y$ such that $(q,y) \succ (q',y')$.

Pick any $(q^*,y^*) \in \Delta(\Omega) \times Y$. Let $(\overline{q},\overline{y}) = (\underline{q},\underline{y}) := (q^*,y^*)$. Clearly, (i) then holds. We also have $I^* = \{(q,y) \in \Delta(\Omega) \times Y | (q,y) \sim (q^*,y^*\} = \Delta(\Omega) \times Y$ where I^* is nonempty since it has (q^*,y^*) as an element. Now, define $f : I^* \to \mathbf{R}$ by $f(q,y) = 0$ for every $(q,y) \in \Delta(\Omega) \times Y$. Since I^* is an indifference set, it follows that f represents the preferences on I^*. f is also clearly affine and onto the interval $[f(\underline{q},\underline{y}), f(\overline{q},\overline{y})]$. Hence (ii) holds. Let g be another affine representation of I^*. We note that for every $(q,y) \in I^*$, $g(q,y) = g(q^*,y^*)$. Let $a := g(q^*,y^*)$ and $b = 1$. Define $h = a + bf$. Then for every $(q,y) \in I^*$, $h(q,y) = g(q^*,y^*) = g(q,y)$. Hence $g = a + bf$ and the moreover remark holds. We now simply note that under the present assumptions $I^* = \Delta(\Omega) \times Y$. Hence there neither exists $(q,y) \in \Delta(\Omega) \times Y$ such that $(q,y) \succ (\overline{q},\overline{y})$ nor $(q,y) \in \Delta(\Omega) \times Y$ such that $(\underline{q},\underline{y}) \succ (q,y)$. Hence it follows that I^* satisfies all the properties for being a a basis for $\Delta(\Omega) \times Y$.

Step 2: Show that when there exists $(q,y), (q',y') \in \Delta(\Omega) \times Y$ such that $(q,y) \succ (q',y')$, then there exists $(\overline{q},\overline{y}), (\underline{q},\underline{y}), [(\overline{q},\overline{y}) \succ (\underline{q},\underline{y})] \in \Delta(\Omega) \times Y$ such that property (ii) and the moreover remark holds.

Let $(q,y) \succ (q',y')$. Then we know from claim 9 that there exists $\omega_1 \in C(q)$ and $\omega_2 \in C(q')$ such that $(i_{\omega_1}, y) \succeq (q,y) \succ (q',y') \succeq (i_{\omega_2}, y')$. Under the present assumptions, it must be that $\omega_1 \neq \omega_2$. From claim 10 it then follows that there exists $y^* \in Y$ such that $(i_{\omega_1}, y^*) \sim (i_{\omega_1}, y)$ and $(i_{\omega_2}, y^*) \sim (i_{\omega_2}, y')$. Let $(\overline{q},\overline{y}) = (i_{\omega_1}, y^*)$, $(\underline{q},\underline{y}) = (i_{\omega_2}, y^*)$, and $I^* := \{(q',y') \in \Delta(\Omega) \times Y | (\overline{q},\overline{y}) \succeq (q',y') \succeq (\underline{q},\underline{y})\}$. Clearly (i) is satisfied. By unique solvability, we know that for every $(q'',y'') \in I^*$, there exists uniquely $\alpha^*_{q'',y''} \in [0,1]$ such that $(\alpha^*_{q'',y''} i_{\omega_1} + (1 - \alpha^*_{q'',y''}) i_{\omega_2}, y^*) \sim (q'',y'')$. Let $f : I^* \to \mathbf{R}$ be the function for which $f(q'',y'') = \alpha^*_{q'',y''}$. From Claim 1, we know that this function represents preferences on I^*. From claim 2, we know that it is affine and onto the interval $[f(\underline{q},\underline{y}), f(\overline{q},\overline{y})]$. Hence (ii) holds. To see that the moreover remark holds, consider another affine representation g of the preferences on I^*. Let $a = g(\underline{q},\underline{y}) - \frac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]} f(\underline{q},\underline{y})$, $b = \frac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]}$ and $h = a + bf$ We note that it is an implication of Claim 3 that h also is an affine representation of the preferences on I^*.

Then we have:

$$\begin{aligned} h(\overline{q},\overline{y}) &= g(\underline{q},\underline{y}) - \frac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]} f(\underline{q},\underline{y}) + \frac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]} f(\overline{q},\overline{y}) \\ &= g(\overline{q},\overline{y}) \end{aligned}$$

Likewise, we have

$$\begin{aligned} h(\underline{q},\underline{y}) &= g(\underline{q},\underline{y}) - \frac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]} f(\underline{q},\underline{y}) + \frac{[g(\overline{q},\overline{y}) - g(\underline{q},\underline{y})]}{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]} f(\underline{q},\underline{y}) \\ &= g(\underline{q},\underline{y}) \end{aligned}$$

Hence h and g coincides at both $(\overline{q},\overline{y})$, and $(\underline{q},\underline{y})$. But then it follows from claim 4 that h and g coincide on $I^* \cap I^* = I^*$. Hence $g = a + bf$ and the moreover remark holds.

Step 3: (i) Let $(q'',y'') \succ (i_{\omega_1},y^*) = (\overline{q},\overline{y}) = (q,y) \succ (q',y') = (\underline{q},\underline{y}) = (i_{\omega_2},y^*)$. We know from Claim 9 that there exists $\omega \in C(q'')$ such that $(i_\omega,y'') \succeq (q'',y'')$. Due to the present assumptions, we know that $\omega \neq \omega_2$. We know from claim 10 that there exist $y_1^* \in Y$ such that $(i_\omega,y_1^*) \sim (i_\omega,y'')$ and $(i_{\omega_2},y_1^*) \sim (i_{\omega_2},y^*)$. Hence, we have

$$(i_\omega,y_1^*) \succeq (q'',y'') \succ (i_{\omega_1},y^*) \succ (i_{\omega_2},y^*) \sim (i_{\omega_2},y_1^*)$$

But then it follows from unique solvability that there exists uniquely $\beta^*_{q'',y''} \in [0,1]$ such that $(\beta^*_{q'',y''} i_\omega + (1-\beta^*_{q'',y''}) i_{\omega_2}, y_1^*) \sim (q'',y'')$.

Hence, if we set $(q_1^*,y_1^*) = (\beta^*_{q'',y''} i_\omega + (1-\beta^*_{q'',y''}) i_{\omega_2}, y_1^*)$ and $(q_1^{**},y_1^*) = (i_{\omega_2},y_1^*)$, we have $(q_1^*,y_1^*) \sim (q'',y'') \succ (\overline{q},\overline{y}) \succ (q_1^{**},y_1^*) \sim (\underline{q},\underline{y})$ as required.

(ii) Let $(i_{\omega_1},y^*) = (\overline{q},\overline{y}) = (q,y) \succ (q',y') = (\underline{q},\underline{y}) = (i_{\omega_2},y^*) \succ (q'',y'')$. We know from Claim 9 that there exists $\omega \in C(q'')$ such that $(q'',y'') \succeq (i_\omega,y'')$. Due to the present assumptions, we know that $\omega \neq \omega_1$. We know from claim 10 that there exist $y_2^* \in Y$ such that $(i_\omega,y_2^*) \sim (i_\omega,y'')$ and $(i_{\omega_1},y_2^*) \sim (i_{\omega_1},y^*)$. Hence, we have

$$(i_{\omega_1},y_2^*) \sim (i_{\omega_1},y^*) \succ (i_{\omega_2},y^*) \succ (q'',y'') \succeq (i_\omega,y_2^*)$$

But then it follows from unique solvability that there exists uniquely $\beta^*_{q'',y''} \in [0,1]$ such that $(\beta^*_{q'',y''} i_{\omega_1} + (1-\beta^*_{q'',y''}) i_\omega, y_2^*) \sim (q'',y'')$.

Hence, if we set $(q_2^{**},y_2^*) = (\beta^*_{q'',y''} i_{\omega_1} + (1-\beta^*_{q'',y''}) i_\omega, y_2^*)$ and $(q_2^*,y_2^*) = (i_{\omega_1},y_2^*)$, we have $(\overline{q},\overline{y}) \sim (q_2^*,y_2^*) \succ (\underline{q},\underline{y}) \succ (q'',y'') \sim (q_2^{**},y_2^*)$ as required.

Q.E.D.

Lemma 16. *If* $\#\{\omega \in \Omega | \exists y_1,y_2 \in Y : (i_\omega,y_1) \succ (i_\omega,y_2)\} = 0$, *then there exists an affine representation* U *of the preferences on* $\Delta(\Omega) \times Y$ *with the further property that for any* $(q,y),(q',y') \in \Delta(\Omega) \times Y$ *for which* $(q,y) \succ (q',y')$ *the function is onto the interval* $[U(q',y'),U(q,y)]$. *Moreover, if* $V : \Delta(\Omega) \times Y \to \mathbf{R}$ *is another affine representation, then there is* a,b $(b > 0)$ *such that* $V = a + bU$.

Proof of Lemma 12: From Lemma 11, we know that there exists a basis interval I^* for $\Delta(\Omega) \times Y$ with an affine representation f^*. The basis interval furthermore has the property that if g^* is another affine representation of I^*, then there exists a,b, $[b > 0]$ such that $g^* = a + bf^*$. From Lemma 3, we know that there exists uniquely an affine extension f of f^* to $\Delta(\Omega) \times Y$ with the properties that f represents preferences on $\Delta(\Omega) \times Y$ and b) that f is onto the interval $[f(q,y),f(q',y')]$ for any arbitrary elements $(q,y),(q',y')$ of $\Delta(\Omega) \times Y$. Hence existence is done.

Consider now any other affine representation g of the preferences on I_1. Denote by g^* the restriction of g to I^*. Then we know that there exists a,b $(b > 0)$ such that $g^* = a + bf^*$. Define $g' := a + bf$. Then g' has the property that it coincides with g^* on I^*. We know from claim 3 that g' also represents preferences on $\Delta(\Omega) \times Y$. Hence g' is an affine extension of g^* to $\Delta(\Omega) \times Y$. But we know from Lemma 3 that only one such extension exists. But then g' must coincide with g since g is the unique extension of g^*. Hence uniqueness has been established. Q.E.D.

4.6.9 Theorem 1:

Claim 13. *At least one of the following three holds:*

(i) There exist $\omega_1, \omega_2, \omega_3, \omega_4$ $[\omega_1 \neq \omega_3, \omega_1 \neq \omega_4, \omega_2 \neq \omega_3, \omega_2 \neq \omega_4] \in S$ *and* $y_1, y_2, y_3, y_4 \in Y$ *such that* $(i_{\omega_1}, y_1) \succ (i_{\omega_2}, y_2)$ *and* $(i_{\omega_3}, y_3) \succ (i_{\omega_4}, y_4)$

(ii) There exists distinct events $\omega_1^*, \omega_2^* \in$ *and* $y^* \in Y$ *such that*

$$[(q_1, y_1) \not\sim (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$$

(iii) There does not exist $\omega \in \Omega$ *and* $y_1, y_2 \in Y$ *such that* $(i_\omega, y_1) \succ (i_\omega, y_2)$.

Proof of Claim: Step 1: Show that if (i) does not hold then $\#\{\omega \in \Omega | \exists y_1, y_2 \in Y : (i_\omega, y_1) \succ (i_\omega, y_2)\} \leq 1$.

Suppose not. Denote by ω_1^*, ω_2^* any two distinct elements of $\{\omega \in \Omega | \exists y_1, y_2 \in Y : (i_\omega, y_1) \succ (i_\omega, y_2)\}$ and by $y_1, y_2, y_3, y_4 \in Y$ any elements of Y such that $(i_{\omega_1^*}, y_1) \succ (i_{\omega_1^*}, y_2)$ and $(i_{\omega_2^*}, y_3) \succ (i_{\omega_2^*}, y_4)$. Now, set $\omega_1 = \omega_2 = \omega_1^*$ and $\omega_3 = \omega_4 = \omega_2^*$. Then $\omega_1 \neq \omega_3, \omega_1 \neq \omega_4, \omega_2 \neq \omega_3$, and $\omega_2 \neq \omega_4$. But we also have,

$(i_{\omega_1}, y_1) \succ (i_{\omega_2}, y_2)$ and $(i_{\omega_3}, y_3) \succ (i_{\omega_4}, y_4)$ which contradicts our present assumption. Hence $\#\{\omega \in \Omega | \exists y_1, y_2 \in Y : (i_\omega, y_1) \succ (i_\omega, y_2)\} \leq 1$.

Step 2: Show that if (i) does not hold and $\#\{\omega \in \Omega | \exists y_1, y_2 \in Y : (i_\omega, y_1) \succ (i_\omega, y_2)\} = 1$ then (ii) must hold.

Denote by ω_1^* the unique element of $\{\omega \in \Omega | \exists y_1, y_2 \in Y : (i_\omega, y_1) \succ (i_\omega, y_2)\}$, by ω_2^* any other element of Ω, and by y^* an element of Y.

Step 2A: Show that for every $\omega \in \Omega \setminus \{\omega_1^*)$ and any $y' \in Y$, it must follow that $(i_\omega, y') \sim (i_{\omega_2^*}, y^*)$.

Suppose not. Consider any such ω. Clearly, it follows from the present assumption that $\omega \neq \omega_2^*$. Let $y_1^*, y_2^*, y' \in Y$ satisfy $(i^*_{\omega_1}, y_1^*) \succ (i_{\omega_1^*}, y_2^*)$ and $(i_\omega, y') \not\sim (i_{\omega_2^*}, y^*)$. (i) If $(i_\omega, y') \succ (i_{\omega_2^*}, y^*)$, set $\omega_1 = \omega_2 = \omega_1^*$, $\omega_3 = \omega$, $\omega_4 = \omega_2^*$, $y_1 = y_1^*, y_2 = y_2^*, y_3 = y'$, and $y_4 = y^*$. Then $\omega_1 \neq \omega_3, \omega_1 \neq \omega_4, \omega_2 \neq \omega_3, \omega_2 \neq \omega_4$. But we also have $(i_{\omega_1}, y_1) \succ (i_{\omega_2}, y_2)$ and $(i_{\omega_3}, y_3) \succ (i_{\omega_4}, y_4)$ which contradicts our present assumption. Hence $(i_{\omega_2^*}, y^*) \succeq (i_\omega, y')$. (ii) Likewise, if $(i_{\omega_2^*}, y^*) \succ (i_\omega, y')$ set $\omega_1 = \omega_2 = \omega_1^*$, $\omega_3 = \omega_2^*$, $\omega_4 = \omega$, $y_1 = y_1^*, y_2 = y_2^*, y_3 = y^*$, and $y_4 = y'$. Then $\omega_1 \neq \omega_3, \omega_1 \neq \omega_4, \omega_2 \neq \omega_3, \omega_2 \neq \omega_4$. But we also have $(i_{\omega_1}, y_1) \succ (i_{\omega_2}, y_2)$ and $(i_{\omega_3}, y_3) \succ (i_{\omega_4}, y_4)$ which contradicts our present assumption. Hence $(i_\omega, y') \succeq (i_{\omega_2^*}, y^*)$. Conclude from (i) and (ii) that $(i_\omega, y') \sim (i_{\omega_2^*}, y^*)$ as required.

Step 2B: Show that for every $(q, y) \in \Delta(\Omega) \times Y$, $[(q, y) \not\sim (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$ which implies that (ii) holds.

Pick any $(q, y) \in \Delta(\Omega) \times Y$ such that $[(q, y) \not\sim (i_{\omega_2^*}, y^*)]$. (i) if $(q, y) \succ (i_{\omega_2^*}, y^*)$, we know from claim 9 that there exists $\omega \in C(q)$ such that $(i_\omega, y) \succeq (q, y)$. Hence we have $(i_\omega, y) \succeq (q, y) \succ (i_{\omega_2^*}, y^*)$. But then it follows from step 2A that $\omega = \omega_1^*$. Hence we have $\omega_1^* \in C(q)$ which means that $q(\omega_1^*) > 0$ as required. (ii) if $(i_{\omega_2^*}, y^*) \succ (q, y)$, we know from claim 9 that there exists $\omega \in C(q)$ such that $(q, y) \succeq (i_\omega, y)$. Hence we have $(i_{\omega_2^*}, y^*) \succ (q, y) \succeq (i_\omega, y)$. But then it follows from step 2A that $\omega = \omega_1^*$. Hence we have $\omega_1^* \in C(q)$ which means that $q(\omega_1^*) > 0$ as required.

Step 3: Show that if (i) is violated then either (ii) or (iii) holds.

Indeed, from step 1 we know that $\#\{\omega \in \Omega | \exists y_1, y_2 \in Y : (i_\omega, y_1) \succ (i_\omega, y_2)\} \leq 1$. If $\#\{\omega \in \Omega | \exists y_1, y_2 \in Y : (i_E, y_1) \succ (i_E, y_2)\} = 0$ then (iii) holds. If $\#\{\omega \in \Omega | \exists y_1, y_2 \in Y : (i_\omega, y_1) \succ (i_\omega, y_2)\} = 1$ then it follows from step 2 that (ii) holds.

Q.E.D.

Lemma 17. *There exists an affine representation U of the preferences on $\Delta(\Omega) \times Y$ with the further property that for any $(q, y), (q', y') \in \Delta(\Omega) \times Y$ for which $(q, y) \succ (q', y')$ the function is onto the interval $[U(q', y'), U(q, y)]$. Moreover,*

(i) if there does not exist distinct elements ω_1^*, ω_2^* of Ω and y^* in Y such that $[(q, y) \not\prec (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$ then it follows that if $V : \Delta(\Omega) \times Y \to \mathbf{R}$ is another affine representation of the preferences on $\Delta(\Omega) \times Y$, then there is a, b $(b > 0)$ such that $V = a + bU$; and

(ii) if there does exist distinct elements ω_1^*, ω_2^* of Ω and y^* in Y such that $[(q, y) \not\prec (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$ then it follows that if $V : \Delta(\Omega) \times Y \to \mathbf{R}$ is another affine representation of the preferences on $\Delta(\Omega) \times Y$, then there is $a, b_1, b_2 (b_1 > 0, b_2 > 0)$ such that:

$$V(q, y) = \begin{cases} a + b_1[U(q, y) - U(i_{\omega_2^*}, y^*)] & \textit{if } (q, y) \succeq (i_{\omega_2^*}, y^*) \\ a + b_2[U(q, y) - U(i_{\omega_2^*}, y^*)] & \textit{otherwise} \end{cases}$$

Proof of Lemma 13: From Claim 12, we know that at least one of the following holds.

(i) There exist $\omega_1, \omega_2, \omega_3, \omega_4$ $[\omega_1 \neq \omega_3, \omega_1 \neq \omega_4, \omega_2 \neq \omega_3, \omega_2 \neq \omega_4] \in S$ and $y_1, y_2, y_3, y_4 \in Y$ such that $(i_{\omega_1}, y_1) \succ (i_{\omega_2}, y_2)$ and $(i_{\omega_3}, y_3) \succ (i_{\omega_4}, y_4)$

(ii) There exists distinct events $\omega_1^*, \omega_2^* \in$ and $y^* \in Y$ such that

$$[(q_1, y_1) \not\prec (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$$

(iii) There does not exist $\omega \in \Omega$ and $y_1, y_2 \in Y$ such that $(i_\omega, y_1) \succ (i_\omega, y_2)$.

If (i) holds, then our Lemma follows from Lemma 4. Likewise if (ii) holds, then it follows from Lemma 10. Finally, if (iii) holds, then our Lemma follows from Lemma 12.

Q.E.D.

Lemma 18. *If axioms $A1 - A5$ are satisfied, then an affine function $U : \Delta(\Omega) \times Y \to \mathbf{R}$ represents the preferences on $\Delta(\Omega) \times Y$ only if there exists a function $u : \cup_{\omega \in \Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ such that for every $(q, y) \in \Delta(\Omega) \times Y$, $U(q, y) = \sum_{\omega \in C(q)} q(\omega) u(\omega, y(\omega))$.*

Proof of Lemma 14: Let $U : \Delta(\Omega) \times Y \to \mathbf{R}$ be an affine representation of the preferences on $\Delta(\Omega) \times Y$.

Step 1: Show that there exists a function $u^* : \Omega \times Y \to \mathbf{R}$ such that for every $(q, y) \in \Delta(\Omega) \times Y$, $U(q, y) = \sum_{\omega \in C(q)} q(\omega) u^*(\omega, y)$.

Define $u^* : \Delta(\Omega) \times Y \to \mathbf{R}$ by $u^*(\omega, y) = U(i_\omega, y)$. We use induction on the number of elements of $C(q)$ to show that this function has all the required properties. Let $(q, y) \in \Delta(\Omega) \times Y$ satisfy the property that $\#C(q) = 1$. Then $U(q, y) = u^*(\omega, y) = \sum_{\omega \in C(q)} q(\omega) u^*(\omega, y)$ by the definition of u^*. Hence the required properties are satisfied whenever $\#C(q) = 1$.

Suppose now that the desired properties are satisfied whenever $\#C(q) = n-1$ for some $n \geq 2$. We claim that it then must hold whenever $\#C(q) = n$ as well. Indeed, let $(q, y) \in \Delta(\Omega) \times Y$ satisfy the property that $\#C(q) = n$. Let $\omega^* \in C(q)$. Define $q^* \in \Delta(\Omega)$ by $q^*(\omega^*) = 0$ and $q^*(\omega) = \frac{q(\omega)}{[1-q(\omega^*)]}$ for every other $\omega \in \Omega$. Then $\#C(q^*) = n-1$ and $q = q(\omega^*)i_{\omega^*} + (1 - q(\omega^*)q^*$. Since U is affine, we have:

$$\begin{aligned} U(q,y) &= q(\omega^*)U(i_{\omega^*}, y) + (1 - q(\omega^*))U(q^*, y) \\ &= q(\omega^*)u^*(\omega^*, y) + (1 - q(\omega^*)) \textstyle\sum_{\omega \in C(q^*)} \frac{q(\omega)}{[1-q(\omega^*)]} u^*(\omega, y) \\ &= \textstyle\sum_{\omega \in C(q)} q(\omega)u^*(\omega, y) \end{aligned}$$

as required. Hence we conclude using induction that the desired property holds for all such n.

Step 2: Show that for every $(\omega', y'(\omega')) \in \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega))$ there exists a number $\alpha^*_{\omega', y'(\omega')}$ such that $[y''(\omega') = y'(\omega')] \Rightarrow [u^*(\omega', y'') = \alpha^*_{\omega', y'(\omega')}]$.

Pick any $(\omega', y'(\omega')) \in \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega))$ and any $y^* \in Y$ for which $y^*(\omega') = y'(\omega')$. Define $\alpha^*_{\omega', y'(\omega')} = u^*(\omega', y^*)$. Then $U(i_{\omega'}, y^*) = \sum_{\omega \in C(i_{\omega'})} i_{\omega'}(\omega)u^*(\omega, y^*) = u^*(\omega, y^*) = \alpha^*_{\omega', y'(\omega')}$. Now, consider any $y'' \in Y$ for which $y''(\omega') = y'(\omega')$. Axiom 1 states that $(i_{\omega'}, y'') \sim (i_{\omega'}, y^*)$. Since U represents the preferences on $\Delta(\Omega) \times Y$, we must have $U(i_{\omega'}, y'') = U(i_{\omega'}, y^*)$. But then we have $u^*(\omega', y'') = U(i_{\omega'}, y'') = U(i_{\omega'}, y^*) = u^*(\omega', y^*) = \alpha^*_{\omega', y'(\omega')}$ as required.

Step 3: Show that there exists a function $u : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ such that for every $(q, y) \in \Delta(\Omega) \times Y$, $U(q, y) = \sum_{\omega \in C(q)} q(\omega)u(\omega, y(\omega))$.

From step 1, we know that there exists a function $u^* : \Omega \times Y \to \mathbf{R}$ such that $U(q, y) = \sum_{\omega \in C(q)} q(\omega)u^*(\omega, y)$. From step 2, we know that for every $(\omega', y(\omega')) \in \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega))$ there exists a number $\alpha^*_{\omega', y'(\omega')}$ such that $[y''(\omega') = y'(\omega')] \Rightarrow [u^*(\omega', y'') = \alpha^*_{\omega', y'(\omega')}]$. Hence, we can define a function $u : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ by $u(\omega', y(\omega')) = \alpha^*_{\omega', y'(\omega')}$. Then we have $U(q, y) = \sum_{\omega \in C(q)} q(\omega)u^*(\omega, y) = \sum_{\omega \in C(q)} q(\omega)\alpha^*_{\omega, y(\omega)} = \sum_{\omega \in C(q)} q(\omega)u(\omega, y(\omega))$ as required. Q.E.D.

Claim 14. *Let $q_1, q_2 \in \Delta(\Omega)$. Then for every $\alpha \in (0, 1)$, $C(\alpha q_1 + (1-\alpha)q_2) = C(q_1) \cup C(q_2)$.*

Proof of Claim: We note that for every $\omega \in C(q_1) \cup C(q_2)$, it follows that $\alpha q_1(\omega) + (1 - \alpha)q_2(\omega) > 0$. Hence $\omega \in C(\alpha q_1 + (1 - \alpha)q_2)$. We also note that $\omega \in C(\alpha q_1 + (1 - \alpha)q_2)$ implies that $\alpha q_1(\omega) + (1 - \alpha)q_2(\omega) > 0$. But then either $q_1(\omega) > 0$ or $q_2(\omega) > 0$. Hence $\omega \in C(q_1) \cup C(q_3)$ and we can conclude that $C(\alpha q_1 + (1 - \alpha)q_2) = C(q_1) \cup C(q_2)$. Q.E.D.

Proof of Theorem 1: Step 1: Show that Axioms A1-A5 implies the existence of the function u.

From Lemma 13, we know that there exists an affine representation U of the preferences on $\Delta(\Omega) \times Y$ with the further property that for any $(q, y), (q', y') \in \Delta(\Omega) \times Y$ for which $(q, y) \succ (q', y')$ the function is onto the interval $[U(q', y'), U(q, y)]$. From Lemma 14, it follows that there exists a function $u : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ such that for every $(q, y) \in \Delta(\Omega) \times Y$, $U(q, y) = \sum_{\omega \in C(q)} q(\omega)u(\omega, y(\omega))$. This function u hence has all the required properties. Hence existence is done.

Step 2: Show that the only if part of the moreover statement holds.

Let $v : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ be another function with the same properties. Define $V : \Delta(\Omega) \times Y \to \mathbf{R}$ by $V(q, y) = \sum_{\omega\in C(q)} q(\omega)v(\omega, y(\omega))$. This function is clearly an affine representation of the preferences on $\Delta(\Omega) \times Y$.

(i) If there does not exist distinct elements ω_1^*, ω_2^* of Ω and y^* in Y such that $[(q, y) \not\sim (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$, then we know from Lemma 13 that there is a, b $(b > 0)$ such that $V = a+bU$. Consider any $(\omega, y(\omega)) \in \cup_{\omega\in\Omega}(\{\omega\}\times Y(\omega))$. Then there exists $y^* \in Y$ such that $y^*(\omega) = y(\omega)$. Then we must have $v(\omega, y(\omega)) = V(i_\omega, y^*) = a + bU(i_\omega, y^*) = a + bu(\omega, y(\omega))$ as required.

(ii) If there does exist distinct elements ω_1^*, ω_2^* of Ω and y^* in Y such that $[(q, y) \not\sim (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$, then we know from Lemma 13 that there is $a, b_1, b_2 (b_1 > 0, b_2 > 0)$ such that:

$$V(q,y) = \begin{cases} a + b_1[U(q, y) - U(i_{\omega_2^*}, y^*)] \text{ if } (q, y) \succeq (i_{\omega_2^*}, y^*) \\ a + b_2[U(q, y) - U(i_{\omega_2^*}, y^*)] \text{ otherwise} \end{cases}$$

Consider any $(\omega, y(\omega)) \in \cup_{\omega\in\Omega}(\{\omega\}\times Y(\omega))$. Then there exists $y^{**} \in Y$ such that $y^{**}(\omega) = y(\omega)$. If $(i_\omega, y^{**}) \succeq (i_{\omega_2^*}, y^*)$ then we must have $v(\omega, y(\omega)) = V(i_\omega, y^{**}) = a + b_1[U(i_\omega, y^*) - U(i_{\omega_2^*}, y^*)] = a + b_1[u(\omega, y(\omega)) - u(\omega^*, y^*(\omega^*))]$ as required. Likewise, if $(i_{\omega_2^*}, y^*) \succeq (i_\omega, y^{**})$, we must have $v(\omega, y(\omega)) = V(i_\omega, y^{**}) = a + b_2[U(i_\omega, y^{**}) - U(i_{\omega_2^*}, y^*)] = a + b_2[u(\omega, y(\omega)) - u(\omega^*, y^*(\omega^*))]$ as required.

Step 3: Show that the if part of the moreover statement holds if there does not exist distinct elements ω_1^*, ω_2^* of Ω and y^* in Y such that $[(q, y) \not\sim (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$.

Consider any function $v : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ for which there exists a, b $(b > 0)$ such that v is defined by $v(\omega, y(\omega)) = a + bu(\omega, y(\omega))$. Define $V : \Delta(\Omega) \times Y \to \mathbf{R}$ by $V(q, y) = \sum_{\omega\in C(q)} q(\omega)v(\omega, y(\omega))$. Then

$$\begin{aligned} V(q, y) &= \textstyle\sum_{\omega\in C(q)} q(\omega)v(\omega, y(\omega)) \\ &= \textstyle\sum_{\omega\in C(q)} q(\omega)(a + bu(\omega, y(\omega))) \\ &= a + \textstyle\sum_{\omega\in C(q)} q(\omega)bu(\omega, y(\omega)) \\ &= a + bU(q, y). \end{aligned}$$

From Claim 3, it follows that V is an affine representation of the preferences on $\Delta(\Omega)\times Y$ with the required onto property. Hence v has all the required properties.

Step 4: Show that if there exists distinct elements ω_1^*, ω_2^* of Ω and y^* in Y such that $[(q, y) \not\sim (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$, then $[\omega \neq \omega_1^*] \Rightarrow [\forall y \in Y, u(\omega, y(\omega)) = u(\omega_2^*, y^*(\omega_2^*))]$.

Let $\omega \neq \omega_1^*$. Then for every $y \in Y$, it follows from the present assumption that $(i_\omega, y(\omega)) \sim (i_{\omega_2^*}, y^*(\omega_2^*))$. But then by representation, we must have

$$\begin{aligned} u(\omega, y(\omega)) &= \textstyle\sum_{\omega'\in C(i_\omega)} i_\omega(\omega')u(\omega', y(\omega')) \\ &= \textstyle\sum_{\omega'\in C(i_{\omega_2^*})} i_{\omega_2^*}(\omega')u(\omega', y^*(\omega')) \\ &= u(\omega_2^*, y^*(\omega_2^*)) \end{aligned}$$

as required.

Step 5: Show that if there exists distinct elements ω_1^*, ω_2^* of Ω and y^* in Y such that $[(q, y) \not\succ (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$, then $[(i_{\omega_1^*}, y(\omega_1^*)) \succeq (i_{\omega_2^*}, y^*(\omega_2^*))] \Rightarrow [u(\omega_1^*, y(\omega_1^*)) \geq u(\omega_2^*, y^*(\omega_2^*))]$.

Let $(i_{\omega_1^*}, y(\omega_1^*)) \succeq (i_{\omega_2^*}, y^*(\omega_2^*))$. Then by representation, we must have

$$\begin{array}{rcl} u(\omega_1^*, y(\omega_1^*)) & = & \sum_{\omega' \in C(i_\omega)} i_{\omega_1^*}(\omega') u(\omega', y(\omega')) \\ & \geq & \sum_{\omega' \in C(i_{\omega_2^*})} i_{\omega_2^*}(\omega') u(\omega', y^*(\omega')) \\ & = & u(\omega_2^*, y^*(\omega_2^*)) \end{array}$$

as required.

Step 6: Show that if there exists distinct elements ω_1^*, ω_2^* of Ω and y^* in Y such that $[(q, y) \not\succ (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$, then $[(i_{\omega_2^*}, y^*(\omega_2^*)) \succeq (i_{\omega_1^*}, y(\omega_1^*))] \Rightarrow [u(\omega_2^*, y^*(\omega_2^*)) \geq u(\omega_1^*, y(\omega_1^*))]$.

Let $(i_{\omega_2^*}, y^*(\omega_2^*)) \succeq (i_{\omega_1^*}, y(\omega_1^*))$ Then by representation, we must have

$$\begin{array}{rcl} u(\omega_1^*, y(\omega_1^*)) & = & \sum_{\omega' \in C(i_{\omega_1^*})} i_{\omega_1^*}(\omega') u(\omega', y(\omega')) \\ & \leq & \sum_{\omega' \in C(i_{\omega_2^*})} i_{\omega_2^*}(\omega') u(\omega', y^*(\omega')) \\ & = & u(\omega_2^*, y^*(\omega_2^*)) \end{array}$$

as required.

Step 7: Show that if there exists distinct elements ω_1^*, ω_2^* of Ω and y^* in Y such that $[(q, y) \not\succ (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$, then $q(\omega_1^*) > 0$ implies that $[(q, y) \succeq (i_{\omega_2^*}, y^*)] \Leftrightarrow [(i_{\omega_1^*}, y(\omega_1^*)) \succeq (i_{\omega_2^*}, y^*(\omega_2^*))]$.

Indeed, from steps 4 it follows that

$$\begin{array}{rcl} \begin{array}{l} [\sum_{\omega \in C(q)} q(\omega) u(\omega, y(\omega))] \\ -\sum_{\omega' \in C(i_{\omega_2^*})} i_{\omega_2^*}(\omega') u(\omega', y^*(\omega')) \end{array} & = & [\sum_{\omega \in C(q)} q(\omega) u(\omega, y(\omega))] - u(\omega_2^*, y^*(\omega_2^*)) \\ & = & \sum_{\omega \in C(q)} q(\omega) [u(\omega, y(\omega)) - u(\omega_2^*, y^*(\omega_2^*))] \\ & = & q(\omega_1^*) [u(\omega_1^*, y(\omega_1^*)) - u(\omega_2^*, y^*(\omega_2^*))]. \end{array}$$

Hence, when $q(\omega_1^*) > 0$ we have

$$\begin{array}{rcl} (q, y) & \succeq & (i_{\omega_2^*}, y^*) \\ & \Updownarrow & \\ \sum_{\omega \in C(q)} q(\omega) u(\omega, y(\omega)) & \geq & \sum_{\omega' \in C(i_{\omega_2^*})} i_{\omega_2^*}(\omega') u(\omega', y^*(\omega')) \\ & \Updownarrow & \\ u(\omega_1^*, y(\omega_1^*)) & \geq & u(\omega_2^*, y^*(\omega_2^*)) \\ & \Updownarrow & \\ \sum_{\omega' \in C(i_{\omega_1^*})} i_{\omega_1^*}(\omega') u(\omega', y(\omega')) & \geq & \sum_{\omega' \in C(i_{\omega_2^*})} i_{\omega_2^*}(\omega') u(\omega', y^*(\omega')) \\ & \Updownarrow & \\ (i_{\omega_1^*}, y(\omega_1^*)) & \succeq & (i_{\omega_2^*}, y^*(\omega_2^*)) \end{array}$$

as required.

Step 8: Show that the if part of the moreover statement holds if there exists distinct elements ω_1^*, ω_2^* of Ω and y^* in Y such that $[(q,y) \not\sim (i_{\omega_2^*}, y^*)] \Rightarrow [q(\omega_1^*) > 0]$.

Let $v : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ be a function with the property that there exists $a, b_1, b_2 (b_1 > 0, b_2 > 0)$ such that:

$$v(\omega, y(\omega)) = \begin{cases} a + b_1[u(\omega, y(\omega)) - u(\omega_2^*, y^*(\omega^*))] \text{ if } (i_\omega, y) \succeq (i_{\omega_2^*}, y^*) \\ a + b_2[u(\omega, y(\omega)) - u(\omega_2^*, y^*(\omega^*))] \text{ otherwise} \end{cases}$$

Define $V : \Delta(\Omega) \times Y \to \mathbf{R}$ by $V(q,y) = \sum_{\omega\in C(q)} q(\omega)v(\omega, y(\omega))$. We claim that

$$V(q,y) = \begin{cases} a + b_1[U(q,y) - U(i_{\omega_2^*}, y^*)] \text{ if } (q,y) \succeq (i_{\omega_2^*}, y^*) \\ a + b_2[U(q,y) - U(i_{\omega_2^*}, y^*)] \text{ otherwise} \end{cases}$$

Indeed, if $(q,y) \succeq (i_{\omega_2^*}, y^*)$, it follows from steps 4,5, and 7 that

$$\begin{aligned} V(q,y) &= \textstyle\sum_{\omega\in C(q)} q(\omega)v(\omega, y(\omega)) \\ &= \textstyle\sum_{\omega\in C(q)} q(\omega)[a + b_1[u(\omega, y(\omega)) - u(\omega_2^*, y^*(\omega^*))]] \\ &= a + b_1 \textstyle\sum_{\omega\in C(q)} q(\omega)[u(\omega, y(\omega)) - u(\omega_2^*, y^*(\omega^*))] \\ &= a + b_1[[\textstyle\sum_{\omega\in C(q)} q(\omega)u(\omega, y(\omega))] - \textstyle\sum_{\omega\in C(q)} u(\omega_2^*, y^*(\omega^*))] \\ &= a + b_1[U(q,y) - U(i_{\omega_2^*}, y^*(\omega^*))] \end{aligned}$$

as required. Likewise, if $(i_{\omega_2^*}, y^*) \succeq (q,y)$, it follows from steps 4,5, and 7 that

$$\begin{aligned} V(q,y) &= \textstyle\sum_{\omega\in C(q)} q(\omega)v(\omega, y(\omega)) \\ &= \textstyle\sum_{\omega\in C(q)} q(\omega)[a + b_2[u(\omega, y(\omega)) - u(\omega_2^*, y^*(\omega^*))]] \\ &= a + b_2 \textstyle\sum_{\omega\in C(q)} q(\omega)[u(\omega, y(\omega)) - u(\omega_2^*, y^*(\omega^*))] \\ &= a + b_2[[\textstyle\sum_{\omega\in C(q)} q(\omega)u(\omega, y(\omega))] - \textstyle\sum_{\omega\in C(q)} u(\omega_2^*, y^*(\omega^*))] \\ &= a + b_2[U(q,y) - U(i_{\omega_2^*}, y^*(\omega^*))] \end{aligned}$$

as required. Hence, we can conclude that

$$V(q,y) = \begin{cases} a + b_1[U(q,y) - U(i_{\omega_2^*}, y^*)] \text{ if } (q,y) \succeq (i_{\omega_2^*}, y^*) \\ a + b_2[U(q,y) - U(i_{\omega_2^*}, y^*)] \text{ otherwise} \end{cases}$$

From Lemma 9 it then follows that V is an affine representation of the preferences on $\Delta(\Omega) \times Y$ with the required onto property. Hence v has all the required properties.

Step 9: Show that if there exists a function $u : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ such that

$$[(q,y) \succeq (q',y')] \Leftrightarrow [\sum_{\omega\in C(q)} q(\omega)u(\omega, y(\omega)) \geq \sum_{\omega\in C(q')} q'(\omega)u(\omega, y'(\omega))],$$

then axioms $A1 - A5$ holds.

Let $u : \cup_{\omega\in\Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ be such that

$$[(q,y) \succeq (q',y')] \Leftrightarrow [\sum_{\omega\in C(q)} q(\omega)u(\omega, y(\omega)) \geq \sum_{\omega\in C(q')} q'(\omega)u(\omega, y'(\omega))].$$

(i) Let $y, y' \in Y$ satisfy the property that $y(\omega) = y'(\omega)$. Clearly, it then follows that

$$\begin{array}{rcl} \sum_{\omega'' \in C(i_\omega)} i_\omega(\omega'')u(\omega'', y(\omega'')) & = & u(\omega, y(\omega)) \\ & = & u(\omega, y'(\omega)) \\ & = & \sum_{\omega'' \in C(i_\omega)} i_\omega(\omega'')u(\omega'', y'(\omega'')). \end{array}$$

By representation, it then follows that $(i_\omega, y) \succeq (i_\omega, y')$ and $(i_\omega, y') \succeq (i_\omega, y)$. In other words, $(i_\omega, y) \sim (i_\omega, y')$. Hence $A1$ is satisfied.

(ii) Note that for any pair $(q, y), (q', y') \in \Delta(\Omega) \times Y$, either $\sum_{\omega \in C(q)} q(\omega)u(\omega, y(\omega)) \geq \sum_{\omega \in C(q')} q'(\omega)u(\omega, y'(\omega))$, or $\sum_{\omega \in C(q')} q'(\omega)u(\omega, y'(\omega)) \geq \sum_{\omega \in C(q)} q(\omega)u(\omega, y(\omega))$. By representation, this implies $(q, y) \succeq (q', y')$ or $(q', y') \succeq (q, y)$. Hence $A2$ holds.

(iii) Let $(q, y), (q', y'), (q'', y'') \in \Delta(\Omega) \times Y$ satisfy the properties that $(q, y) \succeq (q', y)$ and $(q', y') \succeq (q'', y'')$. By representation, it follows that $\sum_{\omega \in C(q)} q(\omega)u(\omega, y(\omega)) \geq \sum_{\omega \in C(q')} q'(\omega)u(\omega, y'(\omega))$ and $\sum_{\omega \in C(q')} q'(\omega)u(\omega, y'(\omega)) \geq \sum_{\omega \in C(q'')} q''(\omega)u(\omega, y''(\omega))$. By the properties of the real number line, it follows that $\sum_{\omega \in C(q)} q(\omega)u(\omega, y(\omega)) \geq \sum_{\omega \in C(q'')} q''(\omega)u(\omega, y''(\omega))$. By representation, it then follows that $(q, y) \succeq (q'', y'')$. Hence $A3$ holds.

(iv) Let $y_1, y_2 \in Y$, and $q_1, q_2, q_3 \in \Delta(\Omega)$ satisfy the property that $(q_1, y_1) \succ (q_2, y_2) \succ (q_3, y_1)$. By representation, it follows that

$$\begin{array}{rcl} \sum_{\omega \in C(q_1)} q_1(\omega)u(\omega, y_1(\omega)) & > & \sum_{\omega \in C(q_2)} q_2(\omega)u(\omega, y_2(\omega)) \\ & > & \sum_{\omega \in C(q_3)} q_3(\omega)u(\omega, y_1(\omega)). \end{array}$$

We know by the properties of the real number line that there is an $\alpha \in (0, 1)$ such that

$$\alpha[\sum_{\omega \in C(q_1)} q_1(\omega)u(\omega, y_1(\omega))] + (1 - \alpha)[\sum_{\omega \in C(q_3)} q_3(\omega)u(\omega, y_1(\omega))] > \sum_{\omega \in C(q_2)} q_2(\omega)u(\omega, y_2(\omega)).$$

We note that for every $\omega \in C(q_1) \cup C(q_3)$, it follows that $\alpha q_1(\omega) + (1 - \alpha)q_3(\omega) > 0$. Hence $\omega \in C(\alpha q_1 + (1 - \alpha)q_3)$. We also note that $\omega \in C(\alpha q_1 + (1 - \alpha)q_3)$ implies that $\alpha q_1(\omega) + (1 - \alpha)q_3(\omega) > 0$. But then either $q_1(\omega) > 0$ or $q_3(\omega) > 0$. Hence $\omega \in C(q_1) \cup C(q_3)$ and we can conclude that $C(\alpha q_1 + (1 - \alpha)q_3) = C(q_1) \cup C(q_3)$. But then we have,

$$\begin{array}{rcl} \begin{array}{l} \alpha[\sum_{\omega \in C(q_1)} q_1(\omega)u(\omega, y_1(\omega))] \\ +(1 - \alpha)[\sum_{\omega \in C(q_3)} q_3(\omega)u(\omega, y_1(\omega))] \end{array} & > & \sum_{\omega \in C(q_2)} q_2(\omega)u(\omega, y_2(\omega)) \\ & \Downarrow & \\ \sum_{\omega \in C(q_1) \cup C(q_3)} [\alpha q_1(\omega) + (1 - \alpha)q_3(\omega)]u(\omega, y_1(\omega)) & > & \sum_{\omega \in C(q_2)} q_2(\omega)u(\omega, y_2(\omega)) \\ & \Downarrow & \\ \sum_{\omega \in C(\alpha q_1 + (1 - \alpha)q_3)} [\alpha q_1(\omega) + (1 - \alpha)q_3(\omega)]u(\omega, y_1(\omega)) & > & \sum_{\omega \in C(q_2)} q_2(\omega)u(\omega, y_2(\omega)) \\ & \Downarrow & \\ (\alpha q_1 + (1 - \alpha)q_3, y_1) & \succ & (q_2, y_2) \end{array}$$

as required for $A4$.

Likewise, by the properties of the real number line that there is an $\beta \in (0, 1)$ such that

$$\beta[\sum_{\omega \in C(q_1)} q_1(\omega)u(\omega, y_1(\omega))] + (1 - \beta)[\sum_{\omega \in C(q_3)} q_3(\omega)u(\omega, y_1(\omega))] < \sum_{\omega \in C(q_2)} q_2(\omega)u(\omega, y_2(\omega)).$$

We note that for every $\omega \in C(q_1) \cup C(q_3)$, it follows that $\beta q_1(\omega) + (1-\beta)q_3(\omega) > 0$. Hence $\omega \in C(\beta q_1 + (1-\beta)q_3)$. We also note that $\omega \in C(\beta q_1 + (1-\beta)q_3)$ implies that $\beta q_1(\omega) + (1-\beta)q_3(\omega) > 0$. But then either $q_1(\omega) > 0$ or $q_3(\omega) > 0$. Hence $\omega \in C(q_1) \cup C(q_3)$ and we can conclude that $C(\beta q_1 + (1-\beta)q_3) = C(q_1) \cup C(q_3)$. But then we have,

$$\begin{array}{rcl}
\begin{array}{l}\beta[\sum_{\omega\in C(q_1)} q_1(\omega)u(\omega, y_1(\omega))] \\ +(1-\beta)[\sum_{\omega\in C(q_3)} q_3(\omega)u(\omega, y_1(\omega))]\end{array} & < & \sum_{\omega\in C(q_2)} q_2(\omega)u(\omega, y_2(\omega)) \\
& \Downarrow & \\
\sum_{\omega\in C(q_1)\cup C(q_3)}[\beta q_1(\omega) + (1-\beta)q_3(\omega)]u(\omega, y_1(\omega)) & < & \sum_{\omega\in C(q_2)} q_2(\omega)u(\omega, y_2(\omega)) \\
& \Downarrow & \\
\sum_{\omega\in C(\beta q_1+(1-\beta)q_3)}[\beta q_1(\omega) + (1-\beta)q_3(\omega)]u(\omega, y_1(\omega)) & < & \sum_{\omega\in C(q_2)} q_2(\omega)u(\omega, y_2(\omega)) \\
& \Downarrow & \\
(q_2, y_2) & \succ & (\beta q_1 + (1-\beta)q_3, y_1)
\end{array}$$

as required for $A4$. Hence $A4$ holds.

(v) Let $(q_1, y_1) \sim (q_2, y_2)$. By representation, we know that $\sum_{\omega\in C(q_1)} q_1(\omega)u(\omega, y_1(\omega)) = \sum_{\omega\in C(q_2)} q_2(\omega)u(\omega, y_2(\omega))$. Consider any $q, q' \in \Delta(\Omega)$ and any $\alpha \in (0,1)$. From Claim [], we know that $C(q)\cup C(q_1) = C(\alpha q + (1-\alpha)q_1))$ and $C(q')\cup C(q_2) = C(\alpha q' + (1-\alpha)q_2))$. Then it follows that

$$\begin{array}{rcl}
(q, y_1) & \succeq & (q', y_2) \\
& \Updownarrow & \\
\sum_{\omega\in C(q)} q(\omega)u(\omega, y_1(\omega)) & \geq & \sum_{\omega\in C(q')} q'(\omega)u(\omega, y_2(\omega)) \\
& \Updownarrow & \\
\begin{array}{l}\alpha\sum_{\omega\in C(q)} q(\omega)u(\omega, y_1(\omega)) \\ +(1-\alpha)\sum_{\omega\in C(q_1)} q_1(\omega)u(\omega, y_1(\omega))\end{array} & \geq & \begin{array}{l}\alpha\sum_{\omega\in C(q')} q'(\omega)u(\omega, y_2(\omega)) \\ +(1-\alpha)\sum_{\omega\in C(q_2)} q_2(\omega)u(\omega, y_2(\omega))\end{array} \\
& \Updownarrow & \\
\begin{array}{l}\sum_{\omega\in C(q)\cup C(q_1)}[\alpha q(\omega) \\ +(1-\alpha)q_1(\omega)]u(\omega, y_1(\omega))\end{array} & \geq & \begin{array}{l}\sum_{\omega\in C(q')\cup C(q_2)}[\alpha q'(\omega) \\ +(1-\alpha)q_2(\omega)]u(\omega, y_2(\omega))\end{array} \\
& \Updownarrow & \\
\begin{array}{l}\sum_{\omega\in C(\alpha q+(1-\alpha)q_1)}[\alpha q(\omega) \\ +(1-\alpha)q_1(\omega)]u(\omega, y_1(\omega))\end{array} & \geq & \begin{array}{l}\sum_{\omega\in C(\alpha q'+(1-\alpha)q_2)}[\alpha q'(\omega) \\ +(1-\alpha)q_2(\omega)]u(\omega, y_2(\omega))\end{array} \\
& \Updownarrow & \\
(\alpha q + (1-\alpha)q_1, y_1) & \succeq & (\alpha q' + (1-\alpha)q_2, y_2).
\end{array}$$

Hence $A5$ holds.

Q.E.D.

4.7 Theorem 2

Throughout this section, we take as given all the assumptions of Theorem 2.

Claim 15. *Let $\omega \in \Omega$ and $y_1, y_2 \in Y$ satisfy $(i_\omega, y_1) \succ (i_\omega, y_2)$ and let $(q^*, y^*) \in \Delta(\Omega) \times Y$ satisfy $(i_\omega, y_1) \succ (q^*, y^*) \succ (i_\omega, y_2)$. Then $\exists y'' \in Y$ such that $(i_\omega, y'') \sim (q^*, y^*)$.*

Proof of Claim 14: Fix $y(\Omega\setminus\{\omega\}) \in \prod_{\omega'\in\Omega\setminus\{\omega\}} Y(\omega')$. We note that it follows from axiom $A1$ that $(i_\omega, y_1(\omega), y(\Omega \setminus \{\omega'\})) \sim (i_\omega, y_1) \succ (q^*, y^*) \succ (i_\omega, y_2) \sim (i_\omega, y_2(\omega), y(\Omega \setminus \{\omega'\}))$.

Let $W_1 := \{y(\omega) \in Y(\omega) | (i_\omega, y) \succ (q^*, y^*)\}$, $W_2 := \{y(\omega) \in Y(\omega) | (q^*, y^*) \succ (i_\omega, y)\}$, and $W_3 := \{y(\omega) \in Y(\omega) | (q^*, y^*) \sim (i_\omega, y)\}$. By definition, these sets are disjoint. We note that W_1 and W_2 are nonempty since $y_1(\omega) \in W_1$ and $y_2(\omega) \in W_2$. From Axiom $A2$ it follows that $Y(\omega) = W_1 \cup W_2 \cup W_3$. It now suffices to show that $W_3 \neq \emptyset$.

Suppose $W_3 = \emptyset$. From Axiom $A6$, it follows that the sets

$$\begin{array}{rcl} Y(\omega) \setminus W_1 & = & \{y(\omega) \in Y(\omega) | (q^*, y^*) \succeq (i_\omega, y)\}; \text{ and} \\ Y(\omega) \setminus W_2 & = & \{y(\omega) \in Y(\omega) | (i_\omega, y) \succeq (q^*, y^*)\} \end{array}$$

are closed. But then we know that W_1 and W_2 are nonempty disjoint open sets such that $Y(\omega) = W_1 \cup W_2$. Hence the pair of sets W_1 and W_2 form a separation for $Y(\omega)$. But $Y(\omega)$ is a connected sets. A contradiction. Hence $W_3 \neq \emptyset$ and there exists $y'' \in Y$ such that $(i_\omega, y'') \sim (q^*, y^*)$. Q.E.D.

Claim 16. *The function u given in Theorem 1 satisfies the property that for every $\omega \in \Omega$ and any $y_1, y_2 \in Y$ for which $(i_\omega, y_1) \succ (i_\omega, y_2)$, it follows that for every $\alpha \in [u(\omega, y_2(\omega)), u(\omega, y_1(\omega))]$ there exists $y^* \in Y$ such that $u(\omega, y^*(\omega)) = \alpha$.*

Proof of Claim 15: Let $\omega \in \Omega$ and $y_1, y_2 \in Y$ satisfy the property that $(i_\omega, y_1) \succ (i_\omega, y_2)$ and let $\alpha \in [u(\omega, y_2(\omega)), u(\omega, y_1(\omega))]$. (i) If $\alpha = u(\omega, y_1(\omega))$, then we can set $y^* = y_1$ and we are done since $\alpha = u(\omega, y^*(\omega))$. (ii) Likewise, if $\alpha = u(\omega, y_2(\omega))$, then we cans set $y^* = y_2$ and we are done since $\alpha = u(\omega, y^*(\omega))$. (iii) If $u(\omega, y_2(\omega)) < \alpha < u(\omega, y_1(\omega))$, we know that $u(\omega, y_1(\omega)) = \sum_{\omega'\in C(i_\omega)} i(\omega')u(\omega', y_1(\omega'))$ and that $u(\omega, y_2(\omega)) = \sum_{\omega'\in C(i_\omega)} i(\omega')u(\omega', y_2(\omega'))$. Hence $\alpha \in (\sum_{\omega'\in C(i_\omega)} i(\omega')u(\omega', y_2(\omega')), \sum_{\omega'\in C(i_\omega)} i(\omega')u(\omega', y_1(\omega')))$. But then we know from the onto property stated in Theorem 1 that there exists $(q^*, y^*) \in \Delta(\Omega) \times Y$ such that $\alpha = \sum_{\omega'\in C(q^*)} q^*(\omega')u(\omega', y^*(\omega'))$. From representation, it then follows that $(i_\omega, y_1) \succ (q^*, y^*) \succ (i_\omega, y_2)$. Then it in turn follows from Claim 14 that there exists $y'' \in Y$ such that $(i_\omega, y'') \sim (q^*, y^*)$. But then we must have

$$\begin{array}{rcl} u(\omega, y''(\omega)) & = & \sum_{\omega'\in C(i_\omega)} u(\omega', y''(\omega')) \\ & = & \sum_{\omega'\in C(q^*)} u(\omega', y^*(\omega')) \\ & = & \alpha \end{array}$$

as required.

Q.E.D.

Proof of Theorem 2: Step 1: Show that the Axioms implies existence.

For each $\omega \in \Omega$, define a preference relation $\succeq_\omega$ on Y such that $[y \succeq_\omega y'] \Leftrightarrow [(i_\omega, y) \succeq (i_\omega, y')]$. Since $\succeq$ satisfies axioms $A2, A4$, and $A6$ it follows that $\succeq_\omega$ is complete, transitive, and closed. From Debreu's [] Theorem, it then follows that there exists an continuous function $v_\omega : Y \to \mathbf{R}$ such that $[v_\omega(y) \geq v_\omega(y')] \Leftrightarrow [y \succeq_\omega y']$.

Step 1A: Show that a continuous expected utility function exists if $\#\Omega = 1$.

Denote by ω^* the unique element of Ω and by i_{ω^*} the unique element of $\Delta(\Omega)$. Define a function $u : \{\omega^*\} \times Y(\omega^*) \to \mathbf{R}$ by $u(\omega^*, y) = v_{\omega^*}(y)$. Since v is continuous, it follows that u is continuous. Then it follows that the function $U : \Delta(\Omega) \times Y$ defined by $U(i_{\omega^*}, y) = \sum_{\omega \in C(i_{\omega^*})} i_{\omega^*}(\omega) u(\omega, y(\omega))$ also is continuous as required. Now, we also have

$$\begin{array}{rcl} (i_{\omega^*}, y) & \succeq & (i_{\omega^*}, y') \\ & \Updownarrow & \\ y & \succeq_{\omega^*} & y' \\ & \Updownarrow & \\ v_{\omega^*}(y) & \geq & v_{\omega^*}(y') \\ & \Updownarrow & \\ u(\omega^*, y) & \geq & u(\omega^*, y') \\ & \Updownarrow & \\ \sum_{\omega' \in C(i_{\omega^*})} i_{\omega^*}(\omega') u(\omega', y) & \geq & \sum_{\omega' \in C(i_{\omega^*})} i_{\omega^*}(\omega') u(\omega', y') \end{array}$$

as required. Hence the desired continuous expected utility function exists.

Step 2: Show that a continuous expected utility function exists if $\#\Omega \geq 2$.

Let $u : \cup_{\omega \in \Omega}(\{\omega\} \times Y(\omega)) \to \mathbf{R}$ be the function given in Theorem 1. It suffices to show that for every $\omega \in \Omega$, $u(\omega, .)$ is continuous.

For each $\omega \in \Omega$ define $A(\omega) := \{r \in \mathbf{R} | \exists y \in Y : v_\omega(y) = r\}$. We note that for every $y \in Y$, there exists uniquely $r_y^* \in \mathbf{R}$ such $u(\omega, y(\omega)) = r_y^*$. Furthermore, whenever $v_\omega(y) = v_\omega(y')$ we know that $r_y^* = r_{y'}^*$. We can hence define uniquely a function $w_\omega : A(\omega) \to \mathbf{R}$ such that $w_\omega(r) = r^*$ if and only if there exists $y \in Y$ such that $v_\omega(y) = r$ and $u(\omega, y(\omega)) = r^*$. We claim that for every ω, the function w_ω is continuous.

Indeed, let $\{r_k\}_{k=1}^\infty$ be a sequence of elements of $A(\omega)$ such that $r_k \to \overline{r}$ where $\overline{r} \in A(\omega)$. It suffices for us to show that $\lim_{k\to\infty} w_\omega(r_k) = w_\omega(\overline{r})$. If $\lim_{k\to\infty} w_\omega(r_k) \neq w_\omega(\overline{r})$, there are two possible cases:

(i) There exists $r^* < w_\omega(\overline{r})$ such that for infinitely many elements of the sequence $\{w_\omega(r_k)\}_{k=1}^\infty$, it follows that $w_\omega(r_k) \leq r^*$. We note that this implies the existence of $\overline{y} \in Y$ and for every k, $y_k \in Y$ such that for infinitely many element of the sequence $\{w_\omega(r_k)\}_{k=1}^\infty$, we have $w_\omega(r_k) = u(\omega, y_k(\omega)) \leq r^* < u(\omega, \overline{y}(\omega)) = w_\omega(\overline{r})$. From claim 15 it follows that there exists $y^* \in Y$ such that $u(\omega, y^*(\omega)) = r^*$. But then we must have that for infinitely many element of the sequence $\{r_k\}_{k=1}^\infty$, $r_k = v_\omega(y_k) \leq v_\omega(y^*) < v_\omega(\overline{y}) = \overline{r}$. But then it can not be that $r_k \to \overline{r}$, a contradiction.

(ii) There exists $r^* > w_\omega(\overline{r})$ such that for infinitely many elements of the sequence $\{w_\omega(r_k)\}_{k=1}^\infty$, it follows that $w_\omega(r_k) \geq r^*$. We note that this implies the existence of $\overline{y} \in Y$ and for every k, $y_k \in Y$ such that for infinitely many element of the sequence $\{w_\omega(r_k)\}_{k=1}^\infty$, we have $w_\omega(r_k) = u(\omega, y_k(\omega)) \geq r^* > u(\omega, \overline{y}(\omega)) = w_\omega(\overline{r})$. From claim 15 it follows that there exists $y^* \in Y$ such that $u(\omega, y^*(\omega)) = r^*$. But then we must have that for infinitely many element of the sequence $\{r_k\}_{k=1}^\infty$, $r_k = v_\omega(y_k) \geq v_\omega(y^*) > v_\omega(\overline{y}) = \overline{r}$. But then it can not be that $r_k \to \overline{r}$, a contradiction.

Since neither (i) nor (ii) can hold, we can hence conclude that $\lim_{k\to\infty} w_\omega(r_k) = w_\omega(\overline{r})$ which in turn implies that for every ω, the associated function w_ω is continuous. We now simply note that for every $\omega \in \Omega$, we have that for every $u(\omega, y(\omega)) = w_\omega(v_\omega(y))$. Since both v_ω and w_ω are continuous functions, it follows that $u(\omega, .)$ also is continuous as required.

Step 2: Show that the existence of a continuous expected utility function implies Axioms A1-A6.

Step2A: Show that if $\#\Omega = 1$, then every preference relation satisfying axioms $A2 - A3$ also satisfies axioms $A1$, $A4$, and $A5$.

Suppose $\#\Omega = 1$ and denote by ω^* the unique element of Ω. We note that $[y(\omega^*) = y'(\omega^*)] \Rightarrow [y = y']$. But then it follows from $A2$ and $A3$ that $(i_{\omega^*}, y) \sim (i_{\omega^*}, y')$ so $A1$ holds. We also not that $\#\Delta(\Omega) = 1$. But then it again follows from $A2$ and $A3$ that there does not exist $y_1 \in Y$ and $q_1, q_3 \in \Delta(\Omega)$ such that $(q_1, y_1) \succ (q_3, y_3)$. Hence $A4$ must be satisfied. Finally, let $(q_1, y_1), (q_2, y_2) \in \Delta(\Omega) \times Y$ satisfy $(q_1, y_1) \sim (q_2, y_2)$. Since $\#\Delta(\Omega) = 1$, it follows that for every $q^*, q^{**} \in \Delta(\Omega)$, $(q^*, y_1) = (q_1, y_1)$ and $(q^{**}, y_2) = (q_2, y_2)$ and it hence follows that $(q^*, y_1) \sim (q^{**}, y_2)$. Consider any $q, q' \in \Delta(\Omega)$ and $\alpha \in (0, 1)$. Then we must have both $(q, y_1) \sim (q', y')$ and $(\alpha q + (1 - \alpha)q_1, y_1) \sim (\alpha q' + (1 - \alpha)q_2, y_2)$. Hence $A5$ is satisfied.

Step 2B: Show that if the preference relation in representable by a continuous utility function representation it follows that axioms $A2, A3$, and $A6$ are satisfied.

(i) Note that for any pair $(q, y), (q', y') \in \Delta(\Omega) \times Y$, either $\sum_{\omega \in C(q)} q(\omega)u(\omega, y(\omega)) \geq \sum_{\omega \in C(q')} q'(\omega)u(\omega, y'(\omega))$, or $\sum_{\omega \in C(q')} q'(\omega)u(\omega, y'(\omega)) \geq \sum_{\omega \in C(q)} q(\omega)u(\omega, y(\omega))$. By representation, this implies $(q, y) \succeq (q', y')$ or $(q', y') \succeq (q, y)$. Hence $A2$ holds.

(ii) Let $(q, y), (q', y'), (q'', y'') \in \Delta(\Omega) \times Y$ satisfy the properties that $(q, y) \succeq (q', y)$ and $(q', y') \succeq (q'', y'')$. By representation, it follows that $\sum_{\omega \in C(q)} q(\omega)u(\omega, y(\omega)) \geq \sum_{\omega \in C(q')} q'(\omega)u(\omega, y'(\omega))$ and $\sum_{\omega \in C(q')} q'(\omega)u(\omega, y'(\omega)) \geq \sum_{\omega \in C(q'')} q''(\omega)u(\omega, y''(\omega))$. By the properties of the real number line, it follows that $\sum_{\omega \in C(q)} q(\omega)u(\omega, y(\omega)) \geq \sum_{\omega \in C(q'')} q''(\omega)u(\omega, y''(\omega))$. By representation, it then follows that $(q, y) \succeq (q'', y'')$. Hence $A3$ holds.

(iiia) Let $\{y_k\}_{k=1}^{\infty}$ be any sequence of elements in the set $\{y \in Y | (i_\omega, y) \succeq (q^*, y^*)\}$ such that $y_k \to \overline{y}$. From representation, we know that for every k, we have $\sum_{\omega' \in \Omega} i_\omega(\omega')u(\omega', y_k(\omega')) \geq \sum_{\omega' \in \Omega} q^*(\omega')u(\omega', y^*(\omega'))$. From continuity, it then follows that $\sum_{\omega' \in \Omega} i_\omega(\omega')u(\omega', \overline{y}(\omega')) \geq \sum_{\omega' \in \Omega} q^*(\omega')u(\omega', y^*(\omega'))$. Then it in turn follows from representation that $(i_\omega, \overline{y}) \succeq (q^*, y^*)$. Hence the set $\{y \in Y | (i_\omega, y) \succeq (q^*, y^*)\}$ is closed.

(iiia) Let $\{y_k\}_{k=1}^{\infty}$ be any sequence of elements in the set $\{y \in Y | (q^*, y^*) \succeq (i_\omega, y)\}$ such that $y_k \to \overline{y}$. From representation, we know that for every k, we have $\sum_{\omega' \in \Omega} i_\omega(\omega')u(\omega', y_k(\omega')) \leq \sum_{\omega' \in \Omega} q^*(\omega')u(\omega', y^*(\omega'))$. From continuity, it then follows that $\sum_{\omega' \in \Omega} i_\omega(\omega')u(\omega', \overline{y}(\omega')) \leq \sum_{\omega' \in \Omega} q^*(\omega')u(\omega', y^*(\omega'))$. Then it in turn follows from representation that $(q^*, y^*) \succeq (i_\omega, \overline{y})$. Hence the set $\{y \in Y | (q^*, y^*) \succeq (i_\omega, y)\}$ is closed.

We can then conclude from steps (i)-(iii) that axioms $A2, A3$, and $A6$ all hold.

Step 2C: Show that axioms $A1, A4$, and $A5$ all hold.

If $\#\Omega = 1$, this follows from step 2A. If $\#\Omega \geq 2$, then it follows from Theorem 1.

Step 3: Show that the moreover remark holds.

This is an immediate consequence of Theorem 1.

Q.E.D.