The Existence and Strong Informational Efficiency of Perfect State-ω Competitive Equilibria in a State-Contingent Claims Model with Differential Information

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1 Introduction

In a recent paper, Karni(2001) combines (i) an introspective preference relation on hypothetical objective lotteries; and (ii) a preference relation on acts by using appropriate axioms and obtains both an expected utility function representation of preferences and a unique subjective probability on states.

The present paper models consumer behavior in differential information economies in much the same way. Each consumer is modelled as having both an introspective preference relation on hypothetical objective lotteries and, for every price in the interior of the price simplex, a preference relation on consumption bundles. Using this approach, we derive both the existence of an expected utility function and a function that maps the interior of the price simplex to unique probabilities on events in the pooled information partition.

Unlike the Radner type rational expectations model, this approach results in a well defined demand correspondence on the interior of the price simplex. As a result, the aggregate demand correspondence and the excess demand correspondence are both well defined for all prices in the interior of the price simplex. We consider further assumptions on preferences (non-satiation and a form of convexity) that guarantee that for a given price, all optimal bundles of a consumer allocates the same proportion of income to the various events in the pooled information partition. Under such assumptions, the existence of an function mapping the interior of the price simplex to an unique optimal income allocation is assured.

We consider a solution concept, perfect competitive equilibria, that are similar in spirit to the perfect equilibrium concept formulated by Selten for extensive form games. Using a synthesis of arguments used by Arrow to prove the existence of a competitive equilibrium and Selten to show the existence of a perfect equilibrium, we show in our first Theorem that additional convexity assumptions when combined with appropriate assumptions on the income allocation rule assures the existence of a perfect competitive equilibrium.

Having thus shown the existence of perfect competitive equilibria, we then turn to address informational efficiency. We define markets to be informationally efficient if, in every perfect equilibrium, every consumer assigns a probability of 1 to the element of the pooled information partition that has actually been realized. Using this definition of informational efficiency, we consider behavioral assumptions for consumers that would guarantee the informational efficiency of markets.

The kind of assumptions we arrive at are outgrowths of the type of logic that goes along with a two state economy with one informed and one uninformed consumer. In such a simple economy, a reasonable assumption would seem to be that the uninformed consumer should refrain from reallocating income between the two states. Since the informed consumer will always want to spend his income on the state that has actually occurred, the uninformed consumer would never, in equilibrium, be able to execute a trade in which his spending on the the state that has actually occurred was higher than his income in that state. Hence the only reasonable probability assignment by the uninformed consumer would seem to be the one that makes it optimal for him to not reallocate income between the two states. We formulate assumptions on consumer behavior that extend this logic to the general differential information economy. In our second theorem, we then show that the derived assumption assures the informational efficiency of markets.

Finally, we consider a stronger assumptions on the income allocation rule that can be derived from a mini-max problem for the consumer. We consider this income allocation in the context where every element of the pooled information partition has only one associated commodity. For this type of economy, we propose a tatonnement process for the price vector and show that the tatonnement process converges to a unique price assign zero prices to all commodities except the one associated with the event actually realized.

2 The Model

We here introduce a differential information economy in state-contingent claims. The following subsections introduce the various ingredients of the differential information economy.

2.1 Consumers

There is a finite (non-empty) set of consumers $M := \{1, \ldots, m\}$.

2.2 States

There is a finite (non-empty) set of states of the world Ω. We denote by $\tilde{\mathcal{F}}$ the set of all fields defined on Ω.

2.3 Type-Spaces

Every consumer i has a finite (non-empty) type-space T^i.

2.4 Type-Profiles

There is a finite (non-empty) set of type-profiles $\tilde{T} \subset \prod_{i \in M} T^i$.

2.5 State-Contingent Claims

There is a finite (non-empty) set of state-contingent claims $L := \{1, \ldots, L\}$. Associated with the claims is a non-empty-valued correspondence $\mathcal{L} : L \to \Omega$ associating with each claim $l \in L$ the set of states $\mathcal{L}(l)$ in which it is deliverable. The set of state-contingent claims deliverable in at least one state contained in an event E will be denoted by $L(E) := \{l \in L | \exists \omega \in E : \mathcal{L}(l) \ni \omega\}$. Throughout, we will use the notation $L(\omega)$ instead of $L(\{\omega\})$. The following assumption will frequently be imposed.

Assumption (B1:). *$\forall \omega \in \Omega$, $\#[L(\omega)] \geq 1$*

2.6 Information

Every consumer i has an information-field function $\mathcal{F}^i : T^i \to \tilde{\mathcal{F}}$ associating with each of his types t^i, a field $\mathcal{F}^i(t^i)$ in $\tilde{\mathcal{F}}$. We denote by $\mathcal{PF}^i(.|t^i)$ the set of all minimal non-empty events in $\mathcal{F}^i(t^i)$, and for every ω in Ω, we denote by $\mathcal{PF}^i(\omega|t^i)$ the unique element of $\mathcal{PF}^i(.|t^i)$ that contains ω. Furthermore, we denote by $\mathcal{PF}^{i*}(.|t^i)$ the partition of Ω for which $\mathcal{PF}^{i*}(\omega|t^i) := \cap_{t' \in \tilde{T}: t'^i = t^i}(\cap_{j \in M} \mathcal{PF}^j(\omega|t'^j))$ and define $\mathcal{F}^{i*}(.|t^i) = 2^{\mathcal{PF}^{i*}(.|t^i)}$. For every pair (l, E) in $L \times \mathcal{PF}^i(.|t^i)$, we now introduce the following notation:

$$\begin{aligned}
\mathcal{PG}^i_{l,E}(.|t^i) &:= \{\mathcal{L}(l) \cap E\} \cup (\cup_{\omega \in E \setminus \mathcal{L}(l)} \{\{\omega\}\}) \\
\mathcal{G}^i_{l,E}(.|t^i) &:= 2^{\mathcal{PG}^i_{l,E}(.|t^i)} \\
\mathcal{PG}^i_E(.|t^i) &:= \{E' \in \mathcal{PF}^{i*}(.|t^i) | E' \subset E\} \\
\mathcal{G}^i_E(.|t^i) &:= 2^{\mathcal{PG}^i_E(.|t^i)} \\
\mathcal{G}^{i*}_E(.|t^i) &:= \mathcal{G}^i_E(.|t^i) \cap (\cap_{l \in L} \mathcal{G}^i_{l,E}(.|t^i))
\end{aligned}$$

We denote by $\mathcal{PG}^{i*}_E(.|t^i)$ the set of minimal non-empty events in $\mathcal{G}^{i*}_E(.|t^i)$ and introduce the following additional notation:

$$\begin{aligned}
\mathcal{PH}^i(.|t^i) &:= \cup_{E \in \mathcal{PF}^i(.|t^i)} \mathcal{PG}^{i*}_E(.|t^i) \\
\mathcal{H}^i(.|t^i) &:= 2^{\mathcal{PH}^i(.|t^i)}
\end{aligned}$$

This notation will be used in the following subsections.

2.7 Consumption Sets

Every consumer i has a consumption set correspondence $X^i : T^i \to \mathbf{R}^{\#[L]}$ associating with each of his types t^i, a consumption set $X^i(t^i)$. For any $x \in X^i(t^i)$, we denote by $x^{L(E)}$

the sub-vector of commodities that are deliverable in event E, and by $x^{L\setminus L(E)}$ the set of commodities not deliverable in event E. The following assumptions will at times be imposed on $X^i(t^i)$.

Assumption (C1: Eventwise Indepence)**.** *For every event* $E \in \mathcal{PF}^{i*}(.|t^i)$*,*

$$[(x^{L(E)}, x^{L\setminus L(E)}) \in X^i(t^i)] \Leftrightarrow [\forall \tilde{x} \in X^i(t^i), (x^{L(E)}, \tilde{x}^{L\setminus L(E)}) \in X^i(t^i)]$$

Assumption (C2: Closedness)**.** $X^i(t^i)$ *is closed.*

Assumption (C3: Boundedness from below)**.** *There exists* b *in* $\mathbf{R}^{\#[L]}$ *such that for every* $x \in X^i(t^i)$*,* $x \gg b$*.*

Assumption (C4: Connectedness)**.** $X^i(t^i)$ *is connected.*

Assumption (C5: Convexity)**.** $X^i(t^i)$ *is convex.*

Assumption (C6: Non-negative Quadrant)**.** $X^i(t^i) = \mathbf{R}_+^{\#[L]}$*.*

2.8 Endowments

Every consumer i has an initial endowment function $e^i : T^i \to \mathbf{R}^{\#[L]}$ associating with each of his types t^i, an initial endowment $e^i(t^i)$. The following assumption will be imposed on the endowment functions throughout this paper.

Assumption (D1: minimal subsistence level)**.** *There exists a bundle* $x \in X^i(t^i)$ *such that* $e^i(t^i) \gg x$ *.*

2.9 Preferences

Before introducing the formal definition of the preference relations that will be used in this paper, we first introduce some additional notation.

For every consumer i, every $t^i \in T^i$, and every $E \in \mathcal{PF}^{i*}(.|t^i)$, we define $Y^i(E|t^i) := \{y^{L(E)} \in \mathbf{R}^{\#[L(E)]} | \exists x \in X^i(t^i) : x^{L(E)} = y^{L(E)}\}$, and for every $E' \in \mathcal{PF}^i(.|t^i)$, we define $Y^{i,t^i}_{E'} := \prod_{E \in \mathcal{PF}^{i*}(.|t^i) : E \subset E'} Y^i(E|t^i)$, $S^{i,t^i}_{E'} := \{E \in \mathcal{PF}^{i*}(.|t^i) | E \subset E'\}$, $\Delta(S^{i,t^i}_{E'}) := \Delta^{\#\{E \in \mathcal{PF}^{i*}(.|t^i) | E \subset E'\} - 1}$, and $D^{i,t^i}_{E'} := \Delta(S^{i,t^i}_{E'}) \times Y^{i,t^i}_{E'}$. Every consumer i has for each of his types $t^i \in T^i$, and every $E' \in \mathcal{PF}^i(.|t^i)$ a preference relation $\succeq^{i,t^i}_{E'}$ on $D^{i,t^i}_{E'}$. Throughout the following we define for any $q \in \Delta(S^{i,t^i}_{E'})$,

$$C(q) := \left\{ E \in \mathcal{PF}^{i*}(.|t^i) \,\middle|\, \begin{array}{ll} a) & E \subset E'; and \\ b) & q_E > 0 \end{array} \right\}$$

Let $(q, y), (q', y')$, and (q'', y'') be any three elements of $D^{i,t^i}_{E'}$. Then the following axioms will be imposed on $\succeq^{i,t^i}_{E'}$.

Assumption (E1: Independence of Irrelevant Events)**.**

$$[(q, y) \succeq^{i,t^i}_{E'} (q', y')] \Leftrightarrow [\forall \tilde{y} \in Y^{i,t^i}_{E'} : (q, y^{C(q)}, \tilde{y}^{E'\setminus C(q)}) \succeq^{i,t^i}_{E'} (q', y')]$$

Assumption (E2: Completeness). $(q,y) \succeq^{i,t^i}_{E'} (q',y')$ *or* $(q',y') \succeq^{i,t^i}_{E'} (q,y)$.

Assumption (E3: Transitivity). $[(q,y) \succeq^{i,t^i}_{E'} (q',y'),\ (q',y') \succeq^{i,t^i}_{E'} (q'',y'')]$ *implies* $[(q,y) \succeq^{i,t^i}_{E'} (q'',y'')]$.

Assumption (E4: Archimedean Axiom). *Let* $y_1, y_2 \in Y^{i,t^i}_{E'}$. *If* $q_1, q_2, q_3 \in \Delta(S^{i,t^i}_{E'})$ *such that* $(q_1,y_1) \succ^{i,t^i}_{E'} (q_2,y_2) \succ^{i,t^i}_{E'} (q_1,y_1)$ *then there exists* $\alpha, \beta \in (0,1)$ *such that* $(\alpha q_1 + (1-\alpha)q_3, y_1) \succ^{i,t^i}_{E'} (q_2,y_2)$, *and* $(q_2,y_2) \succ^{i,t^i}_{E'} (\beta q_1 + (1-\beta)q_3, y_1)$

Assumption (E5: Independence Axiom). *Let* $(q_1,y_1) \sim^{i,t^i}_{E'} (q_2,y_2)$. *Then for all* $q, q' \in \Delta(S^{i,t^i}_{E'})$ *and any* $\alpha \in [0,1]$, $[(q,y_1) \succeq^{i,t^i}_{E'} (q',y_2)] \Leftrightarrow [(\alpha q + (1-\alpha)q_1, y_1) \succeq^{i,t^i}_{E'} (\alpha q' + (1-\alpha)q_2, y_2)]$.

Assumption (E6: Closedness). *For any* $(q,y) \in D^{i,t^i}_{E'}$, *the sets*

$$\{(q',y') \in D^{i,t^i}_{E'} | (q',y') \succeq^{i,t^i}_{E'} (q,y)\}$$

and

$$\{(q',y') \in D^{i,t^i}_{E'} | (q,y) \succeq^{i,t^i}_{E'} (q',y')\}$$

are both closed in $D^{i,t^i}_{E'}$.

Assumption (E4':). *Closedness with respect to* $\Delta(S^{i,t^i}_{E'})$ *]*
For any $q, q' \in \Delta^{\#\{E \in \mathcal{PF}^{i*}(.|t^i) | E \subset E'\}-1}$, *any* $y \in Y^{i,t^i}_{E'}$, *and any* $(q'',y'') \in D^{i,t^i}_{E'}$, *the sets*

$$\{t \in [0,1] | (q'',y'') \succeq^{i,t^i}_{E'} (tq + (1-t)q', y)\}$$

and

$$\{t \in [0,1] | (tq + (1-t)q', y) \succeq^{i,t^i}_{E'} (q'',y'')\}$$

are both closed in the interval $[0,1]$.

Assumption (E5: Substitutability). *For any* $y, y' \in Y^{i,t^i}_{E'}$, *and any* $q_1, q_2, q_3, q_4 \in \Delta^{\#\{E \in \mathcal{PF}^{i*}(.|t^i) | E \subset E'\}-1}$ *for which* $(q_1,y) \sim^{i,t^i}_{E'} (q_3,y')$ *and* $(q_2,y) \sim^{i,t^i}_{E'} (q_4,y')$, *it follows that* $\forall t \in [0,1]$, $(tq_1 + (1-t)q_2, y) \sim^{i,t^i}_{E'} (tq_3 + (1-t)q_4, y')$.

For any q in $\mathcal{M}_f(\cup_{\omega\in\Omega} X^i(\omega) \times \{\omega\})$ and any $(x^i(\omega),\omega)$ in $\cup_{\omega\in\Omega} X^i(\omega) \times \{\omega\}$, denote by $q((x^i(\omega),\omega))$ the probability assigned by q to $(x^i(\omega),\omega)$.

Assumption (E7: Nonsatiation). *For any* $(x^i(\omega),\omega)$ *in* $\cup_{\omega\in\Omega} X^i(\omega) \times \{\omega\}$, *there exists* $y^i(\omega) \in X^i(\omega)$ *such that* $q \succ^i q'$, *where* $q((y^i(\omega),\omega)) = 1$, *and* $q'((x^i(\omega),\omega)) = 1$.

Assumption (E7: Concavity, or Risk Aversion). *For all* $\omega \in \Omega$, *any* $x^i(\omega)$, $y^i(\omega)$ *in* $X^i(\omega)$, *and any* $t \in [0,1]$, *it follows that* $q_\omega \succeq^i q'_\omega$, *where* $q_\omega((tx^i(\omega) + (1-t)y^i(\omega),\omega)) = 1$, $q'_\omega((x^i(\omega),\omega)) = t$, *and* $q'_\omega((y^i(\omega),\omega)) = 1-t$.

Assumption (E8: Strict Concavity, or Strict Risk Aversion). *For every* $\omega \in \Omega$, *any distinct* $x^i(\omega)$, $y^i(\omega)$ *in* $X^i(\omega)$, *and any* $t \in (0,1)$, *it follows that* $q_\omega \succ^i q'_\omega$, *where* $q_\omega((tx^i(\omega) + (1-t)y^i(\omega),\omega)) = 1$, $q'_\omega((x^i(\omega),\omega)) = t$, *and* $q'_\omega((y^i(\omega),\omega)) = 1-t$.

3 Utility Function Representation of Preferences

Definition 1. A function $U_{E'}^{i,t^i} : D_{E'}^{i,t^i} \to \mathbf{R}$ is a utility function representation of the preference relation $\succeq_{E'}^{i,t^i}$ if for any pair of elements $(q, y), (q', y') \in D_{E'}^{i,t^i}$,

$$[(q, y) \succeq_{E'}^{i,t^i} (q', y')] \Leftrightarrow [U_{E'}^{i,t^i}(q, y) \geq U_{E'}^{i,t^i}(q', y')]$$

Definition 2. A function $U_{E'}^{i,t^i} : D_{E'}^{i,t^i} \to \mathbf{R}$ is an expected utility function representation of the preference relation $\succeq_{E'}^{i,t^i}$ if (i) it is a utility function representation of $\succeq_{E'}^{i,t^i}$; and (ii) there exists a function $u_{E'}^{i,t^i} : \cup_{E \in S_{E'}^{i,t^i}} \{E\} \times Y^i(E|t^i) \to \mathbf{R}$ such that for every $(q, y) \in D_{E'}^{i,t^i}$,

$$U_{E'}^{i,t^i}(q, y) = \sum_{E \in S_{E'}^{i,t^i}} q_E * u_E^{i,t^i}(E, y_E)$$

Fact 1. Assume that $X^i(.|T^i)$ satisfies assumption C1. If $\#[S_{E'}^{i,t^i}] = 1$, then an expected utility function representation of $\succeq_{E'}^{i,t^i}$ exists whenever there exists a utility function representation of $\succeq_{E'}^{i,t^i}$.

Theorem 1. *If* $\#[S_{E'}^{i,t^i}] \geq 2$, $X^i(.|t^i)$ *satisfies assumption C1, and* $\succeq_{E'}^{i,t^i}$ *satisfies assumptions* $E1, E2, E3, E4$, *and* $E5$, *then there exists an expected utility function representation of* $\succeq_{E'}^{i,t^i}$.

Theorem 2. *If* $X^i(.|t^i)$ *satisfies assumption* $C1$, *and* $C4$; *and* $\succeq_{E'}^{i,t^i}$ *satisfies assumptions* $E1, E2, E3, E4$, $E5$, *and* $E6$ *then there exists a continuous expected utility function representation of* $\succeq_{E'}^{i,t^i}$.

4 Proofs

4.1 Fact 1:

Proof of Fact 1: Let $U_{E'}^{i,t^i}$ be a utility function representation of $\succeq_{E'}^{i,t^i}$. Denote by E^* the unique element of $S_{E'}^{i,t^i}$. For every $(q, y) \in D_{E'}^{i,t^i}$, set $u_{E'}^{i,t^i}(E^*, y_E) = U_{E'}^{i,t^i}(q, y)$. Then for every $(q, y) \in D_{E'}^{i,t^i}$, $U_{E'}^{i,t^i}(q, y) = u_{E'}^{i,t^i}(E^*, y_{E^*}) = \sum_{E \in S_{E'}^{i,t^i}} q_E * u_{E'}^{i,t^i}(E, y_E)$ demonstrating that $U_{E'}^{i,t^i}$ is an expected utility function representation as required.

4.2 Theorem 1:

4.2.1 Some Preliminary Results

Lemma 1 (Mixture Monotonicity). *Assume that* $\succeq_{E'}^{i,t^i}$ *satisfies assumptions* $E2, E3, E4$, *and* $E5$. *Let* $y \in Y_{E'}^{i,t^i}$. *Then for any* $q_1, q_2 \in \Delta(S_{E'}^{i,t^i})$, *and any* $\alpha, \beta \in (0, 1)$, *where* $(q_1, y) \succ_{E'}^{i,t^i} (q_2, y)$ *and* $\alpha < \beta$, $(\beta q_1 + (1 - \beta)q_2, y) \succ_{E'}^{i,t^i} (\beta q_1 + (1 - \beta)q_2, y)$.

Proof of Lemma 1: (i) Suppose $\alpha = 0$. Note that $q_1 = \beta q_1 + (1-\beta)q_1$ and $q_2 = \beta q_2 + (1-\beta)q_2$ obviously. Now, by $E5$, $(q_1, y) \succ_{E'}^{i,t^i} (q_2, y) \Rightarrow (\beta q_1 + (1-\beta)q_1, y) \succ_{E'}^{i,t^i} (\beta q_1 + (1-\beta)q_2, y)$ as we have βq_1 on both sides. But by $E5$ again, $(\beta q_1 + (1-\beta)q_2, y) \succ_{E'}^{i,t^i} (\beta q_2 + (1-\beta)q_2, y)$ as we now have $(1-\beta)$ in common on both sides. But note that this implies $(\beta q_1 + (1-\beta)q_2, y) \succ_{E'}^{i,t^i} (q_2, y) = (\alpha q_1 + (1-\alpha)q_2, y)$ when $\alpha = 0$, and we are done. (ii) Suppose $\alpha > 0$. Now recall from (i) that $(\beta q_1 + (1-\beta)q_2, y) \succ_{E'}^{i,t^i} (q_2, y)$. Thus defining $q_3 = \beta q_1 + (1-\beta)q_2$, then $(q_3, y) \succ_{E'}^{i,t^i} (q_2, y)$. Now, define $\gamma = \frac{\alpha}{\beta}$. Then $(\gamma q_3 + (1-\gamma)q_3, y) \succ_{E'}^{i,t^i} (q_2, y)$. But as $(q_3, y) \succ_{E'}^{i,t^i} (q_2, y)$, then by $E5$, $(\gamma q_3 + (1-\gamma)q_3, y) \succ_{E'}^{i,t^i} (\gamma q_3 + (1-\gamma)q_2, y)$ where γq_3 is in common on both sides. Or, by definition of q_3, $(\gamma q_3 + (1-\gamma)q_3, y) \succ_{E'}^{i,t^i} (\gamma(\beta q_1 + (1-\beta)q_2) + (1-\gamma)q_2, y)$. Then rearranging, $(\gamma q_3 + (1-\gamma)q_3, y) \succ_{E'}^{i,t^i} (\gamma\beta q_1 + (1-\beta\gamma)q_2, y)$. But by the definition of γ, $\gamma\beta = \alpha$, thus $(\gamma q_3 + (1-\gamma)q_3, y) \succ_{E'}^{i,t^i} (\alpha q_1 + (1-\alpha)q_2, y)$. But as $q_3 = \gamma q_3 + (1-\gamma)q_3 = \beta q_1 + (1-\beta)q_2$ by definition, then $(\beta q_1 + (1-\beta)q_2, y) \succ_{E'}^{i,t^i} (\alpha q_1 + (1-\alpha)q_2, y)$. Q.E.D.

Lemma 2 (Unique Solvability). *Assume that* $\succeq_{E'}^{i,t^i}$ *satisfies assumptions* $E2, E3, E4$,*and* $E5$. *Let* $y_1, y_2 \in Y_{E'}^{i,t^i}$. *If* $q_1, q_2, q_3 \in \Delta(S_{E'}^{i,t^i})$ *such that* $(q_1, y_1) \succeq_{E'}^{i,t^i} (q_2, y_2) \succeq_{E'}^{i,t^i} (q_3, y_1)$ *and* $(q_1, y_1) \succ_{E'}^{i,t^i} (q_3, y_1)$ *then there exists a unique* $\alpha^* \in [0,1]$ *such that* $(q_2, y_2) \sim_{E'}^{i,t^i} (\alpha^* q_1 + (1-\alpha^*)q_3, y_1)$.

Proof of Lemma 2: (i) If $(q_1, y_1) \sim_{E'}^{i,t^i} (q_2, y_2)$, then $\alpha^* = 1$ and we are done. (ii) If $(q_2, y_2) \sim_{E'}^{i,t^i} (q_3, y_1)$, then $\alpha^* = 0$ and we are done. (iii) if $(q_1, y_1) \succ_{E'}^{i,t^i} (q_2, y_2) \succ_{E'}^{i,t^i} (q_3, y_1)$, then define the set $Q = \{\alpha \in (0,1) | (q_2, y_2) \succeq_{E'}^{i,t^i} (\alpha q_1 + (1-\alpha)q_3, y_1)\}$. This set is nonempty because $\alpha = 0$ is an element of it and it is bounded above by $\alpha = 1$. Thus there is a supremum of Q. Let $\alpha^* := \sup Q$. Then we can consider two violating cases. Case 1: $(q_1, y_1) \succ_{E'}^{i,t^i} (q_2, y_2) \succ_{E'}^{i,t^i} (\alpha^* q_1 + (1-\alpha^*)q_3, y_1)$. Then by $E4'$, there is a $\beta \in (0,1)$ such that $(q_1, y_1) \succ_{E'}^{i,t^i} (\beta(\alpha^* q_1 + (1-\alpha^*)q_3) + (1-\beta)q_1, y_1)$. Or rearranging, $(q_1, y_1) \succ_{E'}^{i,t^i} ((1-\beta(1-\alpha^*))q_1 + \beta(1-\alpha^*)q_3, y_1)$. But as $\beta(1-\alpha^*) < (1-\alpha^*)$, then $(1-\beta(1-\alpha^*)) > \alpha^*$. But then α^* is not a supremum of Q. A contradiction. Case 2: $(\alpha^* q_1 + (1-\alpha^*)q_3, y_1) \succ_{E'}^{i,t^i} (q_2, y_2)$. We can proceed the same way, i.e., by $E4'$ we can find some $\gamma \in [0,1]$ such that $([1-\gamma(1-\alpha^*)]q_1 + \gamma(1-\alpha^*)q_3, y_1) \succ_{E'}^{i,t^i} (q_2, y_2)$, which implies that α^* is not a supremum - thus a contradiction. Consequently, it must be that neither Case 1 or Case 2 can apply, thus $(\alpha^* q_1 + (1-\alpha^*)q_3, y_1) \sim_{E'}^{i,t^i} (q_2, y_2)$. Finally, by Lemma 1, α^* is unique. Q.E.D.

Consider any $y_1 \in Y_{E'}^{i,t^i}$, and any $q_1, q_2 \in \Delta(S_{E'}^{i,t^i})$ such that $(q_1, y_1) \succ_{E'}^{i,t^i} (q_2, y_1)$. Define $I := \{(q, y) \in D_{E'}^{i,t^i} | (q_1, y_1) \succeq_{E'}^{i,t^i} (q, y) \succeq_{E'}^{i,t^i} (q_2, y_1)\}$. For each $(q, y) \in I$, define $f(q, y)$ as a number such that $(q, y) \sim_{E'}^{i,t^i} (f(q, y)q_1 + (1-f(p))q_2, y_1)$. If the assumptions of Lemma 2 are satisfied, then such a $f(q, y)$ exists and is unique. We now make two claims:

Claim 1 (Representation on I). $f(.)$ *represents preferences on* I*, i.e., for all* $(q, y), (q, y') \in I$, $f(q, y) \geq f(q', y')$ *if an only if* $(f(q, y)q_1 + (1-f(q, y))q_2, y_1) \succeq_{E'}^{i,t^i} (f(q', y')q_1 + (1-f(q', y'))q_2, y_1)$.

Proof or Claim 1: Consider that by mixture monotonicity (Lemma 1), $(q_1, y_1) \succ_{E'}^{i,t^i} (q_2, y_1)$ and $f(q, y) \geq f(q', y')$ implies that $(f(q, y)q_1 + (1 - f(q, y))q_2, y_1) \succeq_{E'}^{i,t^i} (f(q', y')q_1 + (1 - f(q', y'))q_2, y_1)$. But by the definition of $f(q, y)$ and $f(q', y')$, (i.e., $(q, y) \sim_{E'}^{i,t^i} (f(q, y)q_1 + (1 - f(q, y))q_2, y_1)$ and $(q', y') \sim_{E'}^{i,t^i} (f(q', y')q_1 + (1 - f(q', y'))q_2, y_1)$), we can note immediately by transitivity $E3$ that this implies that $(q, y) \succeq_{E'}^{i,t^i} (q', y')$. The same argument works in reverse. Thus, $[f(q, y) \geq f(q', y')] \Leftrightarrow [(q, y) \succeq_{E'}^{i,t^i} (q', y')]$, i.e., $f(.)$ represents the preferences on I, and we are done. Q.E.D.

Claim 2 (Affinity on I)**.** *Let* $y \in Y_{E'}^{i,t^i}$. *Then* $f(., y)$ *is affine for all* $q, q' \in \Delta(S_{E'}^{i,t^i})$ *such that* $(q, y), (q', y) \in I$, *i.e.* $f(\alpha q + (1 - \alpha)q', y) = \alpha f(q, y) + (1 - \alpha)f(q', y)$. *The function* $f(.)$ *is furthermore onto the interval* $[f(q_2, y_1), f(q_1, y_1)]$.

Proof of Claim 2: Consider any $y \in Y_{E'}^{i,t^i}$ and any $q, q' \in \Delta(S_{E'}^{i,t^i})$ such that $(q, y), (q', y) \in I$ and define $q'' = \alpha q + (1 - \alpha)q'$. We note that it follows from the definition of $f(.)$ and $E5$ that $(\alpha q + (1 - \alpha)q', y) \sim_{E'}^{i,t^i} (\alpha(f(q, y)q_1 + (1 - f(q, y))q_2) + (1 - \alpha)(f(q', y)q_1 + (1 - f(q', y))q_2), y_1)$. Rearranging, we have $(\alpha q + (1 - \alpha)q', y) \sim_{E'}^{i,t^i} ([\alpha f(q, y) + (1 - \alpha)f(q', y)]q_1 + [1 - [\alpha f(q, y) + (1 - \alpha)f(q', y)]]q_2, y_1)$. From mixture monotonicity (Lemma 1), we have $(q_1, y_1) \succeq_{E'}^{i,t^i} ([\alpha f(q, y) + (1 - \alpha)f(q', y)]q_1 + [1 - [\alpha f(q, y) + (1 - \alpha)f(q', y)]]q_2, y_1) \succeq_{E'}^{i,t^i} (q_2, y_1)$. From transitivity, $E3$, it then follows that $(q_1, y_1) \succeq_{E'}^{i,t^i} (\alpha q + (1 - \alpha)q', y) \succeq_{E'}^{i,t^i} (q_2, y_1)$. Hence $(q'', y) \in I$ for any $\alpha \in (0, 1)$. Note from above that $(q'', y) \sim_{E'}^{i,t^i} ([\alpha f(q, y) + (1 - \alpha)f(q', y)]q_1 + [1 - [\alpha f(q, y) + (1 - \alpha)f(q', y)]]q_2, y_1)$. By the unique solvability (Lemma 2), it must be that $\alpha^* := f(q'', y) = [\alpha f(q, y) + (1 - \alpha)f(q', y)]$, or by the definition of q'', $f(\alpha q + (1 - \alpha)q', y) = \alpha f(q, y) + (1 - \alpha)f(q', y)$. This is the definition of affinity. To see that the function is onto, pick any $\alpha \in [f(q_2, y_1), f(q_1, y_1)]$. We note that α is actually in the interval $[0, 1]$, since it follows from how the function was defined that $f(q_2, y_1) = 0$ $f(q_1, y_1) = 1$. Then $f(\alpha q_1 + (1 - \alpha)q_2, y_1) = \alpha$ by the definition of f. Q.E.D.

Let $(q_1, y_1), (q_2, y_2)$ be any two elements of $D_{E'}^{i,t^i}$ such that $(q_1, y_1) \succ_{E'}^{i,t^i} (q_2, y_2)$ and define $I := \{(q, y) \in D_{E'}^{i,t^i} | (q_1, y_1) \succeq_{E'}^{i,t^i} (q_2, y_2)\}$. We make the following claim:

Claim 3 (Order-Preservation on I)**.** *If a function* f *represents the preferences on* I *and is affine, then* $g = a + bf$ *where* $b > 0$ *also (i) represents the preferences on* I *and (ii) is affine. Furthermore, if the function* f *is onto the interval* $[f(q_2, y_2), f(q_1, y_1)]$ *then (iii) the function* g *is onto the interval* $[g(q_2, y_2), g(q_1, y_1)]$.

Proof of Claim 3: For any $(q, y), (q', y') \in D_{E'}^{i,t^i}$, $[(q, y) \succeq_{E'}^{i,t^i} (q', y')] \Leftrightarrow [f(q, y) \geq f(q', y')]$ by the representation of f. Thus, if $b > 0$, then this implies that $[a + bf(q, y) \geq a + bf(q', y')] \Leftrightarrow [(q, y) \succeq_{E'}^{i,t^i} (q', y')]$. Thus $[g(q, y) \geq g(q', y')] \Leftrightarrow [(q, y) \succeq_{E'}^{i,t^i} (q', y')]$ by definition. (ii) As f is affine for any $y \in Y_{E'}^{i,t^i}$, it follows that for any $q, q' \in \Delta(S_{E'}^{i,t^i})$ such that $(q, y), (q', y) \in I$, $f(\alpha q + (1 - \alpha)q', y) = \alpha f(q, y) + (1 - \alpha)f(q', y)$. Now, by definition, $g(\alpha q + (1 - \alpha)q', y) = a + bf(\alpha q + (1 - \alpha)q', y) = a + b[\alpha f(q, y) + (1 - \alpha)f(q', y)] = \alpha a + (1 - \alpha)a + b\alpha f(q, y) + b(1 - \alpha)f(q', y) = \alpha[a + bf(q, y)] + (a - \alpha)[a + bf(q', y)] = \alpha g(q, y) + (1 - \alpha)g(q', y)$. To see that g is onto the interval $[g(q_2, y_2), g(q_1, y_1)]$ whenever f is onto the interval $[f(q_2, y_2), f(q_1, y_1)]$, pick any $\alpha \in [g(q_2, y_2), g(q_1, y_1)]$ and define $\beta := \frac{\alpha - a}{b}$. We note that:

$$\begin{array}{rcl} f(q_2, y_2) & = & \frac{g(q_2,y_2)-a}{b} \\ & \leq & \frac{\alpha-a}{b} \\ & \leq & \frac{g(q_1,y_1)-a}{b} \\ & = & f(q_1, y_1) \end{array}$$

Hence $\beta \in [f(q_2, y_2), f(q_1, y_1)]$. Since f is onto the interval $[f(q_2, y_2), f(q_1, y_1)]$, it follows that there exists $(q, y) \in D^{i,t^i}_{E'}$ such that $f(q, y) = \beta$. Hence it follows from the definition of g that $g(q, y) = a + b * \beta = a + b * \frac{(\alpha - a)}{b} = \alpha$. Q.E.D.

Consider any $y_1, y_2, y_2', \overline{y}, \underline{y} \in Y^{i,t^i}_{E'}$, and any $q_1, q_1', q_2, q_2', \overline{q}, \underline{q} \in \Delta(S^{i,t^i}_{E'})$ such that:

(i) $(q_1, y_1) \succeq^{i,t^i}_{E'} (\overline{q}, \overline{y}) \succ^{i,t^i}_{E'} (\underline{q}, \underline{y}) \succeq^{i,t^i}_{E'} (q_1', y_1)$; and

(ii) $(q_2, y_2) \succeq^{i,t^i}_{E'} (\overline{q}, \overline{y}) \succ^{i,t^i}_{E'} (\underline{q}, \underline{y}) \succeq^{i,t^i}_{E'} (q_2', y_2')$.

Define I_1, and I_2 respectively by $I_1 := \{(q, y) \in D^{i,t^i}_{E'} | (q_1, y_1) \succeq^{i,t^i}_{E'} (q, y) \succeq^{i,t^i}_{E'} (q_1', y_1)\}$ and $I_2 := \{(q, y) \in D^{i,t^i}_{E'} | (q_2, y_2) \succeq^{i,t^i}_{E'} (q, y) \succeq^{i,t^i}_{E'} (q_2', y_2')\}$.

Let f_1 and f_2 be affine representations of the preferences on I_1 and I_2 respectively satisfying the properties that (iii) $f_1(\overline{q}, \overline{y}) = f_2(\overline{q}, \overline{y})$; and (iv) $f_1(\underline{q}, \underline{y}) = f_2(\underline{q}, \underline{y})$. We make the following claim.

Claim 4 (Coincidence on $I_1 \cap I_2$). *f_1 and f_2 coincides on $I_1 \cap I_2$.*

Proof of Claim 4: Let $(q, y) \in I_1 \cap I_2$. Since $(q, y) \in I_1$, it follows from unique solvability and mixture monotonicity that there exists uniquely $\overline{\alpha}, \underline{\alpha}$ ($\overline{\alpha} > \underline{\alpha}$), and $\alpha \in [0, 1]$ such that (a) $(\overline{\alpha} q_1 + (1 - \overline{\alpha}) q_1', y_1) \sim^{i,t^i}_{E'} (\overline{q}, \overline{y})$; (b) $(\underline{\alpha} q_1 + (1 - \underline{\alpha}) q_1', y_1) \sim^{i,t^i}_{E'} (\underline{q}, \underline{y})$; and (c) $(\alpha q_1 + (1 - \alpha) q_1', y_1) \sim^{i,t^i}_{E'} (q, y)$. We now consider each of the three possible cases:

Case 1: $(\overline{\alpha} q_1 + (1 - \overline{\alpha}) q_1', y_1) \succeq^{i,t^i}_{E'} (\alpha q_1 + (1 - \alpha) q_1', y_1) \succeq^{i,t^i}_{E'} (\underline{\alpha} q_1 + (1 - \underline{\alpha}) q_1', y_1)$. Then by unique solvability, there exists $\alpha^* \in [0, 1]$ such that $(\alpha q_1 + (1 - \alpha) q', y_1) \sim^{i,t^i}_{E'} (\alpha^*(\overline{\alpha} q_1 + (1 - \overline{\alpha}) q_1') + (1 - \alpha^*)(\underline{\alpha} q_1 + (1 - \underline{\alpha}) q_1'), y_1)$. It then follows that:

$$\begin{aligned}
f_1(q,y) &= f_1(\alpha q_1 + (1-\alpha)q_1', y_1) && \text{(by representation and (c))}\\
&= f_1(\alpha^*(\overline{\alpha} q_1 + (1-\overline{\alpha})q_1') \\
&\quad +(1-\alpha^*)(\underline{\alpha} q_1 + (1-\underline{\alpha})q_1'), y_1) && \text{(by representation and definition of } \alpha^*)\\
&= \alpha^* f_1(\overline{\alpha} q_1 + (1-\overline{\alpha})q_1', y_1) \\
&\quad +(1-\alpha^*) f_1(\underline{\alpha} q_1 + (1-\underline{\alpha})q_1', y_1) && \text{(by affinity of } f_1)\\
&= \alpha^* f_1(\overline{q}, \overline{y}) + (1-\alpha^*) f_1(\underline{q}, \underline{y}) && \text{(by representation and definitions}\\
& && \text{of } \overline{\alpha} \text{ and } \underline{\alpha})\\
&= \alpha^* f_2(\overline{q}, \overline{y}) + (1-\alpha^*) f_2(\underline{q}, \underline{y}) && \text{(by (iii) and (iv))}\\
&= \alpha^* f_2(\overline{\alpha} q_1 + (1-\overline{\alpha})q_1', y_1) && \text{(by representation and definitions}\\
&\quad +(1-\alpha^*) f_2(\underline{\alpha} q_1 + (1-\underline{\alpha})q_1', y_1) && \text{of } \overline{\alpha} \text{ and } \underline{\alpha})\\
&= f_2(\alpha^*(\overline{\alpha} q_1 + (1-\overline{\alpha})q_1') \\
&\quad +(1-\alpha^*)(\underline{\alpha} q_1 + (1-\underline{\alpha})q_1'), y_1) && \text{(by affinity of } f_2)\\
&= f_2(\alpha q_1 + (1-\alpha)q_1', y_1) && \text{(by representation and definition of } \alpha^*)\\
&= f_2(q,y) && \text{(by representation and (c))}
\end{aligned}$$

Case 2: $(\alpha q_1 + (1-\alpha)q_1', y_1) \succ_{E'}^{i,t^i} (\overline{\alpha} q_1 + (1-\overline{\alpha})q_1', y_1) \succ_{E'}^{i,t^i} (\underline{\alpha} q_1 + (1-\underline{\alpha})q_1', y_1)$. Then again by unique solvability, there exists $\alpha^* \in (0,1)$ such that $(\overline{\alpha} q_1 + (1-\overline{\alpha})q', y_1) \sim_{E'}^{i,t^i} (\alpha^*(\alpha q_1 + (1-\alpha)q_1') + (1-\alpha^*)(\underline{\alpha} q_1 + (1-\underline{\alpha})q_1'), y_1)$. It then follows that:

$$\begin{aligned}
f_1(\overline{q},\overline{y}) &= f_1(\overline{\alpha} q_1 + (1-\overline{\alpha})q', y_1) && \text{(by representation and (a))}\\
&= f_1(\alpha^*(\alpha q_1 + (1-\alpha)q_1') \\
&\quad +(1-\alpha^*)(\underline{\alpha} q_1 + (1-\underline{\alpha})q_1'), y_1) && \text{(by representation and definition of } \alpha^*)\\
&= \alpha^* f_1(\alpha q_1 + (1-\alpha)q_1', y_1) \\
&\quad +(1-\alpha^*) f_1(\underline{\alpha} q_1 + (1-\underline{\alpha})q_1', y_1) && \text{(by affinity of } f_1)\\
&= \alpha^* f_1(q,y) + (1-\alpha^*) f_1(\underline{q}, \underline{y}) && \text{(by representation, (b), and (c))}
\end{aligned}$$

Rearranging, we have

$$f_1(q,y) = (\frac{1}{\alpha^*})[f_1(\overline{q},\overline{y}) - (1-\alpha^*) f_1(\underline{q},\underline{y})].$$

By the same token, we have

$$\begin{aligned}
f_2(\overline{q},\overline{y}) &= f_2(\overline{\alpha} q_1 + (1-\overline{\alpha})q', y_1) && \text{(by representation and (a))}\\
&= f_2(\alpha^*(\alpha q_1 + (1-\alpha)q_1') \\
&\quad +(1-\alpha^*)(\underline{\alpha} q_1 + (1-\underline{\alpha})q_1'), y_1) && \text{(by representation and definition of } \alpha^*)\\
&= \alpha^* f_2(\alpha q_1 + (1-\alpha)q_1', y_1) \\
&\quad +(1-\alpha^*) f_2(\underline{\alpha} q_1 + (1-\underline{\alpha})q_1', y_1) && \text{(by affinity of } f_2)\\
&= \alpha^* f_2(q,y) + (1-\alpha^*) f_2(\underline{q}, \underline{y}) && \text{(by representation, (b), and (c))}
\end{aligned}$$

Again, after rearranging, we have

$$f_2(q,y) = (\frac{1}{\alpha^*})[f_2(\overline{q},\overline{y}) - (1-\alpha^*) f_2(\underline{q},\underline{y})]$$

.

From the two above rearranged equalities, it then follows that:

$$\begin{aligned} f_1(q,y) &= (\tfrac{1}{\alpha^*})[f_1(\overline{q},\overline{y}) - (1-\alpha^*)f_1(\underline{q},\underline{y})] && \text{(from first equality)} \\ &= (\tfrac{1}{\alpha^*})[f_2(\overline{q},\overline{y}) - (1-\alpha^*)f_2(\underline{q},\underline{y})] && \text{(from (iii) and (iv)} \\ &= f_2(q,y) && \text{(from second equality)} \end{aligned}$$

as required.

Case 3: $(\overline{\alpha}q_1 + (1-\overline{\alpha})q_1', y_1) \succ_{E'}^{i,t^i} (\underline{\alpha}q_1 + (1-\underline{\alpha})q_1', y_1) \succ_{E'}^{i,t^i} (\alpha q_1 + (1-\alpha)q_1', y_1)$. Then again by unique solvability, there exists $\alpha^* \in (0,1)$ such that $(\underline{\alpha}q_1 + (1-\underline{\alpha})q', y_1) \sim_{E'}^{i,t^i} (\alpha^*(\overline{\alpha}q_1 + (1-\overline{\alpha})q_1') + (1-\alpha^*)(\alpha q_1 + (1-\alpha)q_1'), y_1)$. It then follows that:

$$\begin{aligned} f_1(\underline{q},\underline{y}) &= f_1(\underline{\alpha}q_1 + (1-\underline{\alpha})q', y_1) && \text{(by representation and (b))} \\ &= f_1(\alpha^*(\overline{\alpha}q_1 + (1-\overline{\alpha})q_1') \\ &\quad +(1-\alpha^*)(\alpha q_1 + (1-\alpha)q_1'), y_1) && \text{(by representation and definition of } \alpha^*) \\ &= \alpha^* f_1(\overline{\alpha}q_1 + (1-\overline{\alpha})q_1', y_1) \\ &\quad +(1-\alpha^*)f_1(\alpha q_1 + (1-\alpha)q_1', y_1) && \text{(by affinity of } f_1) \\ &= \alpha^* f_1(\overline{q},\overline{y}) + (1-\alpha^*)f_1(q,y) && \text{(by representation, (a), and (c))} \end{aligned}$$

Rearranging, we have

$$f_1(q,y) = \frac{1}{(1-\alpha^*)}[f_1(\underline{q},\underline{y}) - \alpha^* f_1(\overline{q},\overline{y})].$$

By the same token, we have

$$\begin{aligned} f_2(\underline{q},\underline{y}) &= f_2(\underline{\alpha}q_1 + (1-\underline{\alpha})q', y_1) && \text{(by representation and (b))} \\ &= f_2(\alpha^*(\overline{\alpha}q_1 + (1-\overline{\alpha})q_1') \\ &\quad +(1-\alpha^*)(\alpha q_1 + (1-\alpha)q_1'), y_1) && \text{(by representation and definition of } \alpha^*) \\ &= \alpha^* f_2(\overline{\alpha}q_1 + (1-\overline{\alpha})q_1', y_1) \\ &\quad +(1-\alpha^*)f_2(\alpha q_1 + (1-\alpha)q_1', y_1) && \text{(by affinity of } f_2) \\ &= \alpha^* f_2(\overline{q},\overline{y}) + (1-\alpha^*)f_2(q,y) && \text{(by representation, (a), and (c))} \end{aligned}$$

Again, after rearranging, we have

$$f_2(q,y) = \frac{1}{(1-\alpha^*)}[f_2(\underline{q},\underline{y}) - \alpha^* f_2(\overline{q},\overline{y})].$$

From the two above rearranged equalities, it then follows that:

$$\begin{aligned} f_1(q,y) &= \tfrac{1}{(1-\alpha^*)}[f_1(\underline{q},\underline{y}) - \alpha^* f_1(\overline{q},\overline{y})] && \text{(from first equality)} \\ &= \tfrac{1}{(1-\alpha^*)}[f_2(\underline{q},\underline{y}) - \alpha^* f_2(\overline{q},\overline{y})] && \text{(from (iii) and (iv)} \\ &= f_2(q,y) && \text{(from second equality)} \end{aligned}$$

as required. Q.E.D.

Consider any $(q_1,y_1),,(q_1',y_1'),(q_2,y_2),(q_2',y_2'),(\overline{q},\overline{y}),(\underline{q},\underline{y}) \in D_{E'}^{i,t^i}$ such that:

(i) $(q_1,y_1) \succeq_{E'}^{i,t^i} (\overline{q},\overline{y}) \succ_{E'}^{i,t^i} (\underline{q},\underline{y}) \succeq_{E'}^{i,t^i} (q_1',y_1')$; and

(ii) $(q_2,y_2) \succeq_{E'}^{i,t^i} (\overline{q},\overline{y}) \succ_{E'}^{i,t^i} (\underline{q},\underline{y}) \succeq_{E'}^{i,t^i} (q_2',y_2')$.

Define I_1, and I_2 respectively by $I_1 := \{(q,y) \in D_{E'}^{i,t^i} | (q_1,y_1) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (q_1',y_1')\}$ and $I_2 := \{(q,y) \in D_{E'}^{i,t^i} | (q_2,y_2) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (q_2',y_2')\}$.

Let f_1 and f_2 be affine representations of the preferences on I_1 and I_2 respectively with the further property that f_1 coincides with f_2 on $I_1 \cap I_2$. We make the following claim.

Claim 5 (Representation on $I_1 \cup I_2$)**.** *The function f that coincides with f_1 on I_1 and f_2 on I_2 represents preferences on $I_1 \cup I_2$.*

Proof of Claim 5: Assume without loss of generality that $(q_1,y_1) \succeq_{E'}^{i,t^i} (q_2,y_2)$. We consider all of three possible cases. (i) If $(q_2',y_2') \succeq_{E'}^{i,t^i} (q_1',y_1')$ then $I_2 \subset I_1$. Thus $I_1 \cup I_2 = I_1$, and $f(q,y) = f_1(q,y)$ for all $(q,y) \in I_1 \cup I_2$. Hence, we are done since f_1 represents preferences on I_1. (ii) If $(q_1',y_1') \succeq_{E'}^{i,t^i} (q_2',y_2')$, and $(q_2,y_2) \succeq_{E'}^{i,t^i} (q_1,y_1)$, then the present assumption implies that $I_1 \subset I_2$. Thus $I_1 \cup I_2 = I_2$, and $f(q,y) = f_2(q,y)$ for all $(q,y) \in I_1 \cup I_2$. Hence, we are done since f_2 represents preferences on I_2.

(iii) $(q_1,y_1) \succ_{E'}^{i,t^i} (q_2,y_2) \succeq_{E'}^{i,t^i} (\overline{q},\overline{y}) \succ_{E'}^{i,t^i} (\underline{q},\underline{y}) \succeq_{E'}^{i,t^i} (q_1',y_1') \succ_{E'}^{i,t^i} (q_2',y_1')$.

Step 1: Show that $[(q,y) \succeq_{E'}^{i,t^i} (q',y')] \Rightarrow [f(q,y) \geq f(q',y')]$. Pick any $(q,y),(q',y') \in I_1 \cup I_2$ such that $(q,y) \succeq_{E'}^{i,t^i} (q',y')$. (1a) If $(q_2,y_2) \succeq_{E'}^{i,t^i} (q,y)$ then (q,y) and (q',y') are both in I_2. Hence it follows from the assumption that f_2 represents preferences on I_2 that $f_2(q,y) \geq f_2(q',y')$. But from the definition of f it then it also follows that $f(q,y) \geq f(q',y')$. (1b) If $(q',y') \succeq_{E'}^{i,t^i} (q_1',y_1')$, then (q,y) and (q',y') are both in I_1. Hence from the assumption that f_1 represents preferences on I_1 it follows that $f_1(q,y) \geq f_1(q',y')$. But from the definition of f it then also follows that $f(q,y) \geq f(q',y')$. (1c) $(q,y) \succeq_{E'}^{i,t^i} (q_2,y_2) \succeq_{E'}^{i,t^i} (\overline{q},\overline{y}) \succ_{E'}^{i,t^i} (\underline{q},\underline{y}) \succeq_{E'}^{i,t^i} (q_1',y_1') \succeq_{E'}^{i,t^i} (q',y')$. Then $(q,y) \in I_1$, and $q',y') \in I_2$. But then it follows that:

$$\begin{aligned}
f(q,y) &= f_1(q,y) && \text{(from definition of } f\text{)} \\
&> f_1(\underline{q},\underline{y}) && \text{(from properties of } f_1\text{)} \\
&= f_2(\underline{q},\underline{y}) && \text{(by assumption)} \\
&> f_2(q',y') && \text{(from properties of } f_2\text{)} \\
&= f(q',y') && \text{(from definition of } f\text{)}
\end{aligned}$$

as required.

Step 2: Show that $[f(q,y) \geq f(q',y_1')] \Rightarrow [(q,y) \in I_1]$.

Pick any $(q,y) \in I_1 \cup I_2$ such that $f(q,y) \geq f(q',y_1')$. Clearly, it suffices to show that $[(q,y) \in I_2] \Rightarrow [(q,y) \in I_1]$. We note that $[(q,y) \in I_2] \Leftrightarrow [(q_2,y_2) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (q_2',y_2')]$. Since f_2 represents preferences on I_2, and f coincides with f_2 on I_2, it follows that $[f(q_2,y_2) \geq f(q,y) \geq f(q_2',y_2')]$. Likewise, we note that since f_1 represents preferences on I_1, and f coincides with f_1 on I_1, it follows from the present assumption that $f(q_2,y_2) \geq f(q_1',y_1') \geq f(q_2',y_2')$. Combining these with the present assumption that $f(q,y) \geq f(q',y_1')$, it follows that $f(q_2,y_2) \geq f(q,y) \geq f(q_1',y_1') \geq f(q_2',y_2')$. But then it follows from the facts that f_2 represents preferences on I_2 and that f coincides with f_2 on I_2 that $(q_2,y_2) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (q_1',y_1')$. Combining this with the fact that $(q_1,y_1) \succeq_{E'}^{i,t^i} (q_2,y_2)$ then implies that $(q_1,y_1) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (q_1',y_1')$. Hence $(q,y) \in I_1$.

Step 3: Show that $[f(q_2, y_2) \geq f(q, y)] \Rightarrow [(q, y) \in I_2]$.

Pick any $(q, y) \in I_1 \cup I_2$ such that $f(q_2, y_2) \geq f(q, y)$. Clearly, it suffices to show that $[(q, y) \in I_1] \Rightarrow [(q, y) \in I_2]$. We note that $[(q, y) \in I_1] \Leftrightarrow [(q_1, y_1) \succeq_{E'}^{i,t^i} (q, y) \succeq_{E'}^{i,t^i} (q'_1, y'_1)]$. Since f_1 represents preferences on I_1, and f coincides with f_1 on I_1, it follows that $[f(q_1, y_1) \geq f(q, y) \geq f(q'_1, y'_1)]$. Likewise, we note that since f_2 represents preferences on I_2, and f coincides with f_2 on I_2, it follows from the present assumption that $f(q_1, y_1) \geq f(q_2, y_2) \geq f(q'_1, y'_1)$. Combining these with the present assumption that $f(q_2, y_2) \geq f(q, y)$, it follows that $f(q_1, y_1) \geq f(q_2, y_2 \geq f(q, y) \geq f(q'_1, y'_1)$. But then it follows from the facts that f_1 represents preferences on I_1 and that f coincides with f_1 on I_1 that $(q_2, y_2) \succeq_{E'}^{i,t^i} (q, y) \succeq_{E'}^{i,t^i} (q'_1, y'_1)$. Combining this with the fact that $(q'_1, y'_1) \succeq_{E'}^{i,t^i} (q'_2, y'_2)$ then implies that $(q_2, y_2) \succeq_{E'}^{i,t^i} (q, y) \succeq_{E'}^{i,t^i} (q'_2, y'_2)$. Hence $(q, y) \in I_2$.

Step 4: Show that $[f(q, y) \geq f(q', y')] \Rightarrow [(q, y) \succeq_{E'}^{i,t^i} (q', y')]$.

Pick any $(q, y), (q', y') \in I_1 \cup I_2$ such that $f(q, y) \geq f(q', y')$. (4a) If $f(q_2, y_2) \geq f(q, y)$ then we know from step 3 that both (q, y), and (q', y') are in I_2. But then we are done since f_2 represents preferences on I_2 and f coincides with f_2 on I_2. (4b) If $f(q', y') \geq f(q'_1, y'_1)$ then we know from step 2 that both (q, y), and (q', y') are in I_1. But then we are done since f_1 represents preferences on I_1 and f coincides with f_1 on I_1. (4c) $f(q, y) > f(q_2, y_2) > f(q'_1, y'_1) > f(q', y')$. Then we know from step 2 that $(q, y) \in I_1$, and from step 3 that $(q', y') \in I_2$. Since f_1 represents preferences on I_1 and f coincides with f_1 on I_1, we have $(q, y) \succ_{E'}^{i,t^i} (q_2, y_2)$. Likewise, since f_2 represents preferences on I_2 and f coincides with f_2 on I_2, we have $(q'_1, y'_1) \succ_{E'}^{i,t^i} (q', y')$. But under the present assumptions it then follows that $(q, y) \succ_{E'}^{i,t^i} (q_2, y_2) \succ_{E'}^{i,t^i} (q'_1, y'_1) \succ_{E'}^{i,t^i} (q', y')$ as required. Q.E.D.

Claim 6 (Affinity on $I_1 \cup I_2$)**.** *Let $y \in Y_{E'}^{i,t^i}$. Then the function f that coincides with f_1 on I_1 and f_2 on I_2 is affine on $I_1 \cup I_2$ for any $q, q' \in \Delta(S_{E'}^{i,t^i})$ such that $(q, y), (q', y) \in I_1 \cup I_2$.*

Proof of Claim 6: Pick any $y \in Y_{E'}^{i,t^i}$, and any $q, q' \in \Delta(S_{E'}^{i,t^i})$ such that $(q, y), (q', y) \in I_1 \cup I_2$. Without loss of generality, assume that $(q, y) \succeq_{E'}^{i,t^i} (q', y)$. (i) If (q, y) and (q', y) are both in I_1, then we are done since f_1 is affine and f coincides with f_1 on I_1. (ii) If (q, y) and (q', y) are both in I_2, then we are done since f_2 is affine and f coincides with f_2 on I_2. (iii) $(q, y) \notin I_1$, $(q', y) \notin I_2$. It follows from completeness, transitivity, and the present assumptions that $(q_2, y_2) \succeq_{E'}^{i,t^i} (q, y) \succ_{E'}^{i,t^i} (q_1, y_1) \succeq_{E'}^{i,t^i} (\overline{q}, \overline{y}) \succ_{E'}^{i,t^i} (\underline{q}, \underline{y}) \succeq_{E'}^{i,t^i} (q'_2, y_2) \succ_{E'}^{i,t^i} (q', y) \succeq_{E'}^{i,t^i} (q'_1, y_1)$. Define $I_3 := \{(q'', y'') \in D_{E'}^{i,t^i} | (q, y) \succeq_{E'}^{i,t^i} (q'', y'') \succeq_{E'}^{i,t^i} (q', y)\}$ and using unique solvability , define a function $f_3 : I_3 \rightarrow \mathbf{R}$ by $f_3(q'', y'') = \alpha^*_{q'', y''}$ where $\alpha^*_{q'', y''}$ is the unique element of $[0, 1]$ for which $(\alpha^*_{q'', y''} q + (1 - \alpha^*_{q'', y''}) q', y) \sim_{E'}^{i,t^i} (q'', y'')$. From Claim 1, we know that f_3 represents preferences on I_3, from Claim 2, we know that f_3 is affine, and from Claim 3, we know that for any a, b where $(b > 0)$, the function $g := a + bf$ both represents preferences on I_3 and is affine. Let $a = f(\overline{q}, \overline{y}) - \frac{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]}{[f_3(\overline{q}, \overline{y}) - f_3(\underline{q}, \underline{y})]} * f_3(\overline{q}, \overline{y})$, and let $b = \frac{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]}{[f_3(\overline{q}, \overline{y}) - f_3(\underline{q}, \underline{y})]} > 0$. Then we have, on the one hand that $g(\overline{q}, \overline{y}) = f(\overline{q}, \overline{y}) - \frac{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]}{[f_3(\overline{q}, \overline{y}) - f_3(\underline{q}, \underline{y})]} * f_3(\overline{q}, \overline{y}) + \frac{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]}{[f_3(\overline{q}, \overline{y}) - f_3(\underline{q}, \underline{y})]} * f_3(\overline{q}, \overline{y}) = f(\overline{q}, \overline{y}) = f_1(\overline{q}, \overline{y}) = f_2(\overline{q}, \overline{y})$. Likewise, we have that $g(\underline{q}, \underline{y}) = f(\overline{q}, \overline{y}) - \frac{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]}{[f_3(\overline{q}, \overline{y}) - f_3(\underline{q}, \underline{y})]} * f_3(\overline{q}, \overline{y}) + \frac{[f(\overline{q}, \overline{y}) - f(\underline{q}, \underline{y})]}{[f_3(\overline{q}, \overline{y}) - f_3(\underline{q}, \underline{y})]} * f_3(\underline{q}, \underline{y}) = f(\underline{q}, \underline{y}) = f_1(\underline{q}, \underline{y}) =$

$f_2(\underline{q}, \underline{y})$. But then it follows from claim 4 that g (a) coincides with f_1, and hence with f, on $I_3 \cap I_1$; and (b) coincides with f_2, and hence with f, on $I_3 \cap I_2$. But $I_3 \subset I_1 \cup I_2$ implies that $I_3 = I_3 \cap (I_1 \cup I_2) = (I_3 \cap I_1) \cup (I_3 \cap I_2)$. Hence we can conclude that f coincides with g on I_3. But then we are done since g is affine on I_3.

(iv) $(q, y) \notin I_2$, $(q', y) \notin I_1$. It follows from completeness, transitivity, and the present assumptions that $(q_1, y_1) \succeq_{E'}^{i,t^i} (q, y) \succ_{E'}^{i,t^i} (q_2, y_2) \succeq_{E'}^{i,t^i} (\overline{q}, \overline{y}) \succ_{E'}^{i,t^i} (\underline{q}, \underline{y}) \succeq_{E'}^{i,t^i} (q_1', y_1) \succ_{E'}^{i,t^i} (q', y) \succeq_{E'}^{i,t^i} (q_2', y_2)$. Define $I_3 := \{(q'', y'') \in D_{E'}^{i,t^i} | (q, y) \succeq_{E'}^{i,t^i} (q'', y'') \succeq_{E'}^{i,t^i} (q', y)\}$ and using unique solvability , define a function $f_3 : I_3 \rightarrow \mathbf{R}$ by $f_3(q'', y'') = \alpha^*_{q'',y''}$ where $\alpha^*_{q'',y''}$ is the unique element of $[0, 1]$ for which $(\alpha^*_{q'',y''} q + (1 - \alpha^*_{q'',y''}) q', y) \sim_{E'}^{i,t^i} (q'', y'')$. From Claim 1, we know that f_3 represents preferences on I_3, from Claim 2 we know that f_3 is affine, and from Claim 3 we know that for any a, b where $(b > 0)$, the function $g := a + bf$ both represents preferences on I_3 and is affine. Let $a = f(\overline{q}, \overline{y}) - \frac{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y}) - f_3(\underline{q},\underline{y})]} * f_3(\overline{q}, \overline{y})$, and let $b = \frac{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y}) - f_3(\underline{q},\underline{y})]} > 0$. Then we have, on the one hand that $g(\overline{q}, \overline{y}) = f(\overline{q}, \overline{y}) - \frac{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y}) - f_3(\underline{q},\underline{y})]} * f_3(\overline{q}, \overline{y}) + \frac{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y}) - f_3(\underline{q},\underline{y})]} * f_3(\overline{q}, \overline{y}) = f(\overline{q}, \overline{y}) = f_1(\overline{q}, \overline{y}) = f_2(\overline{q}, \overline{y})$. Likewise, we have that $g(\underline{q}, \underline{y}) = f(\overline{q}, \overline{y}) - \frac{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y}) - f_3(\underline{q},\underline{y})]} * f_3(\overline{q}, \overline{y}) + \frac{[f(\overline{q},\overline{y}) - f(\underline{q},\underline{y})]}{[f_3(\overline{q},\overline{y}) - f_3(\underline{q},\underline{y})]} * f_3(\underline{q}, \underline{y}) = f(\underline{q}, \underline{y}) = f_1(\underline{q}, \underline{y}) = f_2(\underline{q}, \underline{y})$. But then it follows from claim 4 that g (a) coincides with f_1, and hence with f, on $I_3 \cap I_1$; and (b) coincides with f_2, and hence with f, on $I_3 \cap I_2$. But $I_3 \subset I_1 \cup I_2$ implies that $I_3 = I_3 \cap (I_1 \cup I_2) = (I_3 \cap I_1) \cup (I_3 \cap I_2)$. Hence we can conclude that f coincides with g on I_3. But then we are done since g is affine on I_3. Q.E.D.

Let $(\overline{q}, \overline{y}), (\underline{q}, \underline{y}) \in D_{E'}^{i,t^i}$ satisfy the property that $(\overline{q}, \overline{y}) \succ_{E'}^{i,t^i} (\underline{q}, \underline{y})$. Define $I^* := \{(q, y) \in D_{E'}^{i,t^i} | (\overline{q}, \overline{y}) \succeq_{E'}^{i,t^i} (q, y) \succeq_{E'}^{i,t^i} (\underline{q}, \underline{y})\}$.

Definition 3. I^* is a basis interval if it satisfies the following three criteria:

(i) there exists an affine function $f^* : I^* \rightarrow \mathbf{R}$ that represents the preferences on I^* and is onto the interval $[f^*(\underline{q}, \underline{y}), f^*(\overline{q}, \overline{y})]$;
(ii) for every $(q_1, y_1) \in D_{E'}^{i,t^i}$ such that $(q, y) \succ_{E'}^{i,t^i} (\overline{q}, \overline{y})$, there exists $y_1^* \in Y_{E'}^{i,t^i}$, and $q_1^*, q_1^{**} \in \Delta(S_{E'}^{i,t^i})$ such that $(q_1^*, y_1^*) \sim_{E'}^{i,t^i} (q_1, y_1) \succ_{E'}^{i,t^i} (\overline{q}, \overline{y}) \succ_{E'}^{i,t^i} (q_1^{**}, y_1^*) \succeq_{E'}^{i,t^i} (\underline{q}, \underline{y})$; and
(iii) for every $(q_2, y_2) \in D_{E'}^{i,t^i}$ such that $(\underline{q}, \underline{y}) \succ_{E'}^{i,t^i} (q_2, y_2)$, there exists $y_2^* \in Y_{E'}^{i,t^i}$, and $q_2^*, q_2^{**} \in \Delta(S_{E'}^{i,t^i})$ such that $(\overline{q}, \overline{y}) \succeq_{E'}^{i,t^i} (q_2^*, y_2^*) \succ_{E'}^{i,t^i} (\underline{q}, \underline{y}) \succ_{E'}^{i,t^i} (q_2, y_2) \sim_{E'}^{i,t^i} (q_2^{**}, y_2^*)$.

We now turn to a couple of additional claims. Let $I^* := \{(q, y) \in D_{E'}^{i,t^i} | (\overline{q}, \overline{y}) \succeq_{E'}^{i,t^i} (q, y) \succeq_{E'}^{i,t^i} (\underline{q}, \underline{y})\}$ be a basis interval and let f^* be an affine representation of the preferences on I^* such that f^* is onto the interval $[f^*(\underline{q}, \underline{y}), f^*(\overline{q}, \overline{y})]$. Let $(q_1, y_1), (q_1', y_1') \in D_{E'}^{i,t^i}$ satisfy the property that $(q_1, y_1) \succeq_{E'}^{i,t^i} (\overline{q}, \overline{y}) \succ_{E'}^{i,t^i} (\underline{q}, \underline{y}) \succeq_{E'}^{i,t^i} (q_1', y_1')$ and let $I := \{(q, y) \in D_{E'}^{i,t^i} | (q_1, y_1) \succeq_{E'}^{i,t^i} (q, y) \succeq_{E'}^{i,t^i} (q_1', y_1')\}$.

Claim 7 (Representation of I^*). *There exists an affine representation f of the preferences on I such that a) f coincides with f^* on I^*, and b) f is onto the interval $[f(q_1', y_1'), f(q_1, y_1)]$.*

Proof of Claim 7: Define $I_1' := \{(q,y) \in D_{E'}^{i,t^i} | (q_1, y_1) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (\underline{q}, \underline{y})\}$. We divide the rest of our proof into two steps.

Step 1: Show that there exists an affine representation f_1 of the preferences on I_1' such that a) f_1 coincides with f^* on I^*; and b) f_1 is onto the interval $[f_1(\underline{q}, \underline{y}), f_1(q_1, y_1)]$.

(i) If $(q_1, y_1) \sim_{E'}^{i,t^i} (\overline{q}, \overline{y})$, then it follows from completeness and transitivity that $I_1' = I^*$. Hence we can set $f_1 = f^*$ and we are done since f^* satisfies all of the desired properties.

(ii) If $(q_1, y_1) \succ_{E'}^{i,t^i} (\overline{q}, \overline{y})$ we know from the definition of a basis that there exists $y_1^* \in Y_{E'}^{i,t^i}$ and $q_1^*, q_1^{**} \in \Delta(S_{E'}^{i,t^i})$ such that $(q_1^*, y_1^*) \sim_{E'}^{i,t^i} (q_1, y_1) \succ_{E'}^{i,t^i} (\overline{q}, \overline{y}) \succ_{E'}^{i,t^i} (q_1^{**}, y_1^*) \succeq_{E'}^{i,t^i} (\underline{q}, \underline{y})$. Let $I_2' := \{(q,y) \in D_{E'}^{i,t^i} | (q_1^*, y_1^*) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (q_1^{**}, y_1^*)\} = \{(q,y) \in D_{E'}^{i,t^i} | (q_1, y_1) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (q_1^{**}, y_1^*)\}$. By mixture monotonicity, we know that for every $(q,y) \in I_2'$, there exists uniquely $\alpha_{q,y}^* \in [0,1]$ such that $(\alpha_{q,y}^* q_1^* + (1 - \alpha_{q,y}^*) q_1^{**}, y_1^*) \sim (q,y)$. Hence we can define a function $g : I_2' \to \mathbf{R}$ by $g(q,y) = \alpha_{q,y}^*$. From Claims 1 and 2, we know that g is an affine representation of the preferences on I_2' with the further property that it is onto the interval $[g(q_1^{**}, y_1^*), g(q_1^*, y_1^*)] = [g(q_1^{**}, y_1^*), g(q_1, y_1)]$. Let $a = f^*(q_1^{**} y_1^*) - \frac{[f^*(\overline{q},\overline{y}) - f^*(q_1^{**}, y_1^*)]}{[g(\overline{q},\overline{y}) - g(q_1^{**}, y_1^*)]} * g(q_1^{**}, y_1^*)$, $b = \frac{[f^*(\overline{q},\overline{y}) - f^*(q_1^{**}, y_1^*)]}{[g(\overline{q},\overline{y}) - g(q_1^{**}, y_1^*)]}$, and define $h := a + bg$. From Claim 3, we know that h is an affine representation of the preferences on I_2' that is onto the interval $[h(q_1^{**}, y_1^*), h(q_1, y_1)]$. We note that $h(\overline{q}, \overline{y}) = f^*(q_1^{**}, y_1^*) - \frac{[f^*(\overline{q},\overline{y}) - f^*(q_1^{**}, y_1^*)]}{[g(\overline{q},\overline{y}) - g(q_1^{**}, y_1^*)]} * g(q_1^{**}, y_1^*) + \frac{[f^*(\overline{q},\overline{y}) - f^*(q_1^{**}, y_1^*)]}{[g(\overline{q},\overline{y}) - g(q_1^{**}, y_1^*)]} * g(\overline{q}, \overline{y}) = f^*(\overline{q}, \overline{y})$. Likewise, we have $h(q_1^{**}, y_1^*) = f^*(q_1^{**}, y_1^*) - \frac{[f^*(\overline{q},\overline{y}) - f^*(q_1^{**}, y_1^*)]}{[g(\overline{q},\overline{y}) - g(q_1^{**}, y_1^*)]} * g(q_1^{**}, y_1^*) + \frac{[f^*(\overline{q},\overline{y}) - f^*(q_1^{**}, y_1^*)]}{[g(\overline{q},\overline{y}) - g(q_1^{**}, y_1^*)]} * g(q_1^{**}, y_1^*) = f^*(q_1^{**}, y_1^*)$. From Claim 4, it then follows that f^* and h coincide on $I^* \cap I_2'$. Let $f_1 : I^* \cup I_2' \to \mathbf{R}$ be the function that coincides with f^* on I^* and with h on I_2'. From claims 5 and 6, it follows that f_1 is an affine representation of the preferences on $I^* \cup I_2' = I_1'$. To see that f is also onto the required interval, note that $[f_1(\underline{q}, \underline{y}), f_1(q,y)] = [f^*(\underline{q}, \underline{y}), f^*(\overline{q}, \overline{y})] \cup [h(q_1^{**}, y_1^*), h(q_1, y_1)]$. We know that f^* is onto the interval $[f^*(\underline{q}, \underline{y}), f^*(\overline{q}, \overline{y})]$ and that h is onto the interval $[h(q_1^{**}, y_1^*), h(q_1, y_1)]$. But since f_1 coincides with f^* on I^* and with h on I_2' it must be that f_1 is onto both intervals and hence onto the interval $[f_1(\underline{q}, \underline{y}), f_1(q,y)]$.

Step 2: Show that there exists an affine representation f of the preferences on I such that a) f coincides with f^* on I^*; and b) f is onto the interval $[f(q_1', y_1'), f(q_1, y_1)]$.

(i) If $(q_1', y_1') \sim_{E'}^{i,t^i} (\underline{q}, \underline{y})$, then it follows from completeness and transitivity that $I = I_1'$. Hence we can set $f = f_1$ and we are done since f_1 satisfies all of the desired properties.

(ii) If $(\underline{q}, \underline{y}) \succ_{E'}^{i,t^i} (q_1', y_1')$, we know from the definition of a basis that there exists $y_2^* \in Y_{E'}^{i,t^i}$, and $q_2^*, q_2^{**} \in \Delta(S_{E'}^{i,t^i})$ such that $(\overline{q}, \overline{y}) \succeq_{E'}^{i,t^i} (q_2^*, y_2^*) \succ_{E'}^{i,t^i} (\underline{q}, \underline{y}) \succ_{E'}^{i,t^i} (q_1', y_1') \sim_{E'}^{i,t^i} (q_2^{**}, y_2^*)$. Let $I_3' := \{(q,y) \in D_{E'}^{i,t^i} | (q_2^*, y_2^*) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (q_2^{**}, y_2^*)\} = \{(q,y) \in D_{E'}^{i,t^i} | (q_2^*, y_2^*) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (q_1', y_1')\}$. By mixture monotonicity, we know that for every $(q,y) \in I_3'$, there exists uniquely $\alpha_{q,y}^* \in [0,1]$ such that $(\alpha_{q,y}^* q_2^* + (1 - \alpha_{q,y}^*) q_2^{**}, y_2^*) \sim (q,y)$. Hence we can define a function $g_2 : I_3' \to \mathbf{R}$ by $g_2(q,y) = \alpha_{q,y}^*$. From Claims 1 and 2, we know that g_2 is an affine representation of the preferences on I_3' with the further property that it is onto the interval $[g_2(q_2^{**}, y_2^*), g_2(q_2^*, y_2^*)] = [g_2(q_1', y_2^*), g_2(q_2^*, y_2^*)]$. Let $a = f^*(\underline{q}, \underline{y}) - \frac{[f^*(q_2^*, y_2^*) - f^*(\underline{q}, \underline{y})]}{[g_2(q_2^*, y_2^*) - g_2(\underline{q}, \underline{y})]} * g_2(\underline{q}, \underline{y})$, $b = \frac{[f^*(q_2^*, y_2^*) - f^*(\underline{q}, \underline{y})]}{[g_2(q_2^*, y_2^*) - g_2(\underline{q}, \underline{y})]}$, and define $h_2 := a + bg_2$. From Claim 3, we know that h_2 is an affine representation of the preferences on I_3' that is onto the interval $[h(q_1', y_1'), q_1^*, y_1^*)]$. We note that $h_2(q_2^*, y_2^*) = f^*(\underline{q}, \underline{y}) - \frac{[f^*(q_2^*, y_2^*) - f^*(\underline{q}, \underline{y})]}{[g_2(q_2^*, y_2^*) - g_2(\underline{q}, \underline{y})]} * g_2(\underline{q}, \underline{y}) + \frac{[f^*(q_2^*, y_2^*) - f^*(\underline{q}, \underline{y})]}{[g_2(q_2^*, y_2^*) - g_2(\underline{q}, \underline{y})]} * g_2(q_2^*, y_2^*) == f^*(q_2^*, y_2^*)$.

Likewise, we have $h_2(\underline{q},\underline{y}) = f^*(\underline{q},\underline{y}) - \frac{[f^*(q_2^*,y_2^*)-f^*(\underline{q},\underline{y})]}{[g_2(q_2^*,y_2^*)-g_2(\underline{q},\underline{y})]} * g_2(\underline{q},\underline{y}) + \frac{[f^*(q_2^*,y_2^*)-f^*(\underline{q},\underline{y})]}{[g_2(q_2^*,y_2^*)-g_2(\underline{q},\underline{y})]} * g_2(\underline{q},\underline{y}) == f^*(\underline{q},\underline{y})$. From Claim 4, it then follows that f_1 and h_2 coincide on $I'_2 \cap I'_3$. Let $f : I'_2 \cup I'_3 \to \mathbf{R}$ be the function that coincides with f_1 on I'_2 and with h_2 on I'_3. From claims 5 and 6, it follows that f is an affine representation of the preferences on $I'_2 \cup I'_3 = I$. Note also that it follows from the facts that f coincides with f_1 on I'_2 and f_1 coincides with f^* on I^* that f coincides with f^* on I^*. To see that f is also onto the required interval, note that $[f(q',y'), f(q,y)] = [f_1(\underline{q},\underline{y}), f_1(q,y)] \cup [h_2(q'_1,y'_1), h_2(q_2^*,y_2^*)]$. We know that f_1 is onto the interval $[f_1(\underline{q},\underline{y}), f_1(q,y)]$ and that h_2 is onto the interval $[h_2(q'_1,y'_1), h_2(q_2^*,y_2^*)]$. But since f coincides with f_1 on I'_2 and with h_2 on I'_3 it must be that f is onto both intervals and hence onto the interval $[f(q',y'), f_1(q,y)]$. Q.E.D.

Let $I^* := \{(q,y) \in D_{E'}^{i,t^i} | (\overline{q},\overline{y}) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (\underline{q},\underline{y})\}$ be a basis interval. Let (q_1,y_1), (q'_1,y'_1), (q_2,y_2), $(q'_2,y'_2) \in D_{E'}^{i,t^i}$ satisfy the properties that
(i) $(q_1,y_1) \succeq_{E'}^{i,t^i} (\overline{q},\overline{y}) \succ_{E'}^{i,t^i} (\underline{q},\underline{y}) \succeq_{E'}^{i,t^i} (q'_1,y'_1)$; and
(ii) $(q_2,y_2) \succeq_{E'}^{i,t^i} (\overline{q},\overline{y}) \succ_{E'}^{i,t^i} (\underline{q},\underline{y}) \succeq_{E'}^{i,t^i} (q'_2,y'_2)$.

Define $I_1 := \{(q,y) \in D_{E'}^{i,t^i} | (q_1,y_1) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (q'_1,y'_1)\}$ and $I_2 := \{(q,y) \in D_{E'}^{i,t^i} | (q_2,y_2) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (q'_2,y'_2)\}$. Finally, let f_1 and f_2 be affine representations of I_1 and I_2 respectively such that a) f_1 and f_2 coincide with f^* on I^*; b) f_1 is onto the interval $[f_1(q'_1,y'_1), f_1(q_1,y_1)]$; and c) f_2 is onto the interval $[f_2(q'_2,y'_2), f_2(q_2,y_2)]$.

Claim 8 (Coincidence on $I_1 \cap I_2$)**.** *The functions f_1 and f_2 coincide on $I_1 \cap I_2$*

Proof of Claim 8: Pick any $(q',y') \in I_1 \cap I_2$. (i) If $(q',y') \in I^*$ then we are done since f_1 and f_2 both coincide with f^* on I^*.

(ii) $(q',y') \succ_{E'}^{i,t^i} (\overline{q},\overline{y})$. Then we must have $(q_1,y_1) \succeq_{E'}^{i,t^i} (q',y') \succ_{E'}^{i,t^i} (\overline{q},\overline{y})$ and $(q_2,y_2) \succeq_{E'}^{i,t^i} (q',y') \succ_{E'}^{i,t^i} (\overline{q},\overline{y})$. From the definition of a basis, it then follows that there exists $y_1^*, y_2^* \in Y_{E'}^{i,t^i}$ and $q_1^*, q_1^{**}, q_2^*, q_2^{**} \in \Delta(S_{E'}^{i,t^i})$ such that $(q_1^*,y_1^*) \sim_{E'}^{i,t^i} (q_1,y_1) \succ_{E'}^{i,t^i} (\overline{q},\overline{y}) \succ_{E'}^{i,t^i} (q_1^{**},q_1^*) \succeq_{E'}^{i,t^i} (\underline{q},\underline{y})$ and $(q_2^*,y_2^*) \sim_{E'}^{i,t^i} (q_2,y_2) \succ_{E'}^{i,t^i} (\overline{q},\overline{y}) \succ_{E'}^{i,t^i} (q_2^{**},q_2^*) \succeq_{E'}^{i,t^i} (\underline{q},\underline{y})$. Define $I'_1 := \{(q,y) \in D_{E'}^{i,t^i} | (q_1^*,y_1^*) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (q_1^{**},y_1^*)\}$ and $I'_2 := \{(q,y) \in D_{E'}^{i,t^i} | (q_2^*,y_2^*) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (q_2^{**},y_2^*)\}$. We note that it follows from above that $(q',y') \in I'_1 \cap I'_2$. Define a function $f'_1 : I'_1 \to \mathbf{R}$ by $f'_1(q,y) = f_1(q,y)$ and a function $f'_2 : I'_2 \to \mathbf{R}$ by $f'_2(q,y) = f_2(q,y)$. Suppose without loss of generality that $(q_1^{**},y_1^*) \succeq_{E'}^{i,t^i} (q_2^{**},y_2^*)$. We note the following four properties:

a) $(q_1^*,y_1^*) \succ_{E'}^{i,t^i} (\overline{q},\overline{y}) \succ_{E'}^{i,t^i} (q_1^{**},q_1^*) \succeq_{E'}^{i,t^i} (q_1^{**},q_1^*)$;
b) $(q_2^*,y_2^*) \succ_{E'}^{i,t^i} (\overline{q},\overline{y}) \succ_{E'}^{i,t^i} (q_1^{**},q_1^*) \succeq_{E'}^{i,t^i} (q_2^{**},q_2^*)$;
c) $f'_1(\overline{q},\overline{y}) = f_1(\overline{q},\overline{y}) = f(\overline{q},\overline{y}) = f_2(\overline{q},\overline{y}) = f'_2(\overline{q},\overline{y})$; and
d) $f'_1(q_1^{**},y_1^*) = f_1(q_1^{**},y_1^*) = f(q_1^{**},y_1^*) = f_2(q_1^{**},y_1^*) = f'_2(q_1^{**},y_1^*)$.

It follows from Claim 4 that f'_1 and f'_2 coincide on $I'_1 \cap I'_2$. But then we are done since this implies $f_1(q',y') = f'_1(q',y') = f'_2(q',y') = f_2(q',y')$.

(iii) $(\underline{q},\underline{y}) \succ_{E'}^{i,t^i} (q',y')$. Then we must have $(\underline{q},\underline{y}) \succ_{E'}^{i,t^i} (q',y') \succeq_{E'}^{i,t^i} (q'_1,y'_1)$ and $(\underline{q},\underline{y}) \succ_{E'}^{i,t^i} (q',y') \succeq_{E'}^{i,t^i} (q'_2,y'_2)$. From the definition of a basis, it then follows that there exists $y_1^*, y_2^* \in$

$Y_{E'}^{i,t^i}$ and $q_1^*, q_1^{**}, q_2^*, q_2^{**} \in \Delta(S_{E'}^{i,t^i})$ such that $(\overline{q}, \overline{y}) \succeq_{E'}^{i,t^i} (q_1^*, q_1^*) \succ_{E'}^{i,t^i} (\underline{q}, \underline{y}) \succ_{E'}^{i,t^i} (q_1', y_1') \sim_{E'}^{i,t^i} (q_1^{**}, y_1^*)$ and $(\overline{q}, \overline{y}) \succeq_{E'}^{i,t^i} (q_2^*, q_2^*) \succ_{E'}^{i,t^i} (\underline{q}, \underline{y}) \succ_{E'}^{i,t^i} (q_2', y_2') \sim_{E'}^{i,t^i} (q_2^{**}, y_2^*)$. Define $I_1' := \{(q, y) \in D_{E'}^{i,t^i} | (q_1^*, y_1^*) \succeq_{E'}^{i,t^i} (q, y) \succeq_{E'}^{i,t^i} (q_1^{**}, y_1^*)\}$ and $I_2' := \{(q, y) \in D_{E'}^{i,t^i} | (q_2^*, y_2^*) \succeq_{E'}^{i,t^i} (q, y) \succeq_{E'}^{i,t^i} (q_2^{**}, y_2^*)\}$. We note that it follows from above that $(q', y') \in I_1' \cap I_2'$. Define a function $f_1' : I_1' \to \mathbf{R}$ by $f_1'(q, y) = f_1(q, y)$ and a function $f_2' : I_2' \to \mathbf{R}$ by $f_2'(q, y) = f_2(q, y)$. Suppose without loss of generality that $(q_1^*, y_1^*) \succeq_{E'}^{i,t^i} (q_2^*, y_2^*)$. We note the following four properties:

a) $(q_1^*, y_1^*) \succeq_{E'}^{i,t^i} (q_2^*, y_2^*) \succ_{E'}^{i,t^i} (\underline{q}, \underline{y}) \succ_{E'}^{i,t^i} (q_1^{**}, q_1^*)$;
b) $(q_2^*, y_2^*) \succeq_{E'}^{i,t^i} (q_2^*, y_2^*) \succ_{E'}^{i,t^i} (\underline{q}, \underline{y}) \succ_{E'}^{i,t^i} (q_2^{**}, q_2^*)$;
c) $f_1'(\underline{q}, \underline{y}) = f_1(\underline{q}, \underline{y}) = f(\underline{q}, \underline{y}) = f_2(\underline{q}, \underline{y}) = f_2'(\underline{q}, \underline{y})$; and
d) $f_1'(q_2^*, y_2^*) = f_1(q_2^*, y_2^*) = f(q_2^*, y_2^*) = f_2(q_2^*, y_2^*) = f_2'(q_2^*, y_2^*)$.

It follows from Claim 4 that f_1' and f_2' coincide on $I_1' \cap I_2'$. But then we are done since this implies $f_1(q', y') = f_1'(q', y') = f_2'(q', y') = f_2(q', y')$. Q.E.D.

We are now ready to state a key Lemma that will greatly assist us in the rest of our proof.

Lemma 3. *If a basis exists, then there exists an affine representation f of the preferences on $D_{E'}^{i,t^i}$. Furthermore, for any two elements $(q, y), (q', y') \in D_{E'}^{i,t^i}$ such that $(q, y) \succeq_{E'}^{i,t^i} (q', y')$, the representation f is onto the interval $[f(q', y'), f(q, y)]$.*

Proof of Lemma 3: Suppose there exists a basis $I^* := \{(q, y) \in D_{E'}^{i,t^i} | (\overline{q}, \overline{y}) \succeq_{E'}^{i,t^i} (q, y) \succeq_{E'}^{i,t^i} (\underline{q}, \underline{y})\}$. Define $I_1 := I^*$, and consider an increasing sequence of intervals $I_1 \subset I_2 \subset I_3 \ldots \subset D_{E'}^{i,t^i}$. Let $I_k = \{(q, y) \in D_{E'}^{i,t^i} | (q_k, y_k) \succeq_{E'}^{i,t^i} (q, y) \succeq_{E'}^{i,t^i} (q_k', y_k')\}$. From Claim 7 it follows that at each step, we can define an affine representation f_k of the preferences on I_k such that a) f_k is onto the interval $[f(q_k', y_k'), f(q_k, y_k)]$, and for every $k' < k$, $[(q, y) \in I_{k'}] \Rightarrow [f_k(q, y) = f_{k-1}(q, y) = f_{k-2}(q, y) = \ldots = f_{k'}(q, y)]$. Thus, let us define this common value $f_k(q, y) = f_{k-1}(q, y) := f(q, y)$. We can thereby construct a function f that represents preferences on the entire set $D_{E'}^{i,t^i}$. To see that f must be onto any interval $[f(q', y'), f(q, y)]$, consider any sufficiently large k so that $(q_k, y_k) \succeq_{E'}^{i,t^i} (q, y) \succeq_{E'}^{i,t^i} (q', y') \succeq_{E'}^{i,t^i} (q_k', y_k')$. Then the onto property follows from the fact that f_k is onto the interval $[f(q_k', y_k'), f(q_k, y_k)]$. Q.E.D.

Claim 9. *Let $(q, y) \in D_{E'}^{i,t^i}$. Then there exists $\overline{E}, \underline{E} \in S_{E'}^{i,t^i}$ such that $(i_{\overline{E}}, y) \succeq_{E'}^{i,t^i} (q, y) \succeq_{E'}^{i,t^i} (\underline{E}, y)$.*

Proof of Claim 9: We note that $C(q)$ has only a finite number of elements. An implication of completeness and transitivity is that there exists $\overline{E}, \underline{E} \in C(q)$ such that for all $E \in C(q)$, $(i_{\overline{E}}, y) \succeq_{E'}^{i,t^i} (i_E, y) \succeq_{E'}^{i,t^i} (i_{\underline{E}}, y)$.

(i) if $(i_{\overline{E}}, y) \succ_{E'}^{i,t^i} (i_{\underline{E}}, y)$, then it follows from unique solvability that for every $E \in C(q)$, there exists uniquely $\alpha_E^* \in [0, 1]$ such that $(\alpha_E^* i_{\overline{E}} + (1 - \alpha_E^*) i_{\underline{E}}, y) \sim (i_E, y)$. Now, order the elements of $C(q)$ as $E_1, E_2, \ldots, E_K$ where $K := \#C(q)$. Define for each $k = 1, 2, \ldots, K$ an element $q_{E_k}^*$ by:

$$q_{E_k}^* = \frac{\sum_{k' \leq k} q(E_{k'}) i_{E_{k'}}}{\sum_{k'' \leq k} q(E_{k''})}$$

We claim that for every k, $(i_{\overline{E}}, y) \succeq^{i,t^i}_{E'} (q^*_{E_k}, y) \succeq^{i,t^i}_{E'} (i_{\underline{E}}, y)$. To see this, first note that $(q^*_{E_1}, y) = (i_{E_1}, y)$ which in turn implies that $(i_{\overline{E}}, y) \succeq^{i,t^i}_{E'} (q^*_{E_1}, y) \succeq^{i,t^i}_{E'} (i_{\underline{E}}, y)$. Hence the desired property holds for $k = 1$. Consider now any k such that the desired property holds for $k' = 1, , 2, \ldots, k-1$. For each such k' denote by $\beta_{k'}$ the unique element of $[0, 1]$ (implied by unique solvability) for which $(\beta_{k'} i_{\overline{E}} + (1 - \beta_{k'}) i_{\underline{E}}, y) \sim^{i,t^i} (q^*_{E_{k'}}, y)$. Likewise, for each $k' = 1, 2, \ldots, K$ denote by $\alpha^*_{k'}$ the unique element of $[0, 1]$ (implied by unique solvability) for which $(\alpha_{k'} i_{\overline{E}} + (1 - \alpha_{k'}) i_{\underline{E}}, y) \sim^{i,t^i} (i_{E_{k'}}, y)$. Then we have,

$$\begin{aligned}
[q^*_{E_k}, y] &= [(\frac{\sum_{k' \leq k} q(E_{k'}) i_{E_{k'}}}{\sum_{k'' \leq k} q(E_{k''})}, y] \\
&= [(\frac{q(E_k) i_{E_k}}{\sum_{k'' \leq k} q(E_{k''})}) + \frac{(\sum_{k'' < k} q(E_{k''}))}{(\sum_{k'' \leq k} q(E_{k''}))} * \frac{(\sum_{k' < k} q(E_{k'}) i_{E_{k'}})}{(\sum_{k'' < k} q(E_{k''}))}, y] \\
&= [(\frac{q(E_k) i_{E_k}}{\sum_{k'' \leq k} q(E_{k''})}) + \frac{(\sum_{k'' < k} q(E_{k''}))}{(\sum_{k'' \leq k} q(E_{k''}))} * q^*_{E_{k-1}}, y] \\
&\sim^{i,t^i} [(\frac{q(E_k)}{\sum_{k'' \leq k} q(E_{k''})}) * (\alpha_{E_k} i_{\overline{E}} + (1 - \alpha_{E_k}) i_{\underline{E}}) + \frac{(\sum_{k'' < k} q(E_{k''}))}{(\sum_{k'' \leq k} q(E_{k''}))} * q^*_{E_{k-1}}, y] \\
&\sim^{i,t^i} [(\frac{q(E_k)}{\sum_{k'' \leq k} q(E_{k''})}) * (\alpha_{E_k} i_{\overline{E}} + (1 - \alpha_{E_k}) i_{\underline{E}}) \\
&\quad + \frac{(\sum_{k'' < k} q(E_{k''}))}{(\sum_{k'' \leq k} q(E_{k''}))} * (\beta_{E_{k-1}} i_{\overline{E}} + (1 - \beta_{E_{k-1}}) i_{\underline{E}}), y] \\
&= [[(\frac{q(E_k)}{\sum_{k'' \leq k} q(E_{k''})}) * \alpha_{E_k} + \frac{(\sum_{k'' < k} q(E_{k''}))}{(\sum_{k'' \leq k} q(E_{k''}))} * \beta_{E_{k-1}}] i_{\overline{E}} \\
&\quad + [1 - [(\frac{q(E_k)}{\sum_{k'' \leq k} q(E_{k''})}) * \alpha_{E_k} + \frac{(\sum_{k'' < k} q(E_{k''}))}{(\sum_{k'' \leq k} q(E_{k''}))} * \beta_{E_{k-1}}]] i_{\underline{E}}, y]
\end{aligned}$$

Letting $\beta_k := [(\frac{q(E_k)}{\sum_{k'' \leq k} q(E_{k''})}) * \alpha_{E_k} + \frac{(\sum_{k'' < k} q(E_{k''}))}{(\sum_{k'' \leq k} q(E_{k''}))} * \beta_{E_{k-1}}]$, we note that the desired property holds for any k. Now, simply note that $q_K = q$. Hence, we can let $\beta = \beta_K$. But then it follows that $(q, y) \sim^{i,t^i} (\beta i_{\overline{E}} + (1 - \beta) i_{\underline{E}}, y)$. This in turn implies that $(i_{\overline{E}}, y) \succeq^{i,t^i}_{E'} (q, y) \succeq^{i,t^i}_{E'} (i_{\underline{E}}, y)$ by mixture monotonicity.

(ii) if $(i_{\overline{E}}, y) \sim^{i,t^i}_{E'} (i_{\underline{E}}, y)$, then it follows from completeness and transitivity that for every $E \in C(q)$, $(i_{\overline{E}}, y) \sim^{i,t^i}_{E'} (i_E, y) \sim^{i,t^i}_{E'} (i_{\underline{E}}, y)$.

Now, order the elements of $C(q)$ as $E_1, E_2, \ldots, E_K$ where $K := \#C(q)$. Define for each $k = 1, 2, \ldots, K$ an element $q^*_{E_k}$ by:

$$q^*_{E_k} = \frac{\sum_{k' \leq k} q(E_{k'}) i_{E_{k'}}}{\sum_{k'' \leq k} q(E_{k''})}$$

We claim that for every k, $(i_{\overline{E}}, y) \sim^{i,t^i}_{E'} (q^*_{E_k}, y) \sim^{i,t^i}_{E'} (i_{\underline{E}}, y)$. To see this, first note that $(q^*_{E_1}, y) = (i_{E_1}, y)$ which in turn implies that $(i_{\overline{E}}, y) \sim^{i,t^i}_{E'} (q^*_{E_1}, y) \sim^{i,t^i}_{E'} (i_{\underline{E}}, y)$. Hence the desired property holds for $k = 1$. Consider now any k such that the desired property holds for $k' = 1, , 2, \ldots, k-1$.

Then we have,

$$\begin{array}{rcl}
[q^*_{E_k}, y] & = & [(\frac{\sum_{k'\leq k} q(E_{k'}) i_{E_{k'}}}{\sum_{k''\leq k} q(E_{k''})}, y] \\
& = & [(\frac{q(E_k) i_{E_k}}{\sum_{k''\leq k} q(E_{k''})}) + \frac{(\sum_{k''<k} q(E_{k''}))}{(\sum_{k''\leq k} q(E_{k''}))} * \frac{(\sum_{k'<k} q(E_{k'}) i_{E_{k'}})}{(\sum_{k''<k} q(E_{k''}))}, y] \\
& = & [(\frac{q(E_k) i_{E_k}}{\sum_{k''\leq k} q(E_{k''})}) + \frac{(\sum_{k''<k} q(E_{k''}))}{(\sum_{k''\leq k} q(E_{k''}))} * q^*_{E_{k-1}}, y] \\
& \sim^{i,t^i} & [(\frac{q(E_k)}{\sum_{k''\leq k} q(E_{k''})}) * i_{\underline{E}} + \frac{(\sum_{k''<k} q(E_{k''}))}{(\sum_{k''\leq k} q(E_{k''}))} * q^*_{E_{k-1}}, y] \\
& \sim^{i,t^i} & [(\frac{q(E_k)}{\sum_{k''\leq k} q(E_{k''})}) * i_{\underline{E}} + \frac{(\sum_{k''<k} q(E_{k''}))}{(\sum_{k''\leq k} q(E_{k''}))} * i_{\underline{E}}, y] \\
& = & [i_{\underline{E}}, y]
\end{array}$$

But then it follows from completeness and transitivity that $(i_{\overline{E}}, y) \sim^{i,t^i}_{E'} (q^*_{E_k}, y) \sim^{i,t^i}_{E'} (i_{\underline{E}}, y)$. Hence the desired property holds for every k. Now, simply note that $q_K = q$. Hence $(i_{\overline{E}}, y) \sim^{i,t^i}_{E'} (q, y) \sim^{i,t^i}_{E'} (i_{\underline{E}}, y)$. Q.E.D.

Claim 10. *Let E_1 and E_2 be distinct elements of S^{i,t^i} and let y_1 and y_2 be elements of $Y^{i,t^i}_{E'}$. Then there exists $y^* \in Y^{i,t^i}_{E'}$ such that $(i_{E_1}, y^*) \sim^{i,t^i}_{E'} (i_{E_1}, y_1)$ and $(i_{E_2}, y^*) \sim^{i,t^i}_{E'} (i_{E_2}, y_2)$.*

Proof of Claim 10: Let $y^*_E = y_{2,E_2}$ if $E = E_2$ and let $y^*_E = y_{1,E_1}$ otherwise. Then it follows from $E1$ that $(i_{E_1}, y^*) \sim^{i,t^i}_{E'} (i_{E_1}, y_1)$ and $(i_{E_2}, y^*) \sim^{i,t^i}_{E'} (i_{E_2}, y_2)$. Q.E.D.

Claim 11. *If there exists $E_1, E_2, E_3, E_4 \in S^{i,t^i}_{E'} [E_1 \neq E_3, E_1 \neq E_4, E_2 \neq E_3, E_2 \neq E_4]$ and $y_1, y_2, y_3, y_4 \in Y^{i,t^i}_{E'}$ such that $(i_{E_1}, y_1) \succ^{i,t^i}_{E'} (i_{E_2}, y_2)$ and $(i_{E_3}, y_3) \succ^{i,t^i}_{E'} (i_{E_4}, y_4)$, then there exists $(\overline{q}, \overline{y}), (\underline{q}, \underline{y}) \in D^{i,t^i}_{E'}$ such that:*

(i) $(\overline{q}, \overline{y}) \succeq^{i,t^i}_{E'} (i_{E_1}, y_1) \succ^{i,t^i}_{E'} (i_{E_2}, y_2) \succeq^{i,t^i}_{E'} (\underline{q}, \underline{y})$;

(ii) $(\overline{q}, \overline{y}) \succeq^{i,t^i}_{E'} (i_{E_3}, y_3) \succ^{i,t^i}_{E'} (i_{E_4}, y_4) \succeq^{i,t^i}_{E'} (\underline{q}, \underline{y})$;

(iii) the preferences on the interval $I^* := \{(q, y) \in D^{i,t^i}_{E'} | (\overline{q}, \overline{y}) \succeq^{i,t^i}_{E'} (q, y) \succeq^{i,t^i}_{E'} (\underline{q}, \underline{y})\}$ are representable by an affine function f that is onto the interval $[f(\underline{q}, \underline{y}), f(\overline{q}, \overline{y})]$; and

(iv) the interval I^* is a basis.

Proof of Claim 11: Step 1: Establish (i)-(iii).
Assume without loss of generality that $(i_{E_1}, y_1) \succeq^{i,t^i}_{E'} (i_{E_3}, y_3)$.
1a) If $(i_{E_2}, y_2) \succeq^{i,t^i}_{E'} (i_{E_4}, y_4)$ set $(\overline{q}, \overline{y}) = (i_{E_1}, y_1)$ and $(\underline{q}, \underline{y}) = (i_{E_4}, y_4)$. Then we have,

$$(\overline{q}, \overline{y}) = (i_{E_1}, y_1) \succ^{i,t^i} (i_{E_2}, y_2) \succeq^{i,t^i}_{E'} (i_{E_4}, y_4) = (\underline{q}, \underline{y}); \text{ and}$$

$$(\overline{q}, \overline{y}) = (i_{E_1}, y_1) \succeq^{i,t^i}_{E'} (i_{E_3}, y_3) \succ^{i,t^i}_{E'} (i_{E_4}, y_4) = (\underline{q}, \underline{y}).$$

Hence (i) and (ii) are both satisfied.
Now, since $E_1 \neq E_4$, it follows from claim 10 that there exists $y^* \in Y^{i,t^i}$ such that

$$(i_{E_1}, y^*) \sim_{E'}^{i,t^i} (i_{E_1}, y_1) \succ_{E'}^{i,t^i} (i_{E_2}, y_2) \succeq_{E'}^{i,t^i} (i_{E_4}, y_4) \sim_{E'}^{i,t^i} (i_{E_4}, y^*)$$

Due to the unique solvability property, we can then define a function $f : I^* \Rightarrow \mathbf{R}$ by $f(q,y) = \alpha^*_{q,y}$ where $\alpha^*_{q,y}$ is the unique element of $[0,1]$ for which $(\alpha^*_{q,y} i_{E_1} + (1-\alpha^*_{q,y}) i_{E_4}, y^*) \sim_{E'}^{i,t^i} (q,y)$. From claims 1 and 2 it follows that f is an affine representation of the preferences on I^* that is furthermore onto the interval $[f(i_{E_4}, y^*), f(i_{E_1}, y^*)] = [f(\underline{q}, \underline{y}, f(\overline{q}, \overline{y}]$. Hence (iii) holds.

1b) If $(i_{E_4}, y_4) \succ_{E'}^{i,t^i} (i_{E_2}, y_2)$, set $(\overline{q}, \overline{y}) = (i_{E_1}, y_1)$ and $(\underline{q}, \underline{y}) = (i_{E_2}, y_2)$. Then we have,

$$(\overline{q}, \overline{y}) = (i_{E_1}, y_1) \succeq_{E'}^{i,t^i} (i_{E_3}, y_3) \succ^{i,t^i} (i_{E_4}, y_4) \succeq_{E'}^{i,t^i} (i_{E_2}, y_2) = (\underline{q}, \underline{y})$$

so (i) and (ii) both hold. Now, since $E_1 \neq E_4 and E_2 \neq E_3$, it follows from claim 10 that there exists y_1^* and y_2^* in $Y_{E'}^{i,t^i}$ such that

$$(i_{E_1}, y_1^*) \sim_{E'}^{i,t^i} (i_{E_1}, y_1) \succ_{E'}^{i,t^i} (i_{E_4}, y_4) \sim_{E'}^{i,t^i} (i_{E_4}, y_1^*)$$

and

$$(i_{E_3}, y_2^*) \sim_{E'}^{i,t^i} (i_{E_3}, y_3) \succ_{E'}^{i,t^i} (i_{E_2}, y_2) \sim_{E'}^{i,t^i} (i_{E_2}, y_2^*)$$

.

From unique solvability it follows that for every (q,y) such that $(i_{E_1}, y_1^*) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (i_{E_4}, y_1^*)$ there exist uniquely $\alpha^*_{q,y} \in [0,1]$ such that $(\alpha^*_{q,y} i_{E_1} + (1-\alpha^*_{q,y}), y_1^*) \sim_{E'}^{i,t^i} (q,y)$. Likewise, for every (q,y) such that $(i_{E_1}, y_1^*) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (i_{E_4}, y_1^*)$ there exist uniquely $\alpha^*_{q,y} \in [0,1]$ such that $(\beta^*_{q,y} i_{E_1} + (1-\beta^*_{q,y}), y_1^*) \sim_{E'}^{i,t^i} (q,y)$.

Hence, we can define a function $f_1 : \{(q,y) \in D_{E'}^{i,t^i} | (i_{E_1}, y_1^*) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (i_{E_4}, y_1^*)\} \to \mathbf{R}$ by $f_1(q,y) = \alpha^*_{q,y}$ and a function $f_2 : \{(q,y) \in D_{E'}^{i,t^i} | (i_{E_3}, y_2^*) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (i_{E_2}, y_2^*)\} \to \mathbf{R}$ by $f_1(q,y) = \beta^*_{q,y}$. From claims 1 and 2 it follows that f_1 is an affine representation of the preferences on $I_1 := \{(q,y) \in D_{E'}^{i,t^i} | (i_{E_1}, y_1^*) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (i_{E_4}, y_1^*)\}$ such that f_1 is onto the interval $[f_1(i_{E_4}, y_1^*), f_1(i_{E_1}, y_1^*)]$. Likewise, it follows that f_2 is an affine representation of the preferences on $I_2 := \{(q,y) \in D_{E'}^{i,t^i} | (i_{E_3}, y_2^*) \succeq_{E'}^{i,t^i} (q,y) \succeq_{E'}^{i,t^i} (i_{E_2}, y_2^*)\}$ such that f_2 is onto the interval $[f_2(i_{E_2}, y_2^*), f_2(i_{E_3}, y_2^*)]$.

Let $(q^*, y^*) = (i_{E_3}, y_3)$ and $(q^{**}, y^{**}) = (i_{E_4}, y_4)$. Then we have $(i_{E_1}, y_1^*) \succeq_{E'}^{i,t^i} (i_{E_3}, y_3) \sim_{E'}^{i,t^i} (q^*, y^*) \succ_{E'}^{i,t^i} (q^{**}, y^{**}) \sim_{E'}^{i,t^i} (i_{E_4}, y_1^*)$ and $(i_{E_3}, y_2^*) \sim_{E'}^{i,t^i} (q^*, y^*) \succ_{E'}^{i,t^i} (q^{**}, y^{**}) \sim_{E'}^{i,t^i} (i_{E_4}, y_1^*) \succ_{E'}^{i,t^i} (i_{E_2}, y_2^*)$. Given claim 3, we can find a and $b[b > 0]$ such that $g = a + bf_2$ is an affine representation of the preferences on I_2 which is onto the interval $[g(i_{E_2}, y_2^*), g(i_{E_3}, y_2^*)]$ and that furthermore has the properties that $g(q^*, y^*) = f_1(q^*, y^*)$ and $g(q^{**}, y^{**}) = f_1(q^{**}, y^{**})$.

From Claim 4, we know that g and f_1 coincide on $I_1 \cap I_2$. Now, let f^* be the function that coincides with f_1 in I_1 and with g on I_2. Then we4 know from claims 5 and 6 that f^* is an affine representation of the preferences in $I_1 \cup I_2 = I^*$ as required. Hence (iii) holds.

Step 2: Show that (i)-(iii) implies that I^* is a basis.

2A) Show that property (ii) from the definition of a basis holds.

Pick any $(q_1', y_1') \in D_{E'}^{i,t}$ such that $(q_1', y_1') \succ_{E'}^{i,t^i} (\overline{q}, \overline{y})$.

From Claim 9, we know that there exist $\overline{E} \in S_{E'}^{i,t^i}$ such that $(i_{\overline{E}}, y_1') \succeq_{E'}^{i,t^i} (q_1', y_1')$. Since $E_2 \neq E_4$, we know that either $\overline{E} \neq E_2$ or $\overline{E} \neq E_4$. Without loss of generality, assume that $\overline{E} \neq E_2$. Then we know from Claim 10 that there exists $y^* \in Y_{E'}^{i,t^i}$ such that $(i_{\overline{E}}, y^*) \sim_{E'}^{i,t^i} (i_{\overline{E}}, y_1')$ and $(i_{E_2}, y^*) \sim_{E'}^{i,t^i} (i_{E_2}, y_2)$. Then we have,

$$(i_{\overline{E}}, y^*) \sim_{E'}^{i,t^i} (i_{\overline{E}}, y_1') \succeq_{E'}^{i,t^i} (q_1', y_1') \succ_{E'}^{i,t^i} (\overline{q}, \overline{y}) \succeq_{E'}^{i,t^i} (i_{E_1}, y_1) \succ_{E'}^{i,t^i} (i_{E_2}, y_2) \sim (i_{E_2}, y^*) \succeq_{E'}^{i,t^i} (\underline{q}, \underline{y})$$

From unique solvability, it follows that there exist uniquely $\alpha^* \in [0,1]$ such that $(\alpha^* i_{\overline{E}} + (1-\alpha^*) i_{E_2}, y^*) \sim (q_1', y_1')$ Now, let $(q^*, y^*) = (\alpha^* i_{\overline{E}} + (1-\alpha^*) i_{E_2}, y^*)$ and $(q^{**}, y^{**}) = (i_{E_2}, y^*)$. Then we have

$$(q^*, y^*) \sim_{E'}^{i,t^i} (q_1', y_1') \succ_{E'}^{i,t^i} (\overline{q}, \overline{y}) \succeq_{E'}^{i,t^i} (i_{E_1}, y_1) \succ_{E'}^{i,t^i} (i_{E_2}, y_2) \sim (q^{**}, y^*) \succeq_{E'}^{i,t^i} (\underline{q}, \underline{y})$$

as required.

2B) Show that property (iii) from the definition of a basis holds.

Pick any $(q_2', y_2') \in D_{E'}^{i,t}$ such that $(\underline{q}, \underline{y}) \succ_{E'}^{i,t^i} (q_2', y_2')$.

From Claim 9, we know that there exist $\underline{E} \in S_{E'}^{i,t^i}$ such that $(q_2', y_2') \succeq_{E'}^{i,t^i} (i_{\underline{E}}, y_2')$. Since $E_1 \neq E_3$, we know that either $\underline{E} \neq E_1$ or $\underline{E} \neq E_3$. Without loss of generality, assume that $\underline{E} \neq E_1$. Then we know from Claim 10 that there exists $y^* \in Y_{E'}^{i,t^i}$ such that $(i_{E_1}, y^*) \sim_{E'}^{i,t^i} (i_{E_1}, y_1)$ and $(i_{\underline{E}}, y^*) \sim_{E'}^{i,t^i} (i_{\underline{E}}, y_2')$. Then we have,

$$(\overline{q}, \overline{y}) \succeq_{E'}^{i,t^i} (i_{E_1}, y^*) \sim_{E'}^{i,t^i} (i_{E_1}, y_1) \succ_{E'}^{i,t^i} (\underline{q}, \underline{y}) \succ_{E'}^{i,t^i} (q_2', y_2') \sim (i_{\underline{E}}, y_2') \sim (i_{\underline{E}}, y^*)$$

From unique solvability, it follows that there exist uniquely $\beta^* \in [0,1]$ such that $(\beta^* i_{E_1} + (1-\beta^*) i_{\underline{E}}, y^*) \sim (q_2', y_2')$. Now, let $(q^*, y^*) = (i_{E_1}, y^*)$ and $(q^{**}, y^{**}) = (\beta^* i_{E_1} + (1-\beta^*) i_{\underline{E}}, y^*)$. Then we have

$$(\overline{q}, \overline{y}) \succeq_{E'}^{i,t^i} (q^*, y^*) \succ_{E'}^{i,t^i} (\underline{q}, \underline{y}) \succ_{E'}^{i,t^i} (q_2', y_2') \sim_{E'}^{i,t^i} (q^{**}, y^*)$$

as required.

Q.E.D.

Claim 12. *If $\#[E] \geq 2$ and $\exists (q,y), (q',y')$ such that $(q,y) \succ_{E'}^{i,t^i} (q',y')$ then at least one of the following three holds:*

(i) There exist $E_1, E_2, E_3, E_4[E_1 \neq E_3, E_1 \neq E_4, E_2 \neq E_3, E_2 \neq E_4] \in S_{E'}^{i,t^i}$ and $y_1, y_2, y_3, y_4 \in Y_{E'}^{i,t^i}$ such that $(i_{E_1}, y_1) \succ_{E'}^{i,t^i} (i_{E_2}, y_2)$ and $(i_{E_3}, y_3) \succ_{E'}^{i,t^i} (i_{E_4}, y_4)$

(ii) There exists distinct events $E_1^*, E_2^* \in S_{E'}^{i,t^i}$ and $y^* \in Y_{E'}^{i,t^i}$ such that

$$[(q_1, y_1) \not\succ_{E'}^{i,t^i} (i_{E_2}^*, y^*)] \Rightarrow [q(E_1^*) > 0]$$

(iii) There does not exist $E \in S_{E'}^{i,t^i}$ and $y_1, y_2 \in Y_{E'}^{i,t^i}$ such that $(i_E, y_1) \succ_{E'}^{i,t^i} (i_E, y_2)$.

Proof of Claim: Step 1: Show that if (i) does not hold then $\#\{E \in S^{i,t^i}_{E'} | \exists y_1, y_2 \in Y^{i,t^i}_{E'} : (i_E, y_1) \succ^{i,t^i}_{E'} (i_E, y_2)\} \leq 1$.

Suppose not. Denote by E_1^*, E_2^* any two distinct elements of $\{E \in S^{i,t^i}_{E'} | \exists y_1, y_2 \in Y^{i,t^i}_{E'} : (i_E, y_1) \succ^{i,t^i}_{E'} (i_E, y_2)\}$ and by $y_1, y_2, y_3, y_4 \in Y^{i,t^i}_{E'}$ any elements of $Y^{i,t^i}_{E'}$ such that $(i_{E_1^*}, y_1) \succ^{i,t^i}_{E'} (i_{E_1^*}, y_2)$ and $(i_{E_2^*}, y_3) \succ^{i,t^i}_{E'} (i_{E_2^*}, y_4)$. Now, set $E_1 = E_2 = E_1^*$ and $E_3 = E_4 = E_2^*$. Then $E_1 \neq E_3, E_1 \neq E_4, E_2 \neq E_3$, and $E_2 \neq E_4$. But we also have,

$(i_{E_1}, y_1) \succ^{i,t^i}_{E'} (i_{E_2}, y_2)$ and $(i_{E_3}, y_3) \succ^{i,t^i}_{E'} (i_{E_4}, y_4)$ which contradicts our present assumption. Hence $\#\{E \in S^{i,t^i}_{E'} | \exists y_1, y_2 \in Y^{i,t^i}_{E'} : (i_E, y_1) \succ^{i,t^i}_{E'} (i_E, y_2)\} \leq 1$.

Step 2: Show that if (i) does not hold and $\#\{E \in S^{i,t^i}_{E'} | \exists y_1, y_2 \in Y^{i,t^i}_{E'} : (i_E, y_1) \succ^{i,t^i}_{E'} (i_E, y_2)\} = 1$ then (ii) must hold.

Denote by E_1^* the unique element of $\{E \in S^{i,t^i}_{E'} | \exists y_1, y_2 \in Y^{i,t^i}_{E'} : (i_E, y_1) \succ^{i,t^i}_{E'} (i_E, y_2)\}$ and by E_2^* any other element of $S^{i,t^i}_{E'}$. We claim that for every $E \in S^{i,t^i}_{E'} \setminus \{E_1^*, E_2^*\})$ and any $y_3, y_4 \in Y^{i,t^i}_{E'}$, it must follow that $(i_E, y_3) \sim^{i,t^i}_{E'} (i_{E_2^*}, y_4)$. Suppose not. Consider any such E and let $y_1^*, y_2^*, y_3^*, y_4^* \in Y^{i,t^i}_{E'}$ satisfy $(i^*_{E_1}, y_1^*) \succ^{i,t^i}_{E'} (i_{E_1^*}, y_2^*)$ and $(i_E, y_3^*) \not\sim^{i,t^i}_{E'} (i_{E_2^*}, y_4)$.

2A) If $(i_E, y_3^*) \succ^{i,t^i}_{E'} (i_{E_2^*}, y_4)$, set $E_1 = E_2 = E_1^*$, $E_3 = E$, $E_4 = E_2^*$, $y_1 = y_1^*, y_2 = y_2^*, y_3 = y_3^*$, and $y_4 = y_4^*$. Then $E_1 \neq E_3, E_1 \neq E_4, E_2 \neq E_3, E_2 \neq E_4$. But we also have $(i_{E_1}, y_1) \succ^{i,t^i}_{E'} (i_{E_2}, y_2)$ and $(i_{E_3}, y_3) \succ^{i,t^i}_{E'} (i_{E_4}, y_4)$ which contradicts our present assumption. Hence $(i_{E_2^*}, y_4) \succeq^{i,t^i}_{E'} (i_E, y_3^*)$.

2B) Likewise, if $(i_{E_2^*}, y_4) \succ^{i,t^i}_{E'} (i_E, y_3^*)$ set $E_1 = E_2 = E_1^*$, $E_3 = E_2^*$, $E_4 = E$, $y_1 = y_1^*, y_2 = y_2^*, y_3 = y_4^*$, and $y_4 = y_3^*$. Then $E_1 \neq E_3, E_1 \neq E_4, E_2 \neq E_3, E_2 \neq E_4$. But we also have $(i_{E_1}, y_1) \succ^{i,t^i}_{E'} (i_{E_2}, y_2)$ and $(i_{E_3}, y_3) \succ^{i,t^i}_{E'} (i_{E_4}, y_4)$ which contradicts our present assumption. Hence $(i_E, y_3^*) \succeq^{i,t^i}_{E'} (i_{E_2^*}, y_4)$.

2C) Conclude from 2A and 2B that $(i_{E_2^*}, y_4) \sim^{i,t^i}_{E'} (i_E, y_3^*)$.

Step 3: Show that if (i) is violated then either (ii) or (iii) holds. Indeed, from step (i) we know that $\#\{E \in S^{i,t^i}_{E'} | \exists y_1, y_2 \in Y^{i,t^i}_{E'} : (i_E, y_1) \succ^{i,t^i}_{E'} (i_E, y_2)\} \leq 1$. If $\#\{E \in S^{i,t^i}_{E'} | \exists y_1, y_2 \in Y^{i,t^i}_{E'} : (i_E, y_1) \succ^{i,t^i}_{E'} (i_E, y_2)\} = 0$ then (iii) holds. If $\#\{E \in S^{i,t^i}_{E'} | \exists y_1, y_2 \in Y^{i,t^i}_{E'} : (i_E, y_1) \succ^{i,t^i}_{E'} (i_E, y_2)\} = 1$ then it follows from step 2 that (ii) holds.

Q.E.D.

Claim 13. *Let E_1^*, E_2^* be distinct elements of $S^{i,t^i}_{E'}$ and let y^* be an element of $Y^{i,t^i}_{E'}$ such that $[(q, y) \not\sim^{i,t^i}_{E'} (i_{E_2^*}, y^*)] \Rightarrow [q(E_1^*) > 0]$. Let y_1, y_2 be elements of $Y^{i,t^i}_{E'}$ such that $(i_{E_1^*}, y_1) \succ^{i,t^i}_{E'} (i_{E_2^*}, y^*) \succ^{i,t^i}_{E'} (i_{E_1^*}, y_2)$. Finally, let f_1 and f_2 be affine representations of the preferences on the intervals $I_1 := \{(q, y) \in D^{i,t^i}_{E'} | (i_{E_1^*}, y_1) \succeq^{i,t^i}_{E'} (q, y) \succeq^{i,t^i}_{E'} (i_{E_2^*}, y^*)\}$ and $I_2 := \{(q, y) \in D^{i,t^i}_{E'} | (i_{E_2^*}, y^*) \succeq^{i,t^i}_{E'} (q, y) \succeq^{i,t^i}_{E'} (i_{E_1^*}, y_2)\}$ respectively with the further properties that a) f_1 is onto the interval $[f_1(i_{E_2^*}, y^*), f_1(i_{E_1}, y_1)]$; b) f_2 is onto the interval $[f_2(i_{E_1}, y_2), f_1(i_{E_2^*}, y^*)]$; and c) $f_1(i_{E_2^*}, y^*) = f_2(i_{E_2^*}, y^*)$. Then the function f that coincides with f_1 on I_1 and f_2 on I_2 is an affine representation of the preferences on the interval $I^* := \{(q, y) \in D^{i,t^i}_{E'} | (i_{E_1^*}, y_1) \succeq^{i,t^i}_{E'} (q, y) \succeq^{i,t^i}_{E'} (i_{E_1^*}, y_2)\} = I_1 \cup I_2$. Furthermore the function f is onto the interval $[f(i_{E_1^*}, y_2), f(i_{E_1^*}, y_1)]$.*

Lemma 4. *Assume that $X^i(.|t^i)$ satisfies assumption $C1$. Assume furthermore that $\succeq^{i,t^i}_{E'}$*

satisfies assumptions $E1, E2, E3, E4'$*, and* $E5$*. Let* q_1 *and* q_2*, be any two distinct elements of* $\Delta^{\#\{E \in \mathcal{PF}^{i*}(.|t^i)|E \subset E'\}-1}$*, and let* $y_1 \in Y^{i,t^i}_{E'}$ *satisfy the property that* $(q_1, y_1) \succ^{i,t^i}_{E'} (q_2, y_1)$*. Then for every* $(q, y) \in D^{i,t^i}_{E'}$ *such that* $(q_1, y_1) \succeq^{i,t^i}_{E'} (q, y) \succeq^{i,t^i}_{E'} (q_2, y_1)$*, there exists uniquely a* $t^* \in [0, 1]$ *such that* $(t^* * q_1 + (1 - t^*) * q_2, y_1) \sim^{i,t^i}_{E'} (q, y)$*.*

Proof of Lemma: Existence. Pick any (q, y) satisfying the described property. Define A and B by:

$$A := \{t \in [0, 1]|(t * q_1 + (1 - t^*) * q_2, y_1) \succeq^{i,t^i}_{E'} (q, y)\}$$

$$B := \{t \in [0, 1]|(q, y) \succeq^{i,t^i}_{E'} (t * q_1 + (1 - t^*) * q_2, y_1)\}$$

It suffices to show that $A \cap B \neq \emptyset$. From $D4'$, it follows that A and B are closed sets. From $D2$, it follows that $A \cup B = [0, 1]$. Define $t^* := \sup\{t \in [0, 1]|(t*q_1+(1-t^*)*q_2, y_1) \succeq^{i,t^i}_{E'} (q, y)\}$. From $D4'$, it follows that $t^* \in B$. If $t^* = 1$ then t^* is clearly in A. IF $t^* < 1$, we note that $[t \in [0, 1]|t > t^*] \subset A$. Since $A = \overline{A}$, it therefore follows that t^* is in A. Hence we conclude that regardless of the value of t^*, it is in $A \cap B$ establishing the existence of the desired t^*.

Uniqueness. Suppose not. Pick any $(q, y) \in D^{i,T^i}_{E'}$ such that there exists distinct t^*_1, t^*_2 ($t^*_1 > t^*_2$) in $[0, 1]$, both satisfying the described property. We now establish two properties implied by the existence of t^*_1 and t^*_2.

Property 1: $(q, y) \sim^{i,t^i}_{E'} (q_1, y_1)$.

To establish property 1, we define a sequence $\{t_k\}^{\infty}_{k=0}$ by

$$t_k = 1 - (1 - t^*_2)[(1 - t^*_1)/(1 - t^*_2)]^k.$$

for $k \geq 1$, we have

$$\begin{aligned} t_k &= 1 - (1 - t^*_2)[\tfrac{1-t^*_1}{1-t^*_2}]^{k-1} * [\tfrac{1-t^*_1}{1-t^*_2}] \\ &= 1 - (1 - t_{k-1})[\tfrac{1-t^*_1}{1-t^*_2}] \\ &= [1 - [\tfrac{1-t^*_1}{1-t^*_2}]] + [\tfrac{1-t^*_1}{1-t^*_2}] * t_{k-1} \end{aligned}$$

We also note that $t_0 = 1 - (1 - t^*_2) = t^*_2$. We claim that for every k,

$$(t_k * q_1 + (1 - t_k) * q_2, y_1) \sim^{i,t^i}_{E'} (q, y).$$

Clearly, this holds for t_0. Suppose that it holds for t_{k-1}. We will show it then must hold for t_k hence establishing this claim. We note,

$$\begin{aligned} (t_k * q_1 + (1 - t_k) * q_2, y_1) &= ([[1 - [\tfrac{1-t^*_1}{1-t^*_2}]] + [\tfrac{1-t^*_1}{1-t^*_2}] * t_{k-1}] * q_1 + [[\tfrac{1-t^*_1}{1-t^*_2}] - [\tfrac{1-t^*_1}{1-t^*_2}] * t_{k-1}]q_2, y_1) \\ &= ([1 - [\tfrac{1-t^*_1}{1-t^*_2}]] * q_1 + [\tfrac{1-t^*_1}{1-t^*_2}] * [t_{k-1} * q_1 + (1 - t_{k-1}) * q_2], y_1) \\ &\sim^{i,t^i}_{E'} ([1 - [\tfrac{1-t^*_1}{1-t^*_2}]] * q_1 + [\tfrac{1-t^*_1}{1-t^*_2}] * [t^*_2 * q_1 + (1 - t^*_2) * q_2], y_1) \\ &= ([\tfrac{(1-t^*_2)-(1-t^*_1)+(1-t^*_1)*t^*_2}{1-t^*_2}] * q_1 + (1 - t^*_1) * q_2, y_1) \\ &= (t^*_1 * q_1 + (1 - t^*_1) * q_2, y_1) \\ &\sim^{i,t^i}_{E'} (q, y) \end{aligned}$$

as required. Now, the fact that t_k converges to 1 implies that

$$\sup\{t \in [0,1] | (q,y) \succeq_{E'}^{i,t^i} (t_k * q_1 + (1-tk) * q_2, y_1)\} = 1.$$

But

$$\{t \in [0,1] | (q,y) \succeq_{E'}^{i,t^i} (t_k * q_1 + (1-tk) * q_2, y_1)\}$$

is a closed set. Hence $(q,y) \succeq_{E'}^{i,t^i} (q_1, y_1)$. This together with $(q_1,y_1) \succeq_{E'}^{i,t^i} (q,y)$ then implies property 1.

Property 2: $(q,y) \sim_{E'}^{i,t^i} (q_2, y_1)$.

To establish property 2, we define a sequence $\{t_k\}_{k=0}^{\infty}$ by

$$t_k = t_1^* * (\frac{t_2^*}{t_1^*})^k$$

for $k \geq 1$, we have

$$\begin{aligned} 1 - t_k &= 1 - t_1^* * [\tfrac{t_2^*}{t_1^*}]^{k-1} * [\tfrac{t_2^*}{t_1^*}] \\ &= 1 - t_{k-1} * [\tfrac{t_2^*}{t_1^*}] \\ &= [1 - \tfrac{t_2^*}{t_1^*}] + \tfrac{t_2^*}{t_1^*} * [1 - t_{k-1}] \end{aligned}$$

We also note that $t_0 = t_1^*$. We claim that for every k,

$$(t_k * q_1 + (1-t_k) * q_2, y_1) \sim_{E'}^{i,t^i} (q', y).$$

Clearly, this holds for t_0. Suppose that it holds for t_{k-1}. We will show it then must hold for t_k hence establishing this claim. We note,

$$\begin{aligned} (t_k * q_1 + (1-t_k) * q_2, y_1) &= ([[1 - [1 - \tfrac{t_2^*}{t_1^*}] - \tfrac{t_2^*}{t_1^*}[1 - t_{k-1}]]q_1 + [[1 - \tfrac{t_2^*}{t_1^*}] + \tfrac{t_2^*}{t_1^*}[1 - t_{k-1}]]q_2, y_1) \\ &= ([1 - [1 - \tfrac{t_2^*}{t_1^*}]] * q_2 + [\tfrac{t_2^*}{t_1^*}] * [t_{k-1} * q_1 + (1 - t_{k-1}) * q_2], y_1) \\ &\sim_{E'}^{i,t^i} ([1 - \tfrac{t_2^*}{t_1^*}] * q_2 + [\tfrac{t_2^*}{t_1^*}][t_1^* * q_1 + (1 - t_1^*) * q_2], y_1) \\ &= (\tfrac{t_1^* - t_2^*}{t_1^*} + \tfrac{1}{t_1^*} * [(t_2^* - t_2^* * t_1^*) * q_2 + t_2^* * t_1^* * q_1], y_1) \\ &= (t_2^* * q_1 + (1 - t_2^*) * q_2, y_1) \\ &\sim_{E'}^{i,t^i} (q, y) \end{aligned}$$

as required. Now, the fact that t_k converges to 0 implies that

$$\inf\{t \in [0,1] | (t_k * q_1 + (1-tk) * q_2, y_1) \succeq_{E'}^{i,t^i} (q,y)\} = 0.$$

But

$$\{t \in [0,1] | (t_k * q_1 + (1-tk) * q_2, y_1) \succeq_{E'}^{i,t^i} (q,y)\}$$

is a closed set. Hence $(q_2, y_1) \succeq_{E'}^{i,t^i} (q,y)$. This together with $(q,y) \succeq_{E'}^{i,t^i} (q_2,y_1)$ then implies property 2.

Now, properties 1 and 2 implies that $(q_2, y_1) \succeq_{E'}^{i,t^i} (q_1, y_1)$. But this is a contradiction to the present assumption that $(q_1, y_1) \succ_{E'}^{i,t^i} (q_2, y_1)$. Hence t^* must be unique.

Lemma 5. *Assume that* $X^i(.|t^i)$ *satisfies assumption* $C1$. *Assume furthermore that* $\succeq^{i,t^i}_{E'}$ *satisfies assumptions* $E1, E2, E3, E4'$, *and* $E5$. *Let* a, b $(a > b)$ *be any two real number, let* q_1, q_2 *be any two elements of* $\Delta(S^{i,t^i}_{E'})$, *and let* y *in* $Y^{i,t^i}_{E'}$ *satisfy the property that* $(q_1, y) \succ^{i,t^i}_{E'} (q_2, y)$. *Define a function* $u : \{(q', y') \in D^{i,t^i}_{E'} | (q_1, y) \succeq^{i,t^i}_{E'} (q', y') \succeq^{i,t^i}_{E'} (q_2, y)\} \to \mathbf{R}$ *by* $u(q', y') = t^*_{q',y'} * a + (1 - t^*_{q',y'}) * b$, *where* $t^*_{q',y'}$ *is the unique number associated with* (q', y') *given in Lemma 1. Then the function* u *satisfies the following properties:*

(i) $[(q', y') \succeq^{i,t^i}_{E'} (q'', y'')] \Leftrightarrow [u(q', y') \geq u(q'', y'')]$; *and*

(ii) If for $l = 1, \ldots, K$, $(q_1, y) \succeq^{i,t^i}_{E'} (q^l, y') \succeq^{i,t^i}_{E'} (q_2, y)$, *then for every* $t \in \Delta^{K-1}$, $u(\sum_{l=1}^{K} t_l * q^l, y') = \sum_{l=1}^{K} t_l * u(q^l, y')$.

Proof of Lemma 2: Step 1. Show that

$$[(t^*_{q',y'} * q_1 + (1 - t^*_{q',y'}) * q_2, y) \succeq^{i,t^i}_{E'} (t^*_{q'',y''} * q_1 + (1 - t^*_{q'',y''}) * q_2, y)] \Rightarrow [t^*_{q',y'} \geq t^*_{q'',y''}]$$

. Indeed, if $(t^*_{q',y'} * q_1 + (1 - t^*_{q',y'}) * q_2, y) \sim^{i,t^i}_{E'} (t^*_{q'',y''} * q_1 + (1 - t^*_{q'',y''}) * q_2, y)$, then it follows from the uniqueness property in Lemma 1 that $t^*_{q',y'} = t^*_{q'',y''}$. Therefore, assume that $(t^*_{q',y'} * q_1 + (1 - t^*_{q',y'}) * q_2, y) \succ^{i,t^i}_{E'} (t^*_{q'',y''} * q_1 + (1 - t^*_{q'',y''}) * q_2, y)$. Define $\tilde{q} = t^*_{q'',y''} * q_1 + (1 - t^*_{q'',y''}) * q_2$. Then it follows from Lemma 1 that there exists uniquely $\tilde{t} \in [0, 1]$ such that

$$(t^*_{q',y'} * q_1 + (1 - t^*_{q',y'}) * q_2, y) \sim^{i,t^i}_{E'} (\tilde{t} * q_1 + (1 - \tilde{t}) * \tilde{q}, y)$$

$$= [[\tilde{t} + (1 - \tilde{t}) * t^*_{q'',y''}] * q_1 + [1 - (\tilde{t} + (1 - \tilde{t}) * t^*_{q'',y''})] * q_2, y]$$

From the uniqueness property in Lemma 1 it then follows that

$$t^*_{q',y'} = \tilde{t} + (1 - \tilde{t}) * t^*_{q'',y''} \geq t^*_{q'',y''}$$

as required.

Step 2. Show that

$$[t^*_{q',y'} \geq t^*_{q'',y''}] \Rightarrow [(t^*_{q',y'} * q_1 + (1 - t^*_{q',y'}) * q_2, y) \succeq^{i,t^i}_{E'} (t^*_{q'',y''} * q_1 + (1 - t^*_{q'',y''}) * q_2, y)]$$

Indeed, if $t^*_{q',y'} = 1$, then

$$(t^*_{q',y'} * q_1 + (1 - t^*_{q',y'}) * q_2, y) = (q_1, y)$$

$$\succeq^{i,t^i}_{E'} (t^*_{q'',y''} * q_1 + (1 - t^*_{q'',y''}) * q_2, y)$$

and we are done.

Therefore, suppose $t^*_{q',y'} < 1$. Let $\tilde{q} = t^*_{q',y'} * q_1 + (1 - t^*_{q',y'}) * q_2$. If $(t^*_{q'',y''} * q_1 + (1 - t^*_{q'',y''}) * q_2, y) \succ^{i,t^i}_{E'} (t^*_{q',y'} * q_1 + (1 - t^*_{q',y'}) * q_2, y)$, then it follows from the uniqueness property in Lemma 1 that there exists $\tilde{t} \in [0, 1]$ such that

$$(t^*_{q'',y''} * q_1 + (1 - t^*_{q'',y''}) * q_2, y) \sim^{i,t^i}_{E'} (\tilde{t} * q_1 + (1 - \tilde{t}) * \tilde{q}, y)$$

$$[[\tilde{t} + (1-\tilde{t}) * t^*_{q',y'}] * q_1 + [1 - (\tilde{t} + (1-\tilde{t}) * t^*_{q',y'})] * q_2, y]$$

From the uniqueness property in Lemma 1 it then follows that $t^*_{q'',y''}) = \tilde{t}+(1-\tilde{t})*t^*_{q',y'} > t^*_{q',y'}$. But this is a contradiction. Hence

$$(t^*_{q',y'} * q_1 + (1-t^*_{q',y'}) * q_2, y) \succeq^{i,t^i}_{E'} (t^*_{q'',y''} * q_1 + (1-t^*_{q'',y''}) * q_2, y)$$

.

Step 3. Show that

$$[(q',y') \succeq^{i,t^i}_{E'} (q'',y'')] \Leftrightarrow [t^*_{q',y'} * q_1 + (1-t^*_{q',y'}) * q_2, y]$$

$$\succeq^{i,t^i}_{E'} (t^*_{q'',y''} * q_1 + (1-t^*_{q'',y''}) * q_2, y)$$

Indeed, this can be easily seen to follow from assumptions $E2$ and $E3$.

Step 4. Show that $[(q',y') \succeq^{i,t^i}_{E'} (q'',y'')] \Leftrightarrow [t^*_{q',y'} \geq t^*_{q'',y''}]$.
Indeed, from steps 1 and 3, it follows that

$$[(q',t') \succeq^{i,t^i}_{E'} (q'',y'')] \Leftrightarrow [[t^*_{q',y'} * q_1 + (1-t^*_{q',y'}) * q_2, y]$$

$$\succeq^{i,t^i}_{E'} (t^*_{q'',y''} * q_1 + (1-t^*_{q'',y''}) * q_2, y)]$$

$$\Rightarrow [t^*_{q',y'} \geq t^*_{q'',y''}]$$

From steps 2 and 3, it likewise follows that

$$[t^*_{q',y'} \geq t^*_{q'',y''}] \Rightarrow [(t^*_{q',y'} * q_1 + (1-t^*_{q',y'}) * q_2, y) \succeq^{i,t^i}_{E'} (t^*_{q'',y''} * q_1 + (1-t^*_{q'',y''}) * q_2, y)]$$

$$\Leftrightarrow (q',y') \geq (q'',y'')$$

as required.

Step 5. Show that $[q = \sum_{l=1}^{K} t^l * q^l] \Rightarrow [t^*_{q,y'} = \sum_{l=1}^{K} t^l * t^*_{q^l,y'}]$.

Without loss of generality, assume that $t^1 > 0$ and define $s^r = \sum_{l=1}^{r}(\frac{t^l}{\sum_{l=1}^{r} t^l}) * q^l$. We claim that $t^*_{s^r,y'} = \sum_{l=1}^{r}(\frac{t^l}{\sum_{l=1}^{r} t^l}) * t^*_{q^l,y'}$. Indeed, this is trivial if $r = 1$. Assume the property holds for $r-1$. We note that $s^r = (\frac{t^r}{\sum_{l=1}^{r} t^l}) * q^r + (\frac{\sum_{l=1}^{r-1} t^l}{\sum_{l=1}^{r} t^l}) * s^{r-1}$. It then follows from assumption $E5$ that

$$(s^r, y') = [(\frac{t^r}{\sum_{l=1}^{r} t^l}) * q^r + (\frac{\sum_{l=1}^{r-1} t^l}{\sum_{l=1}^{r} t^l}) * s^{r-1}, y']$$

$$\sim^{i,t^i}_{E'} [(\frac{t^r}{\sum_{l=1}^{r} t^l}) * (t^*_{q^r,y'} * q_1 + (1-t^*_{q^r,y'}) * q_2) + (\frac{\sum_{l=1}^{r-1} t^l}{\sum_{l=1}^{r} t^l}) * (t^*_{s^{r-1},y'} * q_1 + (1-t^*_{s^{r-1},y'}) * q_2), y']$$

$$= [(\frac{t^r}{\sum_{l=1}^{r} t^l}) * (t^*_{q^r,y'} * q_1 + (1 - t^*_{q^r,y'}) * q_2) +$$

$$(\frac{\sum_{l=1}^{r-1} t^l}{\sum_{l=1}^{r} t^l}) * (\sum_{l=1}^{r-1} (\frac{t^l}{\sum_{l=1}^{r} t^l}) * t^*_{q^l,y'} * q_1 + [(\frac{\sum_{l=1}^{r-1} t^l}{\sum_{l=1}^{r-1} t^l}) - \sum_{l=1}^{r-1} (\frac{t^l}{\sum_{l=1}^{r} t^l}) * t^*_{q^l,y'}] * q_2, y']$$

$$= [(\sum_{l=1}^{r} (\frac{t^l}{\sum_{l=1}^{r} t^l}) * t^*_{q^l,y'}) * q_1 + ((\frac{\sum_{l=1}^{r} t^l}{\sum_{l=1}^{r} t^l}) - \sum_{l=1}^{r} (\frac{t^l}{\sum_{l=1}^{r} t^l}) * t^*_{q^l,y'}) * q_2, y']$$

$$= [(\sum_{l=1}^{r} (\frac{t^l}{\sum_{l=1}^{r} t^l}) * t^*_{q^l,y'}) * q_1 + (1 - \sum_{l=1}^{r} (\frac{t^l}{\sum_{l=1}^{r} t^l}) * t^*_{q^l,y'}) * q_2, y']$$

From lemma 1, it then follows that $t^*_{s^r,y'} = (\frac{t^l}{\sum_{l=1}^{r} t^l}) * t^*_{q^l,y'})$. We now note that $q = s^K$, which implies the desired property.

Step 6. Conclude from steps 4 an 5 that the Lemma is established.

Lemma 6. *Assume that $X^i(.|t^i)$ satisfies assumption $C1$. Assume furthermore that $\succeq^{i,t^i}_{E'}$ satisfies assumptions $E1, E2, E3, E4'$, and $E5$. Let $y_1.y_2, \overline{y}, \underline{y}$ in $Y^{i,t^i}_{E'}$, and $q_1, q_1', q_2, q_2', \underline{q}, \overline{q}$ in $\Delta^{\#\{E \in \mathcal{PF}^{i*}(.|t^i) | E \subset E'\}-1}$ satisfy the following:*
(i) $(q_1, y_1) \succeq^{i,t^i}_{E'} (\overline{q}, \overline{y}) \succ^{i,t^i}_{E'} (\underline{q}, \underline{y}) \succeq^{i,t^i}_{E'} (q_1', y_1)$; and
(ii) $(q_2, y_2) \succeq^{i,t^i}_{E'} (\overline{q}, \overline{y}) \succ^{i,t^i}_{E'} (\underline{q}, \underline{y}) \succeq^{i,t^i}_{E'} (q_2', y_2)$.

Furthermore, for any (q, y) such that $(q_1, y_1) \succeq^{i,t^i}_{E'} (q, y) \succeq^{i,t^i}_{E'} (q_1', y_1)$ [$(q_2, y_2) \succeq^{i,t^i}_{E'} (q, y) \succeq^{i,t^i}_{E'} (q_2', y_2)$ respectively], denote by $t^*_{q,y}$ [$\tilde{t}^*_{q,y}$ resp.] the unique element of $[0, 1]$ given in Lemma [] for which $(t^*_{q,y} * q_1 + (1 - t^*_{q,y})q_1', y_1) \sim^{i,t^i}_{E'} (q, y)$ [$(\tilde{t}^*_{q,y} * q_2 + (1 - \tilde{t}^*_{q,y})q_2', y_2) \sim^{i,t^i}_{E'} (q, y)$ resp.]

Finally, let $a_1 > b_1$, $a_2 > b_2$ be real number such that the functions
$u_1 : \{(q, y) \in D^{i,t^i}_{E'} | (q_1, y_1) \succeq^{i,t^i}_{E'} (q, y) \succeq^{i,t^i}_{E'} (q_1', y_1)\} \to \mathbf{R}$; and

$u_2 : \{(q, y) \in D^{i,t^i}_{E'} | (q_2, y_2) \succeq^{i,t^i}_{E'} (q, y) \succeq^{i,t^i}_{E'} (q_2', y_2)\} \to \mathbf{R}$ defined by

$u_1(q, y) = t^*_{q,y} * a_1 + (1 - t^*_{q,y}) * b_1$ and $u_2(q, y) = \tilde{t}^*_{q,y} * a_2 + (1 - \tilde{t}^*_{q,y}) * b_2$ satisfies the properties

that $u_1(\overline{q}, \overline{y}) = u_2(\overline{q}, \overline{y})$ and $u_1(\underline{q}, \underline{y}) = u_2(\underline{q}, \underline{y})$.

Then u_1 coincides with u_2 on their common domain.

Fact 2. Assume $C1, E1 - E3$, $E4'$, and $E5$. Then for every $y \in Y^{i,t^i}_{E'}$, there exists a function $U_y : \Delta^{\#\{E \in \mathcal{PF}^{i*}(.|t^i) | E \subset E'\}-1} \to \mathbf{R}$, and a function $u_y : \{E \in \mathcal{PF}^{i*}(.|t^i) | E \subset E'\} \to \mathbf{R}$ such that

$$(i) \forall q, q' \in \Delta^{\#\{E \in \mathcal{PF}^{i*}(.|t^i) | E \subset E'\}-1}, U_y(q) \geq U_y(q') \Leftrightarrow (q, y) \succeq^{i,t^i}_{E'} (q', y)$$

$$(ii) \forall q \in \Delta(S^{i,t^i}_{E'}), U_y(q) = \textstyle\sum_{E'' \in \{E \in \mathcal{PF}^{i*}(.|t^i) | E \subset E'\}} q_{E''} u_y(E'')$$

Proof of Fact 2: By Mixture Theorem.

Fact 3. Assume $C1, E1-E3$, $E4'$, and $E5$. If for every $E'' \in \{E \in \mathcal{PF}^{i*}(.|t^i)|E \subset E'\}$, and every $y, y' \in Y^{i,t^i}_{E'}$, $(i_{E''}, y) \sim^{i,t^i}_{E'} (i_{E''}, y')$. Then for every $q \in \Delta^{\#\{E \in \mathcal{PF}^{i*}(.|t^i)|E \subset E'\}-1}$, and every $y, y' \in Y^{i,t^i}_{E'}$, $(q, y) \sim^{i,t^i}_{E'} (q, y')$.

Proof of Fact 3: Pick any $q \in \Delta(S^{i,t^i}_{E'})$, and any $y, y' \in Y^{i,t^i}_{E'}$. Order the elements of $\{E \in \mathcal{PF}^{i*}(.|t^i)|E \subset E'\}$ as $\{E_1, \ldots, E_K\}$ is such a way that $q_{E_1} > 0$. For $r = 1, \ldots, K$ define

$$\tilde{q}^r_{E_k} = \begin{cases} \frac{q_{E_k}}{\sum_{s=1}^{r} q_{E_s}} & \text{if } k \leq r \\ 0 & \text{otherwise} \end{cases}$$

We note that for $r \geq 2$, $\tilde{q}^r = (\frac{q_{E_r}}{\sum_{s=1}^{r} q_{E_s}}) * (i_{E_r}) + (\frac{\sum_{s=1}^{r-1} q_{E_s}}{\sum_{s=1}^{r} q_{E_s}})\tilde{q}^{r-1}$. We claim that for every $r = 1, \ldots, K$ we have that $(\tilde{q}^r, y) \sim^{i,t^i}_{E'} (\tilde{q}^r, y')$. Clearly, this is so for $r = 1$. Suppose that it holds for r. From $B4'$, and $\tilde{q}^r = (\frac{q_{E_r}}{\sum_{s=1}^{r} q_{E_s}}) * (i_{E_r}) + (\frac{\sum_{s=1}^{r-1} q_{E_s}}{\sum_{s=1}^{r} q_{E_s}})\tilde{q}^{r-1}$, it then follows that it must hold for $r + 1$. The fact then follows since $\tilde{q}^K = q$.

Lemma 7. *Assume $C1, E1-E3$, $E4'$, and $E5$. Further assume that $(i_{E_3}, y_3) \succeq^{i,t^i}_{E'} (i_{E_2}, y_2) \succeq^{i,t^i}_{E'} (i_{E_1}, y_1) \succ^{i,t^i}_{E'} (i_{E^*}, y_1) \sim^{i,t^i}_{E'} (i_{E^*}, y_2) \sim^{i,t^i}_{E'} (i_{E^*}, y_3)$. Finally, let u_{y_1}, u_{y_2}, and u_{y_3} be functions as given in fact above with the further property that $u_1(i_{E^*}, y_1) = u_2(i_{E^*}, y_2) = u_3(i_{E^*}, y_3) = 0$. Then any of the two of the following implies the third.*

(i) $\exists t^*_2 \in [0, 1]$ such that $(t^*_2 * i_{E_2} + (1 - t^*_2) * i_{E^*}, y_2) \sim^{i,t^i}_{E'} (i_{E_1}, y_1)$, and $t^*_2 * u_{y_2}(i_{E_2}, y_2) = u_{y_1}(i_{E_1}, y_1)$.

(ii) $\exists t^*_3 \in [0, 1]$ such that $(t^*_3 * i_{E_3} + (1 - t^*_3) * i_{E^*}, y_3) \sim^{i,t^i}_{E'} (i_{E_1}, y_1)$, and $t^*_3 * u_{y_3}(i_{E_3}, y_3) = u_{y_1}(i_{E_1}, y_1)$.

(iii) $\exists t^*_4 \in [0, 1]$ such that $(t^*_4 * i_{E_3} + (1 - t^*_4) * i_{E^*}, y_3) \sim^{i,t^i}_{E'} (i_{E_2}, y_2)$, and $t^*_4 * u_{y_3}(i_{E_3}, y_3) = u_{y_2}(i_{E_2}, y_2)$.

Proof of Lemma 2: (i) and (ii) ⇒ (iii).

From Lemma 1, it follows that there exists $t^*_4 \in [0, 1]$ such that $((t^*_4 * i_{E_3} + (1 - t^*_4) * i_{E^*}, y_3) \sim^{i,t^i}_{E'} (i_{E_2}, y_2)$. From assumption $E5$, it then follows that

$$\begin{array}{rcl} (i_{E_1}, y_1) & \sim^{i,t^i}_{E'} & (t^*_2 * [t^*_4 * i_{E_3} + (1 - t^*_4) * i_{E^*}] + (1 - t^*_2) * i_{E^*}, y_3) \\ & = & (t^*_2 * t^*_4 * i_{E_3} + (1 - t^*_2 * t^*_4) * i_{E^*}, y_3) \end{array}$$

From Lemma 1, it therefore follows that $t^*_3 = t^*_2 * t^*_4 \Rightarrow t^*_4 = \frac{t^*_3}{t^*_2}$.

This in turn implies that

$$\begin{array}{rcl} t^*_4 * u_{y_3}(i_{E_3}, y_3) & = & \frac{t^*_3}{t^*_2} * u_{y_3}(i_{E_3}, y_3) \\ & = & \frac{t^*_3}{t^*_2} * \frac{u_{y_1}(i_{E_1}, y_1)}{t^*_3} \\ & = & \frac{u_{y_1}(i_{E_1}, y_1)}{t^*_2} \\ & = & \frac{t^*_2}{t^*_2} * u_{y_2}(i_{E_2}, y_2) \\ & = & u_{y_2}(i_{E_2}, y_2) \end{array}$$

as required.

(i) and (iii) $\Rightarrow$ (ii). From Lemma 1, it follows that there exists $t_3^* \in [0,1]$ such that $t_3^* * i_{E_3} + (1-t_3^*) * i_{E^*}, y_3) \sim_{E'}^{i,t^i} (i_{E_1}, y_1)$. From assumption $E5$, it follows that

$$\begin{array}{rcl} (i_{E_1}, y_1) & \sim_{E'}^{i,t^i} & (t_2^* * [t_4^* * i_{E_3} + (1-t_4^*) * i_{E^*}] + (1-t_2^*) * i_{E^*}, y_3) \\ & = & (t_2^* * t_4^* * i_{E_3} + (1 - t_2^* * t_4^*) * i_{E^*}, y_3) \end{array}$$

From Lemma 1, it therefore follows that $t_3^* = t_2^* * t_4^*$. It then follows that

$$\begin{array}{rcl} t_3^* * u_{y_3}(i_{E_3}, y_3) & = & t_2^* * t_4^* * u_{y_3}(i_{E_3}, y_3) \\ & = & t_2^* * u_{y_2}(i_{E_2}, y_2) \\ & = & u_{y_1}(i_{E_1}, y_1) \end{array}$$

as required.

(ii) and (iii) $\Rightarrow$ (i). From Lemma 1, it follows that there exists $t_2^* \in [0,1]$ such that $(t_2^* * i_{E_2} + (1-t_2^*) * i_{E^*}, y_2) \sim_{E'}^{i,t^i} (i_{E_1}, y_1)$. From assumption $E5$, it then follows that

$$(t_2^*[t_4^* * i_{E_3} + (1-t_4^*) * i_{E^*}] + (1-t_2^*) i_{E^*}, y_3) \sim_{E'}^{i,t^i} (i_{E_1}, y_1)$$

From Lemma 1, it therefore follows that $t_3^* = t_2^* * t_4^* \Rightarrow t_2^* = \frac{t_3^*}{t_4^*}$.
This in turn implies that

$$\begin{array}{rcl} t_2^* * u_{y_2}(i_{E_2}, y_2) & = & \frac{t_3^*}{t_4^*} * u_{y_2}(i_{E_2}, y_2) \\ & = & \frac{t_3^*}{t_4^*} * t_4^* * u_{y_3}(i_{E_3}, y_3) \\ & = & t_3^* * u_{y_3}(i_{E_3}, y_3) \\ & = & u_{y_1}(i_{E_1}, y_1) \end{array}$$

as required.

4.2.2 Special Case 1:

Lemma 8. *Assume* $C1, E1 - E3$, $E4'$ *and* $E5$. *Assume furthermore that there does not exists an event* $E'' \in \{E \in \mathcal{PF}^{i*}(.|t^i) | E \subset E'\}$ *and distinct* $y, y' \in Y_{E'}^{i,t^i}$ *such that* $(i_E, y) \succ_{E'}^{i,t^i} (i_E, y')$. *Then* $\succeq_{E'}^{i,t^i}$ *has an expected utility function representation.*

Proof of Lemma: Pick any $y \in Y_{E'}^{i,t^i}$. Let U_y, and u_y be the functions given in Fact [] above. Define $U : D_{E'}^{i,t^i} \rightarrow \mathbf{R}$ by $U(q, y') = U_y(q)$, and $u : \cup_{E'' \in \{E \in \mathcal{PF}^{i*}(.|t^i) | E \subset E'\}} \{E''\} \times Y^i(E''|t^i) \rightarrow \mathbf{R}$ by $u(E, y'_E) = u_y(E)$. Then for every $(q, y') \in D_{E'}^{i,t^i}$, $U(q, y') = U_y(q) = \sum_{E'' \in \{E \in \mathcal{PF}^{i*}(.|t^i) | E \subset E'\}} q_{E''} u_y(E'') = \sum_{E'' \in \{E \in \mathcal{PF}^{i*}(.|t^i) | E \subset E'\}} q_{E''} u(E'', y'_{E''})$.

We claim that U represents $\succeq_{E'}^{i,t^i}$. Indeed, for any pair $(q', y'), (q'', y'') \in D_{E'}^{i,t^i}$, we have $(q', y') \succeq_{E'}^{i,t^i} (q'', y'') \Leftrightarrow (q', y) \succeq_{E'}^{i,t^i} (q'', y) \Leftrightarrow U_y(q') \geq U_y(q'') \Leftrightarrow U(q', y') \geq U(q'', y'')$. Here the first if and only if statement is a consequence of Fact [] above.

4.2.3 Special Case 2:

Lemma 9. *Assume* $C1, E1 - E3$, $E4'$ *and* $E5$. *Assume furthermore that:*

(i) There exists one and only one $E^* \in \{E \in \mathcal{PF}^{i*}(.|t^i)|E \subset E'\}$ *with the property that there exists* $y, y' \in Y_{E'}^{i,t^i}$ *such that* $(i_{E^*}, y) \succ_{E'}^{i,t^i} (i_{E^*}, y')$*;*

(ii) $[E_1, E_2 \in \{E \in \mathcal{PF}^{i*}(.|t^i)|E \subset E'\} \setminus \{E^*\}] \Rightarrow [\forall y, y' \in Y_{E'}^{i,t^i}, (i_{E^*}, y) \sim_{E'}^{i,t^i} (i_{E^*}, y')]$*; and*

(iii) $\#\{E \in \mathcal{PF}^{i*}(.|t^i)|E \subset E'\} \geq 2$

Then $\succeq_{E'}^{i,t^i}$ *has an expected utility function representation.*

Proof of Lemma: Step 1. Construct a function $u : \cup_{E \in \mathcal{PF}^{i*}(.|t^i):E \subset E'}\{E\} \times Y^i(E|t^i) \rightarrow \mathbf{R}$ as follows:

(i) $\forall E_1 \in \{E \in \mathcal{PF}^{i*}(.|t^i)|E \subset E'\} \setminus \{E^*\}$ and for all $y \in Y_{E'}^{i,t^i}$, set $u(E_1, y_{E_1}) = 0$.

Denote by E_1^* any element of $\{E \in \mathcal{PF}^{i*}(.|t^i)|E \subset E'\} \setminus \{E^*\}$ and by y_1^* any element of $Y_{E'}^{i,t^i}$

(ii) For every $y \in Y_{E'}^{i,t^i}$ such that $(i_{E^*}, y) \sim_{E'}^{i,t^i} (i_{E_1^*}, y_1^*)$, set $u(E^*, y_{E^*}) = 0$.

If there exists $y' \in Y_{E'}^{i,t^i}$ such that $(i_{E^*}, y) \succ_{E'}^{i,t^i} (i_{E_1^*}, y_1^*)$, pick any such element $y_2^* \in Y_{E'}^{i,t^i}$.

(iii) Set $u(E^*, y_{2,E^*}^*) = 1$.

(iv) For every $y \in Y_{E'}^{i,t^i}$ such that $(i_{E^*}, y) \sim_{E'}^{i,t^i} (i_{E^*}, y_2^*)$, set $u(E^*, y_{E^*}) = 1$.

(v) From our Lemma [], it follows that for every $y \in Y_{E'}^{i,t^i}$ such that $(i_{E^*}, y) \succ_{E'}^{i,t^i} (i_{E^*}, y_2^*)$, there exists uniquely $t_y^* \in (0,1)$ such that $(t_y^* * i_{E^*} + (1 - t_y^*) * i_{E_1^*}, y) \sim_{E'}^{i,t^i} (i_{E^*}, y_2^*)$. For any such y, set $u(E^*, y_{E^*}) = \frac{1}{t_y^*}$.

(vi) From our Lemma [], it also follows that for every $y \in Y_{E'}^{i,t^i}$ such that $(i_{E^*}, y_2^*) \succ_{E'}^{i,t^i} (i_{E^*}, y) \succ_{E'}^{i,t^i} (i_{E_1^*}, y_1^*)$, there exists uniquely $t_y^* \in (0,1)$ such that $(t_y^* * i_{E^*} + (1 - t_y^*) * i_{E_1^*}, y_2^*) \sim_{E'}^{i,t^i} (i_{E^*}, y)$. For any such y, set $u(E^*, y_{E^*}) = t_y^*$.

If there exists $y' \in Y_{E'}^{i,t^i}$ such that $(i_{E_1^*}, y_1^*) \succ_{E'}^{i,t^i} (i_{E^*}, y)$, pick any such element $y_3^* \in Y_{E'}^{i,t^i}$.

(vii) Set $u(E^*, y_{3,E^*}^*) = -1$.

(viii) For every $y \in Y_{E'}^{i,t^i}$ such that $(i_{E^*}, y) \sim_{E'}^{i,t^i} (i_{E^*}, y_3^*)$, set $u(E^*, y_{E^*}) = -1$.

(ix) From our Lemma [], it follows that for every $y \in Y^{i,t^i}_{E'}$ such that $(i_{E^*}, y^*_3) \succ^{i,t^i}_{E'} (i_{E^*}, y)$, there exists uniquely $t^*_y \in (0,1)$ such that $(t^*_y * i_{E^*} + (1-t^*_y) * i_{E^*_1}, y) \sim^{i,t^i}_{E'} (i_{E^*}, y^*_3)$. For any such y, set $u(E^*, y_{E^*}) = \frac{-1}{t^*_y}$.

(x) From our Lemma [], it also follows that for every $y \in Y^{i,t^i}_{E'}$ such that $(i_{E^*_1}, y^*_1) \succ^{i,t^i}_{E'} (i_{E^*}, y) \succ^{i,t^i}_{E'} (i_{E^*}, y^*_3)$, there exists uniquely $t^*_y \in (0,1)$ such that $(t^*_y * i_{E^*} + (1-t^*_y) * i_{E^*_1}, y^*_3) \sim^{i,t^i}_{E'} (i_{E^*}, y)$. For any such y, set $u(E^*, y_{E^*}) = -t^*_y$.

It is straightforward to verify that the above procedure in fact uniquely specifies a function.

Step 2. Define a function $U : D^{i,t^i}_{E'} \to \mathbf{R}$ by $U(q,y) := \sum_{E \in \mathcal{PF}^{i*}(.|t^i): E \subset E'} q_E * u(E, y_E)$.

Step 3. Show that the function U defined in step 2 is a utility function representation of $\succeq^{i,t^i}_{E'}$. Indeed, pick any $(q,y), (q',y') \in D^{i,t^i}_{E'}$. We must show that $[(q,y) \succeq^{i,t^i}_{E'} (q',y')] \Leftrightarrow [U(q,y) \geq U(q',y')]$. Without loss of generality assume that $(i_{E^*}, y) \succeq^{i,t^i}_{E'} (i_{E^*}, y')$.

4.3 Theorem 2:

4.4 State-Contingent Claims

There is a finite non-empty set of state-contingent claims $L := \{1, \ldots, L\}$. Associated with the claims is a non-empty-valued correspondence $\mathcal{L} : L \to \Omega$ associating with each claim $l \in L$ the set of states $\mathcal{L}(l)$ in which it is deliverable. The set of state contingent claims deliverable in state ω will be denoted by $L(\omega) := \{l \in L | \omega \in \mathcal{L}(l)\}$. The following assumptions will frequently be imposed.

Assumption (B1:)**.** $\forall \omega \in \Omega$, $\#[L(\omega)] \geq 1$

Assumption (B2:)**.** $\forall (\omega, l) \in \Omega \times L, [\omega \in \mathcal{L}(l) \Rightarrow \mathcal{L}(l) \subset \mathcal{PF}(\omega)]$

4.5 Consumers

There is a finite non-empty set of consumers $M := \{1, \ldots, m\}$.

4.6 Uncertainty

There is a probability space $(\Omega, \mathcal{F}, \mu)$, where Ω is a finite non-empty set of states, $\mathcal{F}$ is the tribe of subsets of Ω that are events, and μ is a probability measure that assigns to any event B in $\mathcal{F}$ its probability $\mu(B)$. Throughout, we will write $\mu(\omega)$ instead of $\mu(\{\omega\})$ whenever $\omega \in \Omega$. The following assumption will often be imposed on μ.

Assumption (A1: Strictly positive probabilities)**.** $\forall \omega \in \Omega : \mu(\omega) > 0$

4.7 Information

Each consumer i has an information field $\mathcal{F}^i \subset \mathcal{F}$. The set of minimal nonempty events in $\mathcal{F}^i$ is denoted $\mathcal{PF}^i$ and forms a partition of Ω. For any ω in Ω, the unique element of $\mathcal{PF}^i$ that contains ω is denoted $\mathcal{PF}^i(\omega)$. We denote by $\mathcal{PF}$ the partition of Ω satisfying the property that for each ω in Ω, the unique element of $\mathcal{PF}$ containing ω, denoted $\mathcal{PF}(\omega)$, is given by $\mathcal{PF}(\omega) := \cap_{i \in M} \mathcal{PF}(\omega)$.

4.8 State-Contingent Claims

There is a finite non-empty set of state-contingent claims $L := \{1, \ldots, L\}$. Associated with the claims is a non-empty-valued correspondence $\mathcal{L} : L \to \Omega$ associating with each claim $l \in L$ the set of states $\mathcal{L}(l)$ in which it is deliverable. The set of state contingent claims deliverable in state ω will be denoted by $L(\omega) := \{l \in L | \omega \in \mathcal{L}(l)\}$. The following assumptions will frequently be imposed.

Assumption (B1:)**.** $\forall(\omega, l) \in \Omega \times L, [\omega \in \mathcal{L}(l) \Rightarrow \mathcal{L}(l) \subset \mathcal{PF}(\omega)]$

Assumption (B2:)**.** $\forall \omega \in \Omega$, $\#[L(\omega)] \geq 1$

4.9 Consumption Sets

Each consumer i has for every state $\omega \in \Omega$ a consumption set $X^i(\omega) \subset \mathbf{R}^{\#[L(\omega)]}$. His consumption set in state contingent claims is given by

$$X^i := \left\{ x^i : \Omega \to \cup_{\omega \in \Omega} X^i(\omega) \} \;\middle|\; \begin{array}{l} (i) \forall \omega \in \Omega : x^i(\omega) \in X^i(\omega) \\ (ii) \forall l \in L : [\omega, \omega' \in \mathcal{L}(l) \Rightarrow x^i_l(\omega) = x^i_l(\omega') \end{array} \right\}$$

The following assumption will often be imposed.

Assumption (C1: Non-negative Quadrant)**.** $\forall(i, \omega) \in M \times \Omega : X^i(\omega) = \mathbf{R}_+^{\#[L(\omega)]}$

4.10 Endowments

Each consumer i has an initial endowment vector $e^i \in \mathbf{R}^{\#[L]}$. The following assumption will be imposed on the endowment functions throughout this paper.

Assumption (D1: Strictly positive endowments)**.** $\forall i \in M : e^i \gg \mathbf{0}$

4.11 Preferences

Denote by $\mathcal{M}_f(\cup_{\omega \in \Omega} X^i(\omega) \times \{\omega\})$ the set of all probability measures on $\cup_{\omega \in \Omega} X^i(\omega) \times \{\omega\}$ with finite support. The next consumer characteristic is a preference relation $\succeq^i$ defined on $\mathcal{M}_f(\cup_{\omega \in \Omega} X^i(\omega) \times \{\omega\})$. For preferences defined on probability measures rather than certain outcomes, the following four assumptions are standard in the literature.

Assumption (E1: Completeness)**.** *For every q and q' in $\mathcal{M}_f(\cup_{\omega \in \Omega} X^i(\omega) \times \{\omega\})$, it follows that $q \succeq^i q'$ or $q' \succeq^i q$.*

Assumption (E2: Transitivity). *For every q, q', and q'' in $\mathcal{M}_f(\cup_{\omega\in\Omega}X^i(\omega)\times\{\omega\})$ such that $q \succeq^i q'$ and $q' \succeq^i q''$, it follows that $q \succeq^i q''$.*

Assumption (E3: Closedness). *For any q, q', and q'' in $\mathcal{M}_f(\cup_{\omega\in\Omega}X^i(\omega)\times\{\omega\})$, the sets $\{t\in[0,1] | tq+(1-t)q' \succeq^i q''\}$ and $\{t\in[0,1] | q'' \succeq^i tq+(1-t)q'\}$ are both closed in the interval $[0,1]$.*

Assumption (E4: Substitutability). *For any q, q', and q'' in $\mathcal{M}_f(\cup_{\omega\in\Omega}X^i(\omega)\times\{\omega\})$ such that $q \sim^i q'$, and for any $t\in[0,1]$, it follows that $tq+(1-t)q'' \sim^i tq'+(1-t)q''$.*

For any q in $\mathcal{M}_f(\cup_{\omega\in\Omega}X^i(\omega)\times\{\omega\})$ and any $(x^i(\omega),\omega)$ in $\cup_{\omega\in\Omega}X^i(\omega)\times\{\omega\}$, denote by $q((x^i(\omega),\omega))$ the probability assigned by q to $(x^i(\omega),\omega)$.

Assumption (E5: Nonsatiation). *For any $(x^i(\omega),\omega)$ in $\cup_{\omega\in\Omega}X^i(\omega)\times\{\omega\}$, there exists $y^i(\omega)\in X^i(\omega)$ such that $q \succ^i q'$, where $q((y^i(\omega),\omega))=1$, and $q'((x^i(\omega),\omega))=1$.*

Assumption (E6: Concavity, or Risk Aversion). *For all $\omega\in\Omega$, any $x^i(\omega)$, $y^i(\omega)$ in $X^i(\omega)$, and any $t\in[0,1]$, it follows that $q_\omega \succeq^i q'_\omega$, where $q_\omega((tx^i(\omega)+(1-t)y^i(\omega),\omega))=1$, $q'_\omega((x^i(\omega),\omega))=t$, and $q'_\omega((y^i(\omega),\omega))=1-t$.*

Assumption (E7: Strict Concavity, or Strict Risk Aversion). *For every $\omega\in\Omega$, any distinct $x^i(\omega)$, $y^i(\omega)$ in $X^i(\omega)$, and any $t\in(0,1)$, it follows that $q_\omega \succ^i q'_\omega$, where $q_\omega((tx^i(\omega)+(1-t)y^i(\omega),\omega))=1$, $q'_\omega((x^i(\omega),\omega))=t$, and $q'_\omega((y^i(\omega),\omega))=1-t$.*

4.12 Belief Function

Each consumer i has a belief function $b^i:\Omega\times\mathbf{R}^{\#[L]}_{++}\times\mathcal{PF}^i\to[0,1]$. This belief function is presumed to be an element of the following set:

$$B^i := \left\{ b^i:\Omega\times\mathbf{R}^{\#[L]}_{++}\times\mathcal{PF}^i\to[0,1] \;\middle|\; \begin{array}{l} \forall(p,E)\in\mathbf{R}^{\#[L]}_{++}\times\mathcal{PF}^i: \\ (i)\sum_{\omega\in\Omega}b^i(\omega,p,E)=\sum_{\omega\in E}b^i(\omega,p,E)=1 \\ (ii)\forall E'\in\mathcal{PF}:E'\subset E: \\ \omega'\in E'\Rightarrow b^i(\omega',p,E)=\frac{\mu(\omega')}{\sum_{\omega\in E'}\mu(\omega)}\sum_{\omega\in E'}b^i(\omega,p,E) \\ (iii)\lambda>0\Rightarrow\forall\omega\in\Omega:b^i(\omega,\lambda p,E)=b^i(\omega,p,E) \end{array} \right\}$$

Abstract

We consider a formulation of a state-contingent claims model with differential information in which each consumer, as part of his exogeneously given characteristics, has a belief function relating his private information and strictly positive prices to beliefs about the true state of nature. A solution concept in the spirit of Selten's perfect equilibrium is formulated for this economy. We state assumptions for each consumer separately under which such equilibria exist and a stronger set of assumptions for each consumer separately under which all such equilibria are strongly informationally efficient.

5 Introduction

In the absence of uncertainty, a competitive equilibrium for a pure exchange economy imposes two conditions. The first is that every consumer needs to choose his best bundle given the equilibrium price. The second is that the market for every commodity is in balance. The set of equilibrium prices are then the only prices at which all the markets can simultaneously be in balance at the same time as consumers are choosing their best bundles.

When there is no uncertainty, consumers only need to know the price vector in order to figure out what their optimal bundle would be. In the presence of uncertainty, however, this is no longer the case if consumers believe prices to contain information about the true state of nature. A consumer's demand at any price will thus depend not only upon his preferences but also upon what he perceives the likelihood of the various states to be at that particular price. The set of stable prices in any one state will thus depend upon the entire relationship between prices and beliefs held by the consumer's.

We attempt to capture this notion in the present paper by including in each consumer's characteristics an exogeneously given belief function relating the set of strictly positive prices and elements of his information partition to beliefs about the true state of nature. Using this formulation, we then formulate a solution concept, a perfect state-ω competitive equilibrium, for a state contingent claims model with differential information that shares several features with Selten's perfect equilibrium for extensive form games. In particular, the perfect state-ω competitive equilibrium is a limiting point of a sequence of competitive equilibria where Arrow's market participant is limited to strictly positive prices.

In economies without differential information, the assumptions required for the existence of an equilibrium and the Two fundamental Theorems of Welfare economics do not involve joint assumptions about the preferences and initial endowments of different consumers. The present paper, seeks similar assumptions for our economy with differential information.

Of interest for any equilibrium concept is the extent to which at least one equilibrium exists. In the present paper, we present one such existence result. The main assumptions in this result include strictly positive endowments, non-satiation, strict risk aversion and beliefs for each consumer that result in continuous expenditure levels on the various elements of the pooled information partition. The last assumption is weaker than requiring beliefs about the elements of the pooled information partition to be continuous in the prices. The method of proof for this results combines techniques used by Arrow-Debreu in their competitive equilibrium existence results with Techniques used by Selten in his proof of a Perfect Equilibrium in extensive form games. If one strengthens the assumption on the belief function to continuity, an existence result under weak risk aversion should also be available. We do not include it here.

An issue of particular interest in economies with differential information is the extent to which markets are informationally efficient. Strong informational efficiency of markets in the present model entails the market prices fully conveying all the information of all the consumers. When this research was first started, the main interest was in identifying conditions under which markets would be strongly informationally efficient. Towards this end, we impose 4 Axioms of behavior on the consumers in our model. These relate as to how income should be allocated by the consumer among the different states. The first of the four axioms requires that .....

$Insert Discussion of Axioms here$

These Axioms are similar in spirit to Axioms formulated by Shapley for his value in side-payment games. We show that there exists uniquely an income allocation rule satisfying the four axioms.

In the special case where each element of the pooled information partition contains only one state and where there is only one commodity in each state, we consider a tatonnement process for the prices. We show that the associated system of differential equations is convergent and that the prices converge to a a unique price vector. This unique price vector has the property that the prices in all states except the true state equal zero.

We then return to the general model and show, under conditions weaker than those imposed by our four axioms, that every perfect state-ω is strongly informationally efficient. The assumption required in our main theorem is as follows: (i) For any completely uninformed consumer, the beliefs are such that the resulting expenditure level on every element of the pooled information partition exactly coincides with the income from that element of the pooled information partition; and (ii) for any consumer that can rule out at least one state, the expenditure on every element of the pooled information partition that the consumer can not rule out strictly exceeds the income from that element of the pooled information partition.

Note that while the belief function required for an individual consumer will depend on his own preferences and initial endowments, it does not depend on the preferences or initial endowments of other consumers. We show through several examples that these assumptions on the beliefs are hard to relax unless one instead introduces a dependence on the preferences or initial endowments of other consumers. While allowing for dependence of beliefs on a consumers own preferences and initial endowments would at first seem like a strong assumption, we argue that it is much weaker than allowing a dependence on other consumer's preferences or initial endowments.

The assumption in the welfare result rules out informationally inefficient equilibria because in the realized element of the pooled information partition, every consumer holds, after trading, a set of state contingent claim with a market value at least as high as the initial endowment. If any price outside of the realized element of the pooled information partition was strictly positive, the market value of the holdings for at least one consumer would strictly exceed the market value of the initial endowment. Hence the market can not balance at such prices. Non-satiated preferences then means that the beliefs of every consumer must assign a probability of 1 to the realized element of the pooled information partition at any equilibrium price.

The rest of the paper proceeds as follows. In Section 2, we introduce the main notation and define a differential information economy. Section 3 introduces the solution concept, states a couple of claims and includes our existence Theorem. Section 4, discusses Pareto and Strong Informational efficiency in the model and states our welfare result concerning the Pareto and Strong Informational efficiency of Perfect competitive equilibria. Section 5 contains some counterexamples to the Strong informational efficiency of perfect competitive equilibria if the key assumption concerning beliefs in the welfare theorem are violated.

5.1 Definition of an Economy

We now have all the ingredients to state the definition of a differential information economy.

Definition 4. A *Differential Information Economy* is a collection of specified data $\mathcal{E} := \{(\Omega, \mathcal{F}, \mu), \{L, \mathcal{L}\}, \{\mathcal{F}^i, X^i, e^i, \succeq^i, b^i\}_{i \in M}\}$, where the data are as specified in the previous sections.

6 Consistent Beliefs

Definition 5. Let $p \in \overset{\circ}{\Delta}^{\#[L]-1}$ be given. Then a bundle $x^{*i} \in X^i$ is state-ω optimal for a consumer i if:
(1) $\sum_{l \in L} p_l x_l^{*i} \leq \sum_{l \in L} p_l e_l^i$; and
(2) There does not exists another bundle $\hat{x}^i \in X^i$ satisfying the properties that:
(i) $\sum_{l \in L} p_l \hat{x}_l^i \leq \sum_{l \in L} p_l e_l^i$; and
(ii) $q \succ^i q'$, where for each ω' in Ω, $q((\hat{x}^i(\omega'), \omega')) = q'((x^{*i}(\omega'), \omega')) = b^i(\omega' | p, \mathcal{PF}(\omega))$.

Definition 6. A consumer i's belief function is state-ω consistent with a function $r^i : \overset{\circ}{\Delta}^{\#[L]-1} \rightarrow \mathbf{R}_+^{\#\mathcal{PF}}$ if for every $p \in \overset{\circ}{\Delta}^{\#[L]-1}$, every state-$\omega$ optimal bundle x^{*i} satisfies the property that for every $E \in \mathcal{PF}$, $\sum_{l \in \mathcal{L}(l) \cap E \neq \emptyset} p_l x_l^{*i} = r_E^i(p)$.

Lemma 10. *If the preferences of a consumer i satisfies assumptions E1-E4, E5, and E7 and the commodities satisfies assumptions* $B1-B2$*, then for every state-*ω *and every* $\tilde{r}^i : \overset{\circ}{\Delta}^{\#[L]-1} \rightarrow \Delta^{\#\{E \in \mathcal{PF} | E \subset \mathcal{PF}^i(\omega)\}-1}$ *there exists a belief function* b^i *that is state-*ω *consistent with the function* $r^i : \overset{\circ}{\Delta}^{\#[L]-1} \rightarrow \mathbf{R}_+^{\mathcal{PF}}$ *defined by:*

$$r_E^i(p) = \begin{cases} \tilde{r}_E^i(p) * \sum_{l \in L} p_l e_l^i & \text{if } E \subset \mathcal{PF}^i(\omega) \\ 0 & \text{otherwise} \end{cases}$$

7 Some Preliminary Results

We now state 3 claims that will be useful for later purposes and that also will acts as a motivation for the assumptions in several theorem.

Claim 14. *Let* $\mathcal{E} := \{(\Omega, \mathcal{F}, \mu), \{X^i, e^i, \succeq^i, \mathcal{F}^i, b^i\}_{i \in M}\}$ *be a differential information economy satisfying assumptions A1,B1,C1, and D1-D7. Let* ω *be the true state. Then for any strictly positive price vector* $p \in \Delta^{\#[\Omega]*\#[L]-1}$*, every consumer* i *has a unique optimal bundle* $\hat{x}^{*i}$ *satisfying the following three properties:*

(i) $\sum_{\omega' \in PF^i(\omega)} p(\omega') \hat{x}^{*i}(\omega') = \sum_{\omega' \in \Omega} p(\omega') e^i(\omega')$

(ii) $\forall \omega' \in \Omega \setminus PF^i(\omega) : \hat{x}^{*i}(\omega') = \mathbf{0}$

(iii) $\forall E \in PF : E \subset PF^i(\omega) : \sum_{\omega' \in E} p(\omega') \hat{x}^{*i}(\omega') \geq 0.$

Proof of Claim 1: The strictly positive prices combined with non-satiation implies that in for any $E \in PF$ for which $b^i(E, p, PF^i(\omega)) = 0$ the consumer must consume zero of every commodity. Suppose now that there existed two optimal bundles. Then the strict risk aversion means that any strictly convex combination is strictly preferred to both and that this strictly convex combination also is affordable. This contradicts both bundled being optimal. Under the present assumption on preferences, it is well known that the preferences of consumer i are representable by a continuous von Neumann-Morgenstern utility function and which attains its maximum on the budget set. Hence there in fact exists a unique optimal bundle. The non-satiation now implies that all income is spent on the elements of PF that get assigned a strictly positive probability and that each such element gets assigned a non-negative income as required. Q.E.D.

Claim 15. *Let* $\mathcal{E} := \{(\Omega, \mathcal{F}, \mu), \{X^i, e^i, \succeq^i, \mathcal{F}^i, b^i\}_{i \in M}\}$ *be a differential information economy satisfying assumptions A1,B1,C1, and D1-D7. Let* ω *be the true state. Then for every consumer* i*, there exists a function* $r^i : \overset{\circ}{\Delta}^{\#[\Omega]*\#[L]-1} \longrightarrow \Delta^{\#[E \in PF : E \subset PF^i(\omega)]-1}$ *such that for every* $p \in \overset{\circ}{\Delta}^{\#[\Omega]*\#[L]-1}$ *and every* $E \in PF : E \subset PF^i(\omega)$*, the unique optimal bundle* $\hat{x}^{*i}$ *associated with* p *given in Claim 1 satisfies the property that*

$$\sum_{\omega' \in E} p(\omega')\hat{x}^{*i}(\omega') = r^i(E) \sum_{\omega' \in \Omega} p(\omega')e^i(\omega')$$

Proof of Claim 2: Follows immediately from claim 1.

Claim 16. *Let* $\mathcal{E} := \{(\Omega, \mathcal{F}, \mu), \{X^i, e^i, \succeq^i, \mathcal{F}^i, b^i\}_{i \in M}\}$ *be a differential information economy satisfying assumptions A1,B1,C1, and D1-D7. Let* ω *be the true state. Then for any function* $r^i : \overset{\circ}{\Delta}^{\#[\Omega]*\#[L]-1} \longrightarrow \Delta^{\#[E \in PF : E \subset PF^i(\omega)]-1}$*, there is a belief function satisfying the property that for every* $p \in \overset{\circ}{\Delta}^{\#[\Omega]*\#[L]-1}$ *and every* $E \in PF : E \subset PF^i(\omega)$*, the unique optimal bundle* $\hat{x}^{*i}$ *associated with* p *given in Claim 1 satisfies the property that*

$$\sum_{\omega' \in E} p(\omega')\hat{x}^{*i}(\omega') = r^i(E) \sum_{\omega' \in \Omega} p(\omega')e^i(\omega')$$

Proof of Claim 3: Can be proved using the Knaster-Kuratowski-Mazurkiewich (KKM) Theorem.

Claim 17. *Let* $\mathcal{E} := \{(\Omega, \mathcal{F}, \mu), \{X^i, e^i, \succeq^i, \mathcal{F}^i, b^i\}_{i \in M}\}$ *be a differential information economy satisfying assumptions A1,B1,C1, and D1-D7. Let* ω *be the true state. Assume furthermore that the belief function of consumer i is continuous. Then there exist a continuous function* $r^i : \overset{\circ}{\Delta}^{\#[\Omega]*\#[L]-1} \longrightarrow \Delta^{\#[E \in PF : E \subset PF^i(\omega)]-1}$ *such that for every* $p \in \overset{\circ}{\Delta}^{\#[\Omega]*\#[L]-1}$ *and every* $E \in PF : E \subset PF^i(\omega)$*, the unique optimal bundle* $\hat{x}^{*i}$ *associated with* p *given in Claim 1 satisfies the property that*

$$\sum_{\omega' \in E} p(\omega')\hat{x}^{*i}(\omega') = r^i(E) \sum_{\omega' \in \Omega} p(\omega')e^i(\omega')$$

Two additional claims are stated here.
Proof of Claim 4: [Insert Proof here]

Claim 18. *Let $\mathcal{E} := \{(\Omega, \mathcal{F}, \mu), \{X^i, e^i, \succeq^i, \mathcal{F}^i, b^i\}_{i \in M}\}$ be a differential information economy satisfying assumptions A1,B1,C1, and D1-D7. Let ω be the true state. Assume furthermore that consumer i's preferences are representable by a differentiable von Neumann-Morgenstern utility function. Then for any continuous function $r^i : \overset{\circ}{\Delta}^{\#[\Omega]*\#[L]-1} \rightarrow \Delta^{\#[E \in PF : E \subset PF^i(\omega)]-1}$, there is one and only one belief function satisfying the property that for every $p \in \overset{\circ}{\Delta}^{\#[\Omega]*\#[L]-1}$ and every $E \in PF : E \subset PF^i(\omega)$, the unique optimal bundle $\hat{x}^{*i}$ associated with p given in Claim 1 satisfies the property that*

$$\sum_{\omega' \in E} p(\omega')\hat{x}^{*i}(\omega') = r^i(E) \sum_{\omega' \in \Omega} p(\omega')e^i(\omega')$$

This belief function is furthermore continuous and is explicitly given by the following formula:

[Insert Formula Here]

Proof of Claim 5: [Insert Proof Here]

8 Axioms of Behavior

Consider a consumer i observing a particular element E of his information partition and a price vector $p \in \mathbf{R}_+^{\#[L]}$. This consumer must then determine how much to spend on each of the state contingent claims. In fact, for any $E \in \mathcal{PF}^i$, there is a well defined function $I^{i,E} : \mathbf{R}_{++}^{\#[\Omega]*\#[L]} \rightarrow \mathbf{R}_{++}^{\#\mathcal{PF}}$ taking the following form:

$$I_{E'}^{i,E}(p) = \sum_{\omega \in E'} p(\omega) x^{i*}(\omega | p, E)$$

for every $E' \in \mathcal{PF}$ such that $E' \subset E$.

Instead of restricting the domain of the function to the set of strictly positive prices, we will here consider a rule $I^{i,E} : \mathbf{R}_+^{\#[L]} \rightarrow \mathbf{R}_+^{\#[\mathcal{PF}]}$ that determines how a consumer allocates his income in the presence of differential information.

The following four Axioms will be imposed on the rule $I^{i,E}$:

Axiom (A1: Walras' Law)**.** $\forall p \in \mathbf{R}_+^{\#[L]}$,

$$\sum_{E' \in \mathcal{PF}} I_{E'}^{i,E}(p) = \sum_{l \in L} p_l e_l^i$$

Axiom 1 states that all income should be completely spent. As long as preferences are non-satiated, this would seem to be an obvious condition.

Axiom (A2)**.** *For every* $p \in \mathbf{R}_+^{\#[L]}$*, and for every* $E' \in \mathcal{PF} : E' \subset \Omega \setminus E$, $I_{E'}^{i,E}(p) = 0$.

In order to state our third axiom, we first introduce the following additional notation. Define by $G_{\#[\mathcal{PF}]} := \{\sigma : \mathcal{PF} \to \mathcal{PF} | \sigma \text{ is a bijection}\}$. That is, $G_{\#[\mathcal{PF}]}$ is the set of all linear orders on $\mathcal{PF}$. We now have the required notation to state our third axiom.

Axiom (A3: Equal Treatment of Events)**.** *Let* $\sigma \in G_{\#[\mathcal{PF}]}$ *be any linear order on* $\mathcal{PF}$ *with the property that* $E' \subset E$ *if and only if* $\sigma(E') \subset E$*. Then* $\sum_{l \in L : \mathcal{L}(l) \subset E'} p_l e_l^i = \sum_{l \in L : \mathcal{L}(l) \subset \sigma(E')} p'_l e_l^i$ *for every* $E' \in \mathcal{PF}$ *implies that* $I_{\sigma(E')}^{i,E}(p') = I_{E'}^{i,E}(p)$ *for every* $E' \in \mathcal{PF}$.

Axiom (A4)**.** *For every* $E' \in \mathcal{PF}$, $[\sum_{l \in L : \mathcal{L}(l) \subset E'} p_l = \sum_{l \in L : \mathcal{L}(l) \subset \Omega \setminus E} p_l = 0] \Rightarrow [I_{E'}^{i,E}(p) = 0]$.

Axiom (A5 Linearity)**.** *The function* $I^{i,E}$ *is linear.*

An *income* is a function $I^{i,E} : \mathbf{R}_+^{\#[L]} \to \mathbf{R}_+^{\#[\mathcal{PF}]}$ satisfying axioms $A1 - A5$.

Theorem 3. *Let* $\mathcal{E} := \{(\Omega, \mathcal{F}, \mu), \{L, \mathcal{L}\}, \{\mathcal{F}^i, X^i, e^i, \succeq^i, b^i\}_{i \in M}\}$ *be a differential information economy satisfying satisfying assumptions* $B1$ *and* $D1$*. Then for every consumer* i*, there exists one and only one income on* $\mathbf{R}_+^{\#[L]}$*. It is explicitly written as*

$$I_{E'}^{i,E}(p) = \begin{cases} \sum_{l \in L : \mathcal{L}(l) \subset E'} p_l e_l^i + \sum_{l \in L : \mathcal{L}(l) \subset \Omega \setminus E} \frac{p_l e_l^i}{\#\{E'' \in \mathcal{PF} | E'' \subset E\}} & \textit{if } E' \subset E \\ 0 & \textit{otherwise} \end{cases}$$

9 A tatonnement process and its convergence

Suppose that for every $\omega \in \Omega$, $\#[L(\omega)] = 1$, $\#[\mathcal{PF}] = \#[\Omega]$ and that every consumer i's belief function is consistent with the *income.* Then for every $\omega \in \Omega$,

$$x_l^{i*}(p, \mathcal{PF}^i(\omega)) = \begin{cases} e_l^i + \frac{1}{p_l} \sum_{l' \in L : \mathcal{L}(l) \subset \Omega \setminus E} \frac{p_{l'} e_{l'}^i}{\#\{E' \in \mathcal{PF} | E' \subset \mathcal{PF}^i(\omega)\}} & \text{if } \mathcal{L}(l) \subset \mathcal{PF}^i(\omega) \\ 0 & \text{otherwise} \end{cases}$$

Denote by ω^* the true state of the world. Then the value of excess demand for the lth state contingent claim is given by

$$p_l * [\sum_{i \in N} x_l^{i*}(p, \mathcal{PF}^i(\omega^*))] - p_l * [\sum_{i \in N} e_l^i] =$$

$$= \sum_{i \in N : \mathcal{L}(l) \subset \mathcal{PF}^i(\omega^*)} \sum_{l' \in L : \mathcal{L}(l') \subset \Omega \setminus \mathcal{PF}^i(\omega^*)} \frac{p_{l'} e_{l'}^i}{\#\{E' \in \mathcal{PF} | E' \subset \mathcal{PF}^i(\omega^*)\}} - p_l * [\sum_{i \in N : \mathcal{L}(l) \subset \Omega \setminus \mathcal{PF}^i(\omega^*)} e_l^i]$$

We now consider the following disequilibrium process which determines p as a function of time t.

$$\frac{d}{dt} p_l(t) = \sum_{i \in N : \mathcal{L}(l) \subset \mathcal{PF}^i(\omega^*)} \sum_{l' \in L : \mathcal{L}(l') \subset \Omega \setminus \mathcal{PF}^i(\omega^*)} \frac{p_{l'}(t) e_{l'}^i}{\#\{E' \in \mathcal{PF} | E' \subset \mathcal{PF}^i(\omega^*)\}} - p_l(t) * \sum_{i \in N : \mathcal{L}(l) \subset \Omega \setminus \mathcal{PF}^i(\omega^*)} e_l^i$$

given $p_l(0) = p_l^0$.

Note that this is a system of $\#[\Omega]$ linear differential equations. Also note that under the present assumptions, there is a unique state-contingent claim l^* for which $\mathcal{L}(l) \subset \mathcal{PF}(\omega^*)$. For this commodity,

$$\frac{d}{dt} p_{l^*}(t) = \sum_{i \in N} \sum_{l' \in L : \mathcal{L}(l') \subset \Omega \setminus \mathcal{PF}^i(\omega^*)} \frac{p_{l'}(t) e^i_{l'}}{\#\{E' \in \mathcal{PF} | E' \subset \mathcal{PF}^i(\omega^*)\}}$$

We now state the main Theorem of this section.

Theorem 4. *Assume D1. Then for any* $p^0 \in \mathbf{R}_+^{\#[L]}$, *the solution* $\{p(t)\}_t$ *converges to a price vector* p^*, *where* p^* *is explicitly given by:*

$$p_l^* = \begin{cases} \sum_{l' \in L} p^0_{l'} & \textit{if } \mathcal{L}(l) \subset \mathcal{PF}(\omega^*) \\ 0 & \textit{otherwise} \end{cases}$$

10 Perfect State-ω Competitive Equilibria and their existence

The perfect equilibrium concept of Selten was developed in order to deal with behavior at information sets that occur with a probability off zero in equilibrium. Selten's method was to define an equilibrium for an ϵ-perturbed game and then to define a perfect equilibrium for the original game to be perfect if it is the limiting point of a sequence of equilibria associated with a sequence of ϵ-perturbed games as the perturbations get small. We will here consider ϵ-perturbed economies and define equilibria for such economies. Our equilibrium concept will then be the limiting point of a sequence of perturbed economies as the perturbations get small. Formally, the definition of an ϵ pertubed economy will be the following:

Definition 7. An *ϵ-perturbed differentical information economy* is a pair $(\mathcal{E}, \omega)$, where $\mathcal{E}$ is a differential information economy and $\epsilon : L \to (0, 1)$ is a function satisfying the property that $\sum_{l \in L} \epsilon(l) \leq 1$.

Our equilibrium solution concept for the ϵ perturbed economy will be one in which Arrow's market participant is restricted to set prices falling belonging to a subset of the set of strictly positive prices. As a result of this, the market participant that seeks to maximize the value of excess demand in Arrow's proof may not be able to force the value of excess demand to zero in the equilibrium. Markets will hence not be in complete balance in an ϵ perturbed economy. Note however that when ϵ is small, the markets with the largest excess demand will be assigned the highest prices while the rest of the markets will be assigned a price equal to ϵ if the market participant behaves optimally. Symmetric *epsilon*-perturbed economies will be used in the proof of our main existence theorem and we therefore define them here.

Definition 8. An ϵ-perturbed differentical information economy $(\mathcal{E}, \omega)$ is *symmetric* if there exists a number δ such that $\forall(\omega, l) \in \Omega \times L$, $\epsilon(\omega, l) = \delta$.

The equilibrium for the ϵ perturbed economy is defined as follows:

Definition 9. A *state-ω competitive equilibrium for an ϵ-perturbed differential information* $(\mathcal{E}, \omega)$ is a triplet $((x^{i*})_{i\in M}, \{b^{*i}\}_{i\in M}, p^*)$ of elements of $\prod_{i\in M} X^i$, $\prod_{i\in M} \Delta^{\#[\mathcal{PF}]-1}$, and $\Delta^{\#[L]-1}$ respectively satisfying the following four properties:

(1) For every consumer i, $\sum_{l\in L} p_l x_l^{i*} \leq \sum_{l\in L} p_l e_l^i$.

(2) There does not exist a consumer i and contingent claims $\hat{x}^i \in X^i$ satisfying the properties that:
  (i) $\sum_{l\in L} p_l \hat{x}_l^i \leq \sum_{l\in L} p_l e_l^i$; and
  (ii) $q \succ^i q'$, where for each ω' in Ω, $q((\hat{x}^i(\omega'), \omega')) = q'((x^i(\omega'), \omega')) = \frac{\mu(\omega')}{\sum_{\omega''\in PF(\omega')} \mu(\omega'')} b^{*i}(\mathcal{PF}(\omega'))$.

(3) p^* solves

$$Max_{p\in\Delta^{\#[L]-1}} \sum_{l\in L} p_l \sum_{i\in M} (x_l^{i*} - e_l^i)$$

subject to the constraint that for each $l \in L$, $p_l \geq \epsilon(l)$.

(4) $\forall (i, \omega') \in M \times \Omega : b^{*i}(\mathcal{PF}(\omega')) = b^i(\mathcal{PF}(\omega'), p^*, \mathcal{PF}^i(\omega'))$.

This nearly corresponds to the definition of a competitive equilibrium in economies without differential information, the key difference being the replacement of the standard market clearing condition by the solution of the maximization problem in (3). We are now ready to define our equilibrium concept as follows.

Definition 10. A *perfect state-ω competitive equilibrium* for a differential information $\mathcal{E}$ is a triplet $((x^{i*})_{i\in M}, \{b^{*i}\}_{i\in M}, p^*)$ of elements of $\prod_{i\in M} X^i$, $\prod_{i\in M} \Delta^{\#[PF]-1}$, and $\Delta^{\#[L]-1}$ respectively satisfying the following four properties:

(1) For every consumer i, $\sum_{l\in L} p_l x_l^{i*} \leq \sum_{l\in L} p_l e_l^i$.

(2) There does not exist a consumer i and contingent claims $\hat{x}^i \in X^i$ satisfying the properties that:
  (i) $\sum_{l\in L} p_l \hat{x}_l^i \leq \sum_{l\in L} p_l e_l^i$; and
  (ii) $q \succ^i q'$, where for each ω' in Ω, $q((\hat{x}^i(\omega'), \omega')) = q'((x^i(\omega'), \omega')) = \frac{\mu(\omega')}{\sum_{\omega''\in \mathcal{PF}(\omega')} \mu(\omega'')} b^{*i}(\mathcal{PF}(\omega'))$.

(3) $\forall l \in L : \sum_{i\in M} x_l^{*i} \leq \sum_{i\in M} e_l^i$.
(4) There exists a sequence of ϵ-perturbed economies $\{(\mathcal{E}, \epsilon_k)\}_{k=1}^{\infty}$ and an associated sequence of competitive equilibria $\{((x_{\epsilon_k}^{i*})_{i\in M}, \{b_{\epsilon_k}^{i*}\}_{i\in M}, p_{\epsilon_k}^*)\}_{k=1}^{\infty}$ of the ϵ-perturbed economies such that

$$(\mathcal{E}, \epsilon_k, (x_{\epsilon_k}^{i*})_{i\in M}, \{b_{\epsilon_k}^{i*}\}_{i\in M}, p_{\epsilon_k}^*) \to (\mathcal{E}, \mathbf{0}, (x^{i*})_{i\in M}, \{b^{i*}\}_{i\in M}, p^*)$$

The following Existence Theorem is now established.

Theorem 5. *Let* $\mathcal{E} := \{(\Omega, \mathcal{F}, \mu), \{L, \mathcal{L}\}, \{\mathcal{F}^i, X^i, e^i, \succeq^i, b^i\}_{i\in M}\}$ *be a differential information economy satisfying assumptions A1,C1,D1, and E1-E7. Let* ω *be the true state. Assume furthermore, that for every consumer* i *and every* $E \in \mathcal{PF}^i$*, there exists a continuous function* $r^{i,E} :\overset{\circ}{\Delta}{}^{\#[L]-1} \longrightarrow \Delta^{\#[E\in\mathcal{PF}:E\subset\mathcal{PF}^i(\omega)]-1}$ *such that consumer* i*'s belief function is consistent with the function* $\tilde{r}^i :\overset{\circ}{\Delta}{}^{\#[L]-1} \times \mathcal{PF}^i \to \mathbf{R}_+^{\#\mathcal{PF}}$ *defined by*

$$\tilde{r}^i_{E'}(p, E) = \begin{cases} r^{i,E}_{E'}(p)[\sum_{l\in L} p_l e^i_l] & \textit{if } E' \subset E \\ 0 & \textit{otherwise} \end{cases}$$

Then the economy $\mathcal{E}$ *has a perfect state-*ω *competitive equilibrium.*

11 Definitions of Efficiency

We here propose a definition of efficiency relying on the notion that it should not be possible to improve for every consumer if the consumers share all the available information.

Definition 11. An allocation $(x^{i*})_{i\in M}$ in $\prod_{i\in M} X^i$ is *state-*ω *efficient* if

(i) $\forall l \in L : \sum_{i\in M} x^{*i}_l \leq \sum_{i\in M} e^i_l$; and

(ii) $\neg\exists(\tilde{x}^{i*})_{i\in M} \in \prod_{i\in M} X^i$:
    (a) $\forall l \in L : \sum_{i\in M} \tilde{x}^{*i}_l \leq \sum_{i\in M} e^i_l$
    (b) $\forall i \in M : \tilde{q}^i \succ^i q^i$

where $\tilde{q}^i$ and q^i are defined by:

$$\tilde{q}^i(\tilde{x}^i(\omega'), \omega') = \begin{cases} \frac{\mu(\omega')}{\sum_{\omega''\in PF(\omega)}\mu(\omega'')} & \text{if } \omega' \in PF(\omega) \\ 0 & \text{otherwise} \end{cases}$$

$$q^i(x^i(\omega'), \omega') = \begin{cases} \frac{\mu(\omega')}{\sum_{\omega''\in PF(\omega)}\mu(\omega'')} & \text{if } \omega' \in PF(\omega) \\ 0 & \text{otherwise} \end{cases}$$

The following definition of strong informational efficiency seems to be the must natural.

Definition 12. A state-ω competitive equilibrium $((x^{i*})_{i\in M}, \{b^{*i}\}_{i\in M}, p^*)$ in $\prod_{i\in M} X^i$, $\prod_{i\in M} \Delta^{\#[\mathcal{PF}]-1}$, and $\Delta^{\#[L]-1}$ respectively is *strongly informationally efficient* if for every consumer $i \in M$, $b^{*i}(PF(\omega)) = 1$.

12 A Welfare Theorem

We now state the following theorem showing conditions on the beliefs under which one could expect strong informational efficiency. Essentially, it relies on each consumer assigning a large enough probability to each element of the fine information partition that they can not rule out to cause them to want to increase the market value of their holdings in every such element (if they can rule out at least one state) or hold claims with the same market value as their initial endowment in every such element (if they can not rule out at least one state). Formally, the Theorem with assumptions is as follows.

Theorem 6. *Let* $\mathcal{E} := \{(\Omega, \mathcal{F}, \mu), \{L, \mathcal{L}\}, \{\mathcal{F}^i, X^i, e^i, \succeq^i, b^i\}_{i \in M}\}$ *be a differential information economy satisfying assumptions A1,C1,D1, and E1-E7. Assume furthermore, that for every consumer* i *and every* $E \in \mathcal{PF}^i$*, there exists a continuous function* $r^{i,E} : \Delta^{\#[L]-1} \to \overset{\circ}{\Delta}^{\#[E \in \mathcal{PF} : E \subset \mathcal{PF}^i(\omega)]-1}$ *such that consumer* i*'s belief function is consistent with the function* $\tilde{r}^i : \overset{\circ}{\Delta}^{\#[L]-1} \times \mathcal{PF}^i \to \mathbf{R}_+^{\#\mathcal{PF}}$ *defined by*

$$\tilde{r}^i_{E'}(p, E) = \begin{cases} \sum_{l \in L : \mathcal{L}(l) \subset E'} p_l e^i_l + r^{i,E}_{E'}(p) * [\sum_{l \in L : \mathcal{L}(l) \subset \Omega \setminus E} p_l e^i_l] & \text{if } E' \subset E \\ 0 & \text{otherwise} \end{cases}$$

Then in every state ω*, every state-*ω *perfect competitive equilibrium for economy* $\mathcal{E}$ *is both efficient and informationally efficient.*

13 Motivating Examples

In this section, we use several examples in order to develop our notion of efficiency, our solution concepts, and assumptions on beliefs under which (a) equilibria exist; and (b) they satisfy our notion of efficiency. Throughout this paper, several assumptions will be made about the rest of the consumer characteristics. They will consistently include A1,B1,C1, and D1-D7. In economies without differential information, the assumptions on consumer characteristics are usually stated for each consumer separately. Our goal here is to develop assumptions very much in the same spirit. The one dependence on other consumers that we will allow is that each consumer knows what the pooled information partition would look like. This said, we turn to our first example.

Example 1. Suppose that there are 2 consumers $M = \{1, 2\}$, 1 commodity $L = \{1\}$, and two states $\Omega = \{\omega_1, \omega_2\}$. Suppose furthermore that $PF^1 = \{\{\omega_1\}, \{\omega_2\}\}$ and $PF^2 = \{\{\omega_1, \omega_2\}$. If the true state realized is ω_1, it can be shown that for strictly positive prices, consumer 1 has a unique optimal bundle given by $x^{1*}(\omega_1|p) = e^1(\omega_1) + \frac{p(\omega_2}{p(\omega_1)} e^1(\omega_2)$, $x^{1*}(\omega_2|p) = 0$. Furthermore, it can be shown that for given beliefs, consumer 2's optimal bundle would be unique as long as prices are strictly positive. Let us denote consumer 2's optimal bundle by x^{*2}.

Suppose for a second that we defined equilibrium prices to be the prices at which (i) both consumers were in equilibrium given their beliefs and (ii) both markets balanced. There are two properties that it would seem desirable for equilibria to possess in this instance. The first would be that the commodities are allocated efficiently in the true state (given that there is only one commodity and preferences in fact are monotonic in the true state for consumer 1, this would indeed be satisfied in the present example for any such equilibrium). A second property relates to the informational efficiency of markets. It would seem to be desirable that in any equilibrium occurring in state ω_1, the uniformed consumer behaved as if he new the true state to be ω_1. This would require that in any equilibrium, his only expenditure would be on state ω_1. Let us see if there would be any assumptions on his beliefs that would guarantee this property.

Suppose $x^{2*}(\omega_1|p) < e^2(\omega_1)$. Then if $\frac{p(\omega_2)}{p(\omega_1)}e^1(\omega_2) = e^2(\omega_1) - x^{2*}(\omega_1|p)$, p would in fact be an equilibrium in state ω_1. By a similar token, it could be shown that if $x^{2*}(\omega_1|p) > e^2(\omega_1)$, p could be an equilibrium price in state ω_2 for some strictly positive endowment of consumer 1. Hence the only demand function for which neither one of these could occur is the demand function where $x^{2*}(\omega_1|p) = e^2(\omega_1)$, $x^{2*}(\omega_2|p) = e^2(\omega_2)$. The only beliefs ruling out strictly positive informationally inefficient prices would hence be the beliefs that generate such a demand function. Suppose that the belief function was such that it generated such a demand function and that furthermore the beliefs when one of the prices was zero was the limiting beliefs of a sequence of beliefs associated with strictly positive prices that converge to that price. Then in this example, the optimal bundles at that price would include the limiting bundle of the sequence of unique bundles associated with the strictly positive prices (note that there could be other optimal bundles as well in the limit). It is straightforward to verify that if the beliefs of consumer 2 satisfied these properties, the unique equilibrium price in state ω_1 would be assign a zero price to the state ω_2 commodity and the unique equilibrium price in ω_2 would assign a zero price to the ω_1 commodity. Hence if the beliefs satisfied these properties, it would be the case that in every state, every competitive equilibrium would be informationally efficient. It is straightforward to verify that if requiring consumer 2 to have the expenditure on a state equal to the income in that state would work if there was more than one commodity in each of the two states. That is, for any strictly positive prices, consumer 2's beliefs should yield a demand function satisfying:

$$p(\omega_1)x^{*2}(\omega_1|p) = p(\omega_1)e^2(\omega_1)$$

and for every price, the beliefs should be the limit of a sequence of beliefs associated with strictly positive prices converging to that price.

One may ask what an explicit formula for consumer 2's beliefs would look like. For general preferences, the belief function generating the desired demand function is not unique. If the preferences of consumer 2 were representable by a differentiable von Neumann-Morgenstern expected utility function, however, the beliefs in this example would in fact uniquely be given by the following formula:

$$b^2(\omega_1, p, \{\omega_1, \omega_2\}) = \frac{\frac{p(\omega_1)}{\frac{\partial u^2(e^2(\omega_1)|\omega_1)}{\partial x^2(\omega_1)}}}{\frac{p(\omega_1)}{\frac{\partial u^2(e^2(\omega_1)|\omega_1)}{\partial x^2(\omega_1)}} + \frac{p(\omega_2)}{\frac{\partial u^2(e^2(\omega_2)|\omega_2)}{\partial x^2(\omega_2)}}}$$

Example 2. Continue to assume that there are 2 consumers and 1 commodity in each state, but let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $PF^1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$, and $PF^2 = \{\{\omega_1, \omega_3\}, \{\omega_2\}\}$. This in turn implies that $PF = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$.

Suppose that prices are strictly positive and that the true state is ω_1. It would seem natural that one should allow consumer 2 to spend the income from state ω_1 (ω_3) plus any strictly positive fraction of the income from ω_2 on ω_1 (ω_3). That is, one should allow:

$$p(\omega_1)x^{*2}(\omega_1|p) = p(\omega_1)e^2(\omega_1) + tp(\omega_2)e^2(\omega_2)$$

$$p(\omega_1)x^{*2}(\omega_3|p) = p(\omega_3)e^2(\omega_3) + (1-t)p(\omega_2)e^2(\omega_2)$$

where $0 < t < 1$.

Suppose now that consumer 1's demand function satisfied:

$$p(\omega_1)x^{*1}(\omega_1|p) = p(\omega_1)e^1(\omega_1) + (1-r)p(\omega_3)e^1(\omega_3)$$

$$p(\omega_1)x^{*1}(\omega_2|p) = p(\omega_2)e^2(\omega_3) + rp(\omega_3)e^1(\omega_3)$$

where $r > 1$ and $p(\omega_1)x^{*1}(\omega_1|p) \geq 0$.

If this is the case, it can be shown that p is a competitive equilibrium in state ω_1 when:

$$e^2(\omega_2) = r\frac{p(\omega_3)}{p(\omega_2)}e^1(\omega_3)$$

and $t = \frac{(r-1)}{r}$.

Hence the only assumption that would rule this out would be that beliefs generated a demand function satisfying:

$$p(\omega_1)x^{*1}(\omega_1|p) \geq p(\omega_1)e^1(\omega_1)$$

for any strictly positive prices. If one on the other hand considers the economy where instead $\overline{PF}^2 = \{\{\omega_2, \omega_3\}, \{\omega_1\}\}$ one would need to assume that

$$p(\omega_2)x^{*1}(\omega_2|p) \geq p(\omega_2)e^1(\omega_2)$$

for any strictly positive. Since PF is the same in both instances, one would need to assume both unless one makes the belief function depend also on consumer 2's information partition. By a symmetric argument, one can motivate the same type of assumptions for consumer 2.

Suppose now that

$$p(\omega_1)x^{*1}(\omega_1|p) = p(\omega_1)e^1(\omega_1)$$

and that

$$p(\omega_1)x^{*2}(\omega_1|p) = p(\omega_1)e^2(\omega_1)$$

In this instance, one can show that p is an equilibrium in state ω_1 when

$$e^2(\omega_2) = \frac{p(\omega_3)e^1(\omega_3)}{p(\omega_2)}$$

Hence we need to assume that either

$$p(\omega_1)x^{*1}(\omega_1|p) > p(\omega_1)e^1(\omega_1)$$

or

$$p(\omega_1)x^{*2}(\omega_1|p) > p(\omega_1)e^2(\omega_1)$$

The most natural assumption would seem to assume both. By symmetry, one can then motivate the assumption that the belief function for consumer 1 should generate a demand function for which:

$$p(\omega_1)x^{*1}(\omega_1|p) = p(\omega_1)e^1(\omega_1) + t(p)p(\omega_3)e^1(\omega_3)$$

$$p(\omega_1)x^{*1}(\omega_2|p) = p(\omega_2)e^1(\omega_2) + (1 - t(p))p(\omega_3)e^1(\omega_3)$$

where $0 < t(p) < 1$ for all strictly positive prices.

This type of assumption hence would preclude strictly positive prices as equilibrium prices in state ω_1. The reason for this is that the value of excess demand for the state ω_1 would be strictly positive for any strictly positive prices.

Suppose that we furthermore imposed that the beliefs at an arbitrary price p should be the limit of a sequence of beliefs associated with strictly positive prices. Would the value of excess demand still be strictly positive in state ω_1 for any price assigning a positive price to a state other than state ω_1. Unfortunately this is not the case. The following two assumption, however, would guarantee the desired result.

(i) There is a continuous function $t : \Delta^2 \to (0, 1)$ such that for any strictly positive prices the beliefs generate a demand function for which:

$$p(\omega_1)x^{*1}(\omega_1|p) = p(\omega_1)e^1(\omega_1) + t(p)p(\omega_3)e^1(\omega_3)$$

$$p(\omega_1)x^{*1}(\omega_2|p) = p(\omega_2)e^1(\omega_2) + (1 - t(p))p(\omega_3)e^1(\omega_3)$$

(ii) For every price p, the beliefs are the limiting beliefs of a sequence of beliefs associated with a sequence of strictly positive prices that converges to p.

We use this type of assumption in a later section to prove one of our Welfare Theorems.

14 Proofs

14.1 Theorem 1

Proof of Theorem 1: Existence. We will show that the function given in our theorem satisfies all four axioms.

Axiom A1: Consider any $p \in \mathbf{R}^{\#[\mathbf{L}]}{}_{+}$. Then

$$\sum_{E' \in \mathcal{PF}} I^{i,E}_{E'}(p) = \sum_{E' \in \mathcal{PF}: E' \subset E} I^{i,E}_{E'}(p)$$

$$= \sum_{E' \in \mathcal{PF}: E' \subset E} [\sum_{l \in L: \mathcal{L}(l) \subset E'} p_l e^i_l + \sum_{l \in L: \mathcal{L}(l) \subset \Omega \backslash E} \frac{p_l e^i_l}{\#\{E'' \in \mathcal{PF} | E'' \subset E\}}]$$

$$= \sum_{E' \in \mathcal{PF}: E' \subset E} \sum_{l \in L: \mathcal{L}(l) \subset E'} p_l e^i_l + \sum_{E' \in \mathcal{PF}: E' \subset E} \sum_{l \in L: \mathcal{L}(l) \subset \Omega \backslash E} \frac{p_l e^i_l}{\#\{E'' \in \mathcal{PF} | E'' \subset E\}}$$

$$= \sum_{l \in L: \mathcal{L}(l) \subset E} p_l e^i_l + \sum_{l \in L: \mathcal{L}(l) \subset \Omega \backslash E} \sum_{E' \in PF: E' \subset E} \frac{p_l e^i_l}{\#\{E'' \in \mathcal{PF} | E'' \subset E\}}$$

$$= \sum_{l \in L: \mathcal{L}(l) \subset E} p_l e^i_l + \sum_{l \in L: \mathcal{L}(l) \subset \Omega \backslash E} p_l e^i_l$$

$$= \sum_{l \in L} p_l e^i_l$$

as required.

Axiom A2: Follows immediately from how $I^{i,E}$ is defined.

Axiom A3: Let $\sigma \in G_{\#\mathcal{PF}}$ be any linear order with the property that $E' \subset E$ if an only if $\sigma(E') \subset E$. Suppose furthermore that for every $E' \in \mathcal{PF}$:

$$\sum_{l \in L: \mathcal{L}(l) \subset E'} p_l e^i_l = \sum_{l \in L: \mathcal{L}(l) \subset \sigma(E')} p'_l e^i_l$$

Pick any $E' \in \mathcal{PF}$. If $E' \subset \Omega \backslash E$ then it immediately follows that $I^{i,E}_{\sigma(E')}(p') = 0 = I^{i,E}_{E'}(p)$. If $E' \subset E$ then,

$$I^{i,E}_{\sigma(E')}(p') = \sum_{l \in L: \mathcal{L}(l) \subset \sigma(E')} p'_l e^i_l + \sum_{l \in L: \mathcal{L}(l) \subset \Omega \backslash E} \frac{p'_l e^i_l}{\#\{E'' \in \mathcal{PF} | E'' \subset E\}}$$

$$= \sum_{l \in L:\mathcal{L}(l) \subset \sigma(E')} p'_l e^i_l + \sum_{E''' \in \mathcal{PF}:\sigma(E''') \subset \Omega \backslash E} \sum_{l \in L:\mathcal{L}(l) \subset \sigma(E''')} \frac{p'_l e^i_l}{\#\{E'' \in \mathcal{PF} | E'' \subset E\}}$$

$$= \sum_{l \in L:\mathcal{L}(l) \subset E'} p_l e^i_l + \sum_{E''' \in \mathcal{PF}:\sigma(E''') \subset \Omega \backslash E} \sum_{l \in L:\mathcal{L}(l) \subset E'''} \frac{p_l e^i_l}{\#\{E'' \in \mathcal{PF} | E'' \subset E\}}$$

$$= \sum_{l \in L:\mathcal{L}(l) \subset E'} p_l e^i_l + \sum_{E''' \in \mathcal{PF}:E''' \subset \Omega \backslash E} \sum_{l \in L:\mathcal{L}(l) \subset E'''} \frac{p_l e^i_l}{\#\{E'' \in \mathcal{PF} | E'' \subset E\}}$$

$$= \sum_{l \in L:\mathcal{L}(l) \subset E'} p_l e^i_l + \sum_{l \in L:\mathcal{L}(l) \subset \Omega \backslash E} \frac{p_l e^i_l}{\#\{E'' \in \mathcal{PF} | E'' \subset E\}}$$

$$= I^{i,E}_{E'}(p)$$

as required.

Axiom A4: Pick any $E' \in \mathcal{PF}$. Suppose $\sum_{l \in L:\mathcal{L}(l) \subset E'} p_l = \sum_{l \in L:\mathcal{L}(l) \subset \Omega \backslash E} p_l = 0$. Then:

$$I^{i,E}_{E'}(p) = \sum_{l \in L:\mathcal{L}(l) \subset E'} p_l e^i_l + \sum_{l \in L:\mathcal{L}(l) \subset \Omega \backslash E} \frac{p_l e^i_l}{\#\{E'' \in \mathcal{PF} | E'' \subset E\}}$$

$$= \sum_{l \in L:\mathcal{L}(l) \subset E'} 0 * e^i_l + \sum_{l \in L:\mathcal{L}(l) \subset \Omega \backslash E} \frac{0 * e^i_l}{\#\{E'' \in \mathcal{PF} | E'' \subset E\}}$$

$$= 0$$

as required.

Axiom A5: Is easily seen to be satisfied due to the functional form of $I^{i,E}$.

Uniqueness: For every $l \in L$, define p^{*l} by:

$$p^{*l}_{l'} = \left\{ \begin{array}{ll} 1 & \text{if } l' = l) \\ 0 & \text{otherwise} \end{array} \right.$$

Then for every $E' \in \mathcal{PF} : E' \subset E$,

$$I^{i,E}_{E'}(p^{*l}) = \begin{cases} e^i_l(\omega) & \text{if } \mathcal{L}(l) \subset E' \\ 0 & \text{if } \mathcal{L}(l) \subset E \setminus E' \\ \frac{e^i_l}{\#\{E'' \in PF | E'' \subset E\}} & \text{otherwise} \end{cases}$$

Indeed if $\mathcal{L}(l) \subset E$ then $E'' \in \mathcal{PF} \setminus \{E'\}$ for which $E'' \in E$, it follows from axiom A4 that $I^{i,E}_{E''}(p^{*l}) = 0$. Axioms A1 and A2 combined then implies that $I^{i,E}_{E'}(p^{*l}) = e^i_l$. If $\mathcal{L}(l) \subset E \setminus E'$, it follows from axiom A4 that $I^{i,E}_{E'}(p^{*l)}) = 0$. Finally, suppose $\mathcal{L}(l) \subset \Omega \setminus E$. Then it follows from axiom A3 that for any two E', E'' in $\mathcal{PF}$ for which $E' \subset E$ and $E'' \subset E$, $I^{i,E}_{E'}(p^{*l}) = I^{i,E}_{E''}(p^{*l})$. From Axioms A1 and A2 it then follows that $I^{i,E}_{E'}(p^{*l}) = \frac{e^i_l(\omega)}{\#\{E'' \in \mathcal{PF} | E'' \subset E\}}$.

We also note that for any $E' \in \mathcal{PF} : E' \subset \Omega \setminus E$, $I^{i,E}_{E'}(p^{*l}) = 0$.

We have hence shown that for any $l \in L$, the value of $I^{i.E}(p^{*l})$ does not depend on the particular choice of $I^{i,E}$. We now note that the collection of vectors $\{p^{*l}\}_{l) \in L}$ forms a linearly independent basis for the space $\mathbf{R}^{\#[L]}_+$. The linearity of an *income* (Axiom $A5$) then implies that for any $p \in \mathbf{R}^{\#[L]}_+$, the income allocation $I^{i,E}(p)$ does not depend on the specific choice of $I^{i,E}$. Q.E.D.

14.2 Theorem 2

Lemma 11. *Pick any* $p^0 \in \mathbf{R}^{\#[L]}_+$. *Then the solution* $\{p(t)\}_t$ *of the system satisfies the property that for every* $t \in \mathbf{R}_+$, $p(t) \geq \mathbf{0}$.

Proof of Lemma 1: Suppose the solution is not non-negative for every $t \in \mathbf{R}_+$. Define $t^* := \sup\{r \in \mathbf{R} | \forall t \in [0,r] : p(t) \geq \mathbf{0}\}$. Because the solution is not nonnegative for every $t \in \mathbf{R}_+$, t^* is a real non-negative number. Now, define

$$K := \left\{ l \in L \;\middle|\; \begin{array}{l} (i) p_l(t^*) = 0 \\ (ii) \exists \epsilon > 0 : \forall t \in (t^*, t^* + \epsilon] : \frac{d}{dt} p_l(t) < 0 \end{array} \right\}.$$

We claim that the set K is non-empty. Suppose not. Then for every $l \in L$, either $p_l(t^*) > 0$ or $\exists \epsilon > 0 : \forall t \in (t^*, t^* + \epsilon] : \frac{d}{dt} p_l(t) \geq 0$. It is easily seen that in either case, there exists $\epsilon_l > 0$ such that for all $t \in (t^*, t^* + \epsilon_l)$, $p_l(t) \geq 0$. Define $\epsilon^* := \min\{\epsilon_l | l \in L\}$. Note that $\epsilon^* > 0$. Then for all $t \in (t^*, t^* + \epsilon^*)$, $p(t) \geq \mathbf{0}$. But this contradicts the definition of t^*. Hence K is nonempty.

Now, for each $l \in K$, define

$$t_l := \sup \left\{ t \in \mathbf{R} \;\middle|\; \begin{array}{l} (i) p_l(t) < 0 \\ (ii) \forall t' \in (t^*, t) : \\ (a) \frac{d}{dt} p_l(t') < 0; and \\ (b) \forall l' \in L \setminus K : p_{l'}(t') \geq 0 \end{array} \right\}$$

as long as the supremum is well defined. Otherwise define $t_l := \infty$.

From the definition of K, it follows that $t_l > t^*$. Define $t_1 := \inf\{t_\omega | \omega \in K\}$ as long as the infimum is well defined. Otherwise define $t_1 := \infty$. Clearly, $t_1 > t^*$.

Now, pick any $t \in (t^*, t_1)$. Then for every $l \in K$, $p_l(t) < 0$ and $\frac{d}{dt} p_l(t) < 0$ while for every $l \in L \setminus K$, $p_l(t) \geq 0$.

Then,

$$\sum_{l\in K}\frac{d}{dt}p_l(t) = [\sum_{l\in K}\sum_{i\in M:\mathcal{L}(l)\subset\mathcal{PF}^i(\omega^*)}\sum_{l'\in L:\mathcal{L}(l')\subset\Omega\setminus\mathcal{PF}^i(\omega^*)}\frac{p_{l'}(t)e^i_{l'}}{\#PF^i(\omega^*)}]-[\sum_{l\in K}\sum_{i\in M:\mathcal{L}(l)\subset\Omega\setminus\mathcal{PF}^i(\omega^*)}p_l(t)e^i_l]$$

$$=\sum_{i\in M}[[\sum_{l\in K:\mathcal{L}(l)\subset\in\mathcal{PF}^i(\omega^*)}\sum_{l'\in L:\mathcal{L}(l')\subset\Omega\setminus\mathcal{PF}^i(\omega^*)}\frac{p_{l'}(t)e^i_{l'}}{\#\mathcal{PF}^i(\omega^*)}]-[\sum_{l\in K:\mathcal{L}(l)\subset\Omega\setminus\mathcal{PF}^i(\omega^*)}p_l(t)e^i_l]]$$

$$\geq\sum_{i\in M}[[\sum_{l\in K:\mathcal{L}(l)\subset\mathcal{PF}^i(\omega^*)}\sum_{l'\in K:\mathcal{L}(l')\subset K\setminus\mathcal{PF}^i(\omega^*)}\frac{p_{l'}(t)e^i_{l'}}{\#PF^i(\omega^*)}]-[\sum_{l\in K:\mathcal{L}(l)\subset K\setminus\mathcal{PF}^i(\omega^*)}p_l(t)e^i_l]]$$

$$=\sum_{i\in M}\sum_{l\in K:\mathcal{L}(l)\subset K\setminus\mathcal{PF}^i(\omega^*)}[[\sum_{l'\in K:\mathcal{L}(l)\in K\cap\mathcal{PF}^i(\omega^*)}\frac{p_{l'}(t)e^i_{l'}}{\#\mathcal{PF}^i(\omega^*)}]-[p_l(t)e^i_l]]$$

$$=\sum_{i\in M}\sum_{l\in K:\mathcal{L}(l)\subset K\setminus\mathcal{PF}^i(\omega^*)}(-1)p_l(t)e^i_l[1-\frac{\#[K\cap\mathcal{PF}^i(\omega^*)]}{\#[\mathcal{PF}^i(\omega^*)]}]$$

$$\geq 0$$

But we know from how t was selected that $\sum_{l\in K}\frac{d}{dt}p_l(t)<0$. A contradiction. Q.E.D.

Lemma 12. *For every* $t\in\mathbf{R}_+$, $\sum_{l\in L}p_l(t)=\sum_{l\in L}p^0_l$.

Proof of Lemma 2: It suffices to show that $\sum_{l\in L}\frac{d}{dt}p_l(t)=0$. But under the present assumptions, this is an immediate consequence of Walras' Law.

Claim 19. *Pick any* $p^0\in\mathbf{R}^{\#[L]}_+$. *Then the solution* $\{p(t)\}_t$ *satisfies the property that* $\inf_{t\in[0,\infty)}\sum_{l\in L:\mathcal{L}(l)\subset\Omega\setminus\{\omega^*\}}p_l(t)=0$.

Proof of Claim 6: From lemmas 1 and 2, it follows that $\inf_{t\in[0,\infty)}\sum_{l\in L:\mathcal{L}(l)\subset\Omega\setminus\{\omega^*\}}p_l(t)\geq 0$. Suppose $\inf_{t\in[0,\infty)}\sum_{l\in L:\mathcal{L}(l)\subset\Omega\setminus\{\omega^*\}}p_l(t)=\delta>0$. Let

$$t^*>\frac{\sum_{l\in L:\mathcal{L}(l)\subset\Omega\setminus\{\omega^*\}}p_l(0)}{\delta[\min_{l\in L:\mathcal{L}(l)\subset\Omega\setminus\{\omega^*\}}\sum_{i\in M:\mathcal{L}(l)\subset\Omega\setminus\mathcal{PF}^i(\omega^*)}\frac{e^i_l}{\#\mathcal{PF}^i(\omega^*)}]}$$

Denote by l^* the unique claim for which $\mathcal{L}(l^*)\subset\{\omega^*\}$.
Then,

$$p_{l^*}(t^*) = p_{l^*}(0) + \int_0^{t^*} \frac{d}{dt} p_{l^*}(t)dt$$

$$= p_{l^*}(0) + \int_0^{t^*} [\sum_{l \in L : \mathcal{L}(l) \subset \Omega \backslash \{\omega^*\}} p_l(t)[\sum_{i \in M : \mathcal{L}(l) \subset \Omega \backslash \mathcal{PF}^i(\omega^*)} \frac{e_l^i}{\#\mathcal{PF}^i(\omega^*)}]]dt$$

$$\geq p_{l^*}(0) + \min_{l \in L : \mathcal{L}(l) \subset \Omega \backslash \{\omega^*\}} [\sum_{i \in M : \mathcal{L}(l) \subset \Omega \backslash \mathcal{PF}^i(\omega^*)} \frac{e_l^i}{\#\mathcal{PF}^i(\omega^*)}] \int_0^{t^*} [\sum_{l' \in L : \mathcal{L}(l') \subset \Omega \backslash \{\omega^*\}} p_{l'}(t)dt$$

$$\geq p_{l^*}(0) + \min_{l \in L : \mathcal{L}(l) \subset \Omega \backslash \{\omega^*\}} [\sum_{i \in M : \mathcal{L}(l) \subset \Omega \backslash \mathcal{PF}^i(\omega^*)} \frac{e_l^i}{\#\mathcal{PF}^i(\omega^*)}] t^* \delta$$

$$> p_{l^*}(0) + \sum_{l \in L : \mathcal{L}(l) \subset \Omega \backslash \{\omega^*\}} p_l(0)$$

$$= \sum_{l \in L} p(\omega)(0)$$

.

But Lemma's 1 and 2 imply that $p_{l^*}(t^*) \leq \sum_{l \in L} p_l(0)$. A contradiction. Q.E.D.

Proof of Theorem: Denote by l^* the unique claim for which $\mathcal{L}(l^*) \subset \{\omega^*\}$. From Lemma 2 and Claim 6, it follows that

$$\sup_{t \in [0,\infty)} p_{l^*}(t) = \sum_{l \in L} p_l(0) - \inf_{t \in [0,\infty)} \sum_{l \in L : \mathcal{L}(l) \subset \Omega \backslash \{\omega^*\}} p_l(t) = \sum_{l \in L} p_l(0)$$

We claim that this in turn implies that $\lim_{t \to \infty} p_{l^*}(t) = \sum_{l \in L} p_l(0)$. From Lemmas 1 and 2, it follows that for every $t \in [0, \infty)$, $p_{l^*}(t) \leq \sum_{l \in L} p_l(0)$. Suppose $p_{l^*}(t)$ does not converge to $\sum_{l \in L} p_l(0)$. Then there exist $\epsilon > 0$ such that for every $t \in [0, \infty)$, there exist $t' \geq t$ such that $p_{l^*}(t') \leq \sum_{l \in L} p_l(0) - \epsilon$. But it follows from Lemma 1 and the definition of the system that $p_{l^*}(t)$ is non-decreasing. Hence for every $t \in [0, \infty)$, $p_{l^*}(t) \leq \sum_{l \in L} p_l(0) - \epsilon$. But this contradicts $\sup_{t \in [0,\infty)} p_{l^*}(t) = \sum_{l \in L} p_l(0)$. Hence $\lim_{t \to \infty} p_{l^*}(t) = \sum_{l \in L} p_l(0)$.

This combined with Lemmas 1 and 2 then implies that for every $l \in L \backslash \{l^*\}$, $\lim_{t \to \infty} p_l(t) = 0$. Hence our Theorem is established. Q.E.D.

14.3 Theorem 3

Lemma 13. *Let $\mathcal{E} := \{(\Omega, \mathcal{F}, \mu), \{X^i, e^i, \succeq^i, \mathcal{F}^i, b^i\}_{i\in M}\}$ be a differential information economy satisfying assumptions A1,B1,C1, and D1-D7. Let ω be the true state. Assume furthermore, that for every consumer i, the belief function satisfies the property that there exists a continuous function $r^i : \overset{\circ}{\Delta}^{\#[\Omega]*\#[L]-1} \to \Delta^{\#[E\in PF:E\subset PF^i(\omega)]-1}$ such that for every $p \in \overset{\circ}{\Delta}^{\#[\Omega]*\#[L]-1}$ and every $E \in PF : E \subset PF^i(\omega)$, the unique optimal bundle $\hat{x}^{*i}$ associated with p given in Claim 1 satisfies the property that*

$$\sum_{\omega'\in E} p(\omega')\hat{x}^{*i}(\omega') = r^i(E) \sum_{\omega'\in\Omega} p(\omega')e^i(\omega')$$

Let $0 < \overline{\epsilon} < \frac{1}{\#\Omega\#L}$ and let $(\mathcal{E}, \epsilon)$ be any symmetric ϵ perturbed economy satisfying the property that $\forall(\omega, l) \in \Omega \times L$, $\epsilon(\omega, l) < \overline{\epsilon}$. Then the ϵ-perturbed economy has a competitive equilibrium $((\tilde{x}^{i*})_{i\in M}, p^*)$ satisfying the property that

$$\forall(i, \omega, l) \in M \times \Omega \times L : \tilde{x}^{i*}_l(\omega) < \frac{\sum_{(\omega',l')\in\Omega\times L} \sum_{i\in M} e^i_{l'}(\omega')}{[1 - (\#L - 1)\overline{\epsilon}]}$$

Proof: We divide the proof into three steps as follows:

Step 1: Define Y^i by... Show that $\forall(i, E) \in M \times PF$, the correspondence $\phi^i_E : \overset{\circ}{\Delta}^{\#[\Omega]*\#[L]-1} \times \mathbf{R}_+ \to \times_{i\in E} Y^i$ defined by

$$\phi^i_E(p, w) = \{\hat{x}^i_E \in \times_{\omega'\in E} Y^i | \sum_{\omega'\in E} p(\omega')\hat{x}^i_E(\omega') \leq w\}$$

is a non-empty-valued, convex-valued, upper semi-continous, and lower semi-continuous correspondence. The non-empty-valuedness, and convex-valuedness have been demonstrated before in the literature (see e.g.,). For $w > 0$, it is also well known that upper and lower semi-continuity holds as well. It remains to show that for all strictly positive prices, the correspondence is upper and lower semi-continuous at $w = 0$. Thus, we pick any $\overline{p}\ \overset{\circ}{\Delta}^{\#[\Omega]*\#[L]-1}$. Note that due to the fact that $\overline{p}$ is strictly positive, $\phi^i_E(\overline{p}, 0) = \{\mathbf{0}\}$. Now, pick any sequence $(p_k, w_k)_{k=1}^{\infty}$ of elements in $\overset{\circ}{\Delta}^{\#[\Omega]*\#[L]-1} \times \mathbf{R}_+$ for which $(p_k, w_k) \to (\overline{p}, 0)$. Let $(\hat{x}^i_{E,k})_{k=1}^{\infty}$ be any associated sequence of elements of $\hat{Y}^i_E$ satisfying the property that for every k, $\hat{x}^i_{E,k} \in \phi^i_E(p_k, w_k$. Then $0 \leq lim_{k\to\infty} \sum_{\omega'\in E} p_k(\omega')\hat{x}^i_{E,k} \leq lim_{k\to\infty} w_k = 0$. But $lim_{k\to\infty} p_k = \overline{p} \gg 0$ then implies $lim_{k\to\infty} \hat{x}^i_{E,k} = \mathbf{0}$. This in turn implies the required upper and lower semi-continuity.

Step 2: Construct an Abstract economy as follows:

Define $N := \{0\} \cup \{(i, E) | (i, E) \in M \times PF\}$. For $(i, E) \in M \times PF$, the set $Y^i_E := \times_{\omega'\in E} Y^i$ is the strategy set. For player 0, the strategy set is now $Y^0 := \{p \in \overset{\circ}{\Delta}^{\#[\Omega]*\#[L]-1} | \forall(l, \omega') \in L \times \Omega : p_l(\omega') \geq \epsilon\}$. It is well known that under the present assumptions, consumer i's preference relation is representable by a continuous von Neumann-Morgenstern expected utility function $v^i : X^i \times \Omega \to \mathbf{R}$. Strict risk aversion of $\succeq^i$ is equivalent to strict concavity of v^i. Given $(p, (\hat{y}^i_E)_{(i,E)\in M\times PF}) \in Y^0 \times (\times_{(i,E)\in M\times PF} \hat{Y}^i_E)$, define $F^i_E(p, (\hat{y}^i_E)_{(i,E)\in M\times PF}) :=$

$\phi^i_E(p, r^i_E \sum_{\omega' \in \Omega \setminus PF^i(\omega)} p(\omega') e^i(\omega'))$ if $E \in PF^i(\omega)$; $F^i_E(p, (\hat{y}^i_E)_{(i,E) \in M \times PF}) := \phi^i_E(p, \mathbf{0})$ otherwise; $F^0 := Y^0$; $u^i_E(p, (\hat{y}^i_E)_{(i,E) \in M \times PF}, \zeta^i_E) := \sum_{\omega' \in E} \frac{\mu(\omega')}{\sum_{\omega'' \in E}} v^i(\zeta^i_E(\omega'), \omega')$ for every $(i, E) \in M \times PF$; $u^0((p, (\hat{y}^i_E)_{(i,E) \in M \times PF}, \zeta^0) := \sum_{\omega' \in \Omega} \zeta^0(\omega') \sum_{i \in E} (\hat{x}^i_{PF^i(\omega')}(\omega') - e^i(\omega'))$. The abstract economy $\{Y^0, \{Y^i_E\}_{(i,E) \in M \times PF}, F^0, \{F^i_E\}_{(i,E) \in M \times PF}, u^0, \{u^i_E\}_{(i,E) \in M \times PF}\}$ satisfies all the assumptions of Arrow's social equilibrium existence theorem and hence it has a social equilibrium $(p^*, \{\hat{x}^{i*}_E\}_{(i,E) \in M \times PF}$.

Step 3: Show that the social equilibrium from step 2 is a state-ω competitive equilibrium for the ϵ-perturbed economy. We first show that any social equilibrium from step 2 satisfies the property that:

$$\forall (i, \omega', l') \in M \times \Omega \times L : \hat{x}^{i*}_{l'}(\omega') < \frac{\sum_{(\omega'', l'') \in \Omega \times L} \sum_{i \in M} e^i_{l''}(\omega'')}{[1 - (\#L - 1)\overline{\epsilon}]}$$

To see this, note that

$$\forall (l', \omega') \in L \times \Omega : (1 - \epsilon(\#[\Omega]\#[L] - 1)) \sum_{i \in M} (\hat{x}^{i*}_{PF^i(\omega'), l'} - e^i_{l'}(\omega')) + \epsilon \sum_{(l'', \omega'') \in L \times \Omega \setminus \{(l', \omega')\}} \sum_{i \in M} (\hat{x}^{i*}_{PF^i(\omega''), l''} - e^i_{l''}(\omega''$$

$$\leq Max_{(l^*, \omega^*) \in L \times \Omega} (1 - \epsilon(\#[\Omega]\#[L] - 1)) \sum_{i \in M} (\hat{x}^{i*}_{PF^i(\omega^*), l^*} - e^i_{l^*}(\omega^*)) + \epsilon \sum_{(l'', \omega'') \in L \times \Omega \setminus \{(l^*, \omega^*)\}} \sum_{i \in M} (\hat{x}^{i*}_{PF^i(\omega''), l''} - e^i_{l''}($$

$$= \sum_{\omega'' \in \Omega} p^*(\omega'') \sum_{i \in M} \sum_{i \in M} (\hat{x}^{i*}_{PF^i(\omega'')} - e^i(\omega'')) \leq 0$$

This in turn implies that

$$\forall (l', \omega') \in L \times \Omega : (1 - \epsilon(\#[\Omega]\#[L] - 1)) \sum_{i \in M} \hat{x}^{i*}_{PF^i(\omega'), l'}$$

$$\leq \epsilon \sum_{(l'', \omega'') \in L \times \Omega \setminus \{(l', \omega')\}} \sum_{i \in M} e^i_{l''}(\omega'') - \epsilon \sum_{(l'', \omega'') \in L \times \Omega \setminus \{(l', \omega')\}} \sum_{i \in M} \hat{x}^{i*}_{PF^i(\omega''), l''} + (1 - \epsilon(\#[\Omega]\#[L] - 1)) \sum_{i \in M} e^i_{l'}(\omega')$$

$$\leq \sum_{(l'', \omega'') \in L \times \Omega} \sum_{i \in M} e^i_{l''}(\omega'')$$

which implies that $\forall (j, \omega', l') \in M \times \Omega \times L$:

$$\hat{x}^{j*}_{l'}(\omega') \leq \sum_{i \in M} \hat{x}^{i*}_{PF^i(\omega'), l'} \leq \frac{\sum_{(l'', \omega'') \in L \times \Omega} \sum_{i \in M} e^i_{l''}(\omega'')}{(1 - \epsilon(\#[\Omega]\#[L] - 1))}$$

$$< \frac{\sum_{(l'', \omega'') \in L \times \Omega} \sum_{i \in M} e^i_{l''}(\omega'')}{(1 - \overline{\epsilon}(\#[\Omega]\#[L] - 1))}$$

as required.

Suppose now that the social equilibrium was not a competitive equilibrium. Then there exists some consumer for which the unique bundle given in Claim 1 is strictly better than $\hat{x}^{i*}$. From the definition of the social equilibrium, it follows that the unique bundle lies outside of $\times_{\omega' \in \Omega} Y^i$. Consider any strictly convex combination of this bundle and $\hat{x}^{i*}$. Then this bundle is also strictly better than $\hat{x}^{i*}$. Our above result then implies that there exists such a strictly convex combination inside $\times_{\omega' \in \Omega} Y^i$. But this contradicts the definition of the social equilibrium. Q.E.D.

Proof of Theorem 1: Consider a sequence of symmetric ϵ-perturbed economies satisfying the properties of the lemma and the property that ϵ converges to zero. Pick any sequence of state-ω competitive equilibria associated with the ϵ-perturbed economies. A consequence of the lemma is that this sequence is contained in a compact set and hence has a convergent subsequence. Consider any subsequence of ϵ-perturbed economies for which the competitive equilibria converges. It is straightforward to verify that the limiting point of the sequence of state-ω competitive equilibria associated with this subsequence in fact is a perfect state-ω competitive equilibrium. Q.E.D.

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Proof of Theorem 3: Consider any perfect state-ω competitive equilibrium. Then there exists a sequence of ϵ-perturbed economies and an associated sequence of state-ω competitive equilibria that converges to the perfect state-ω competitive equilibrium. Pick any such sequence $(p^*_k, \{x^{i*}_{E,k}\}_{(i,E) \in M \times \mathcal{PF}})_{k=1}^{\infty}$. Then the limit of the values of excess demands for every $E \in \mathcal{PF}$ must equal zero. In particular,

$$lim_{k \to \infty} \sum_{i \in M} \sum_{\omega' \in PF(\omega)} p^*_k(\omega')(\hat{x}^{i*}_{PF(\omega'),k}(\omega') - e^i(\omega')) =$$

$$= \lim_{k \to \infty} \sum_{i \in M} r^i_{PF(\omega)}(p^*_k) \sum_{\omega'' \in \Omega \setminus PF^i(\omega)} p^*_k(\omega'') e^i(\omega'') = 0$$

This can only happen if for every $\omega' \in \Omega \setminus PF(\omega)$, $lim_{k \to \infty} p^*_k(\omega') = 0$. Suppose now that there exists some consumer for which $b^{*i}(PF(\omega)) \neq 1$. Consider any $E \in PF \setminus PF(\omega)$ for which $b^*(PF(\omega)) > 0$. Then non-satiation implies that $\hat{x}^{i*}$ can not be optimal for consumer i which contradicts the definition of the perfect equilibrium. Hence $b^{*i}(PF(\omega)) = 1$. From this and the first fundamental Theorem of Welfare economics, efficiency then follows. Q.E.D.

Proof of Theorem 2: Existence. We will show that the function given in our theorem satisfies all four axioms.

Axiom A1: Consider any $p \in \mathbf{R}_+^{\#[\Omega] * \#[L]}$. Then

$$\sum_{E' \in PF : E' \subset E} I^{i,E}_{E'}(p) = \sum_{E' \in PF : E' \subset E} [\sum_{\omega' \in E'} p(\omega') e^i(\omega') + \sum_{\omega \in \Omega \setminus E} \frac{p(\omega) e^i(\omega)}{\#\{E'' \in PF | E'' \subset E\}}]$$

$$= \sum_{E' \in PF : E' \subset E} \sum_{\omega' \in E'} p(\omega')e^i(\omega') + \sum_{E' \in PF : E' \subset E} \sum_{\omega \in \Omega \setminus E} \frac{p(\omega)e^i(\omega)}{\#\{E'' \in PF | E'' \subset E\}}$$

$$= \sum_{\omega' \in E} \sum_{E' \in PF : a) E' \subset E; b) \omega' \in E'} p(\omega')e^i(\omega') + \sum_{\omega \in \Omega \setminus E} \sum_{E' \in PF : E' \subset E} \frac{p(\omega)e^i(\omega)}{\#\{E'' \in PF | E'' \subset E\}}$$

$$= \sum_{\omega' \in E} p(\omega')e^i(\omega') + \sum_{\omega \in \Omega \setminus E} p(\omega)e^i(\omega)$$

$$= \sum_{\omega \in \Omega} p(\omega)e^i(\omega)$$

as required.

Axiom A2: LEt $\sigma \in G_{\#PF}$ be any linear order with the property that $E' \subset E$ if an only if $\sigma(E') \subset E$. Suppose furthermore that for every $E' \in PF$:

$$\sum_{\omega' \in E'} p(\omega')e^i(\omega') = \sum_{\omega' \in \sigma(E')} p'(\omega')e^i(\omega')$$

Then it follows that:

$$I^{i,E}_{\sigma(E')}(p') = \sum_{\omega' \in \sigma(E')} p'(\omega')e^i(\omega') + \sum_{\omega \in \Omega \setminus E} \frac{p'(\omega)e^i(\omega)}{\#\{E'' \in PF | E'' \subset E\}}$$

$$= \sum_{\omega' \in \sigma(E')} p'(\omega')e^i(\omega') + \sum_{E''' \in PF : \sigma(E''') \subset \Omega \setminus E} \sum_{\omega \in \sigma(E''')} \frac{p'(\omega)e^i(\omega)}{\#\{E'' \in PF | E'' \subset E\}}$$

$$= \sum_{\omega' \in E'} p(\omega')e^i(\omega') + \sum_{E''' \in PF : \sigma(E''') \subset \Omega \setminus E} \sum_{\omega \in E'''} \frac{p(\omega)e^i(\omega)}{\#\{E'' \in PF | E'' \subset E\}}$$

$$= \sum_{\omega' \in E'} p(\omega')e^i(\omega') + \sum_{E''' \in PF : E''' \subset \Omega \setminus E} \sum_{\omega \in E'''} \frac{p(\omega)e^i(\omega)}{\#\{E'' \in PF | E'' \subset E\}}$$

$$= \sum_{\omega' \in E'} p(\omega')e^i(\omega') + \sum_{\omega \in \Omega \setminus E} \frac{p(\omega)e^i(\omega)}{\#\{E'' \in PF | E'' \subset E\}}$$

$$= I^{i,E}_{E'}(p)$$

as required.

Axiom A3: Suppose $p(\Omega \setminus E) = \mathbf{0}$ and $p(E') = 0$. Then:

$$I^{i,E}_{E'}(p) = \sum_{\omega' \in E'} p(\omega')e^i(\omega') + \sum_{\omega \in \Omega \setminus E} \frac{p(\omega)e^i(\omega)}{\#\{E'' \in PF | E'' \subset E\}}$$

$$= \sum_{\omega' \in E'} \mathbf{0}e^i(\omega') + \sum_{\omega \in \Omega \setminus E} \frac{\mathbf{0}e^i(\omega)}{\#\{E'' \in PF | E'' \subset E\}}$$

$$= 0$$

as required.

Axiom A4: Is easily seen to be satisfied due to the functional form of $I^{i,E}$.

Uniqueness: For every $(\omega', l') \in \Omega \times L$, define $p^{*(\omega,l)}$ by:

$$p^{*(\omega,l)}_{l'}(\omega') = \begin{cases} 1 & \text{if } (\omega', l) = (\omega, l) \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$I^{i,E}_{E'}(p^{*(\omega,l)}) = \begin{cases} e^i_l(\omega) & \text{if } \omega \in E' \\ 0 & \text{if } \omega \in E \setminus E' \\ \frac{e^i_l(\omega)}{\#\{E'' \in PF | E'' \subset E\}} & \text{otherwise} \end{cases}$$

Indeed if $\omega \in E$ then for every $E'' \in PF \backslash \{E'\}$ for which $E'' \in E$, it follows from axiom A3 that $I^{i,E}_{E''}(p^{*(\omega,l)}) = 0$. Axiom A1 then implies that $I^{i,E}_{E'}(p^{*(\omega,l)}) = e^i_l(\omega)$. If $\omega \in E \backslash E'$, it follows from axiom A3 that $I^{i,E}_{E'}(p^{*(\omega,l)}) = 0$. Finally, suppose $\omega \notin E$. Then it follows from axiom A2 that for any two E', E'' in PF for which $E' \subset E$ and $E'' \subset E$, $I^{i,E}_{E'}(p^{*(\omega,l)}) = I^{i,E}_{E''}(p^{*(\omega,l)})$. From Axiom A1 it then follows that $I^{i,E}_{E'}(p^{*(\omega,l)}) = \frac{e^i_l(\omega)}{\#\{E'' \in PF | E'' \subset E\}}$.

We have hence shown that for any $(\omega', l') \in \Omega \times L$, the value of $I^{i.E}(p^{*(\omega',l')})$ does not depend on the particular choice of $I^{i,E}$. We now note that the collection of vectors $\{p^{*(\omega,l)}\}_{(\omega,l) \in \Omega \times L}$ forms a linearly independent basis for the space $\mathbf{R}^{\#[\Omega]*\#[L]}_+$. The linearity of an *income* (Axiom *A*4) then implies that for any $p \in \mathbf{R}^{\#[\Omega]*\#[L]}_+$, the income allocation $I^{i,E}(p)$ does not depend on the specific choice of $I^{i,E}$. Q.E.D.

Lemma 14. *Pick any* $p^0 \in \mathbf{R}^{\#[\Omega]*\#[L]}_+$. *Then the solution* $\{p(t)\}_t$ *given in Theorem 3 satisfies the property that for every* $t \in \mathbf{R}_+$, $p(t) \geq 0$.

Proof of Lemma 2: Suppose the solution is not non-negative for every $t \in \mathbf{R}_+$. Define $t^* := \sup\{r \ \in \mathbf{R} | \forall t \in [0,r] : p(t) \geq 0\}$. Because the solution is not nonnegative for every $t \in \mathbf{R}_+$, t^* is a real non-negative number. Now, define $K := \left\{\omega \in \Omega \,\middle|\, \begin{array}{l} (i)p(\omega)(t^*) = 0 \\ (ii)\exists\epsilon > 0 : \forall t \in (t^*, t^* + \epsilon] : \frac{d}{dt}p(\omega)(t) < 0 \end{array} \right\}$. We claim that the set K is nonempty. Suppose not. Then for every $\omega \in \Omega$, either $p(\omega)(t^*) > 0$ or $\exists\epsilon > 0 : \forall t \in (t^*, t^* + \epsilon] : \frac{d}{dt}p(\omega)(t) \geq 0$. It is easily seen that in either case, there exists $\epsilon_\omega > 0$ such that for all $t \in (t^*, t^* + \epsilon_\omega)$, $p(\omega)(t) \geq 0$. Define $\epsilon^* := \min\{\epsilon_\omega | \omega \in \Omega\}$. Note that $\epsilon^* > 0$. Then for all $t \in (t^*, t^* + \epsilon^*)$, $p(t) \geq 0$. But this contradicts the definition of t^*. Hence K is nonempty.

Now, for each $\omega \in K$, define

$$t_\omega := \sup\left\{ t \in \mathbf{R} \,\middle|\, \begin{array}{l} (i)p(\omega)(t) < 0 \\ (ii)\forall t' \in (t^*, t) : \\ (a)\frac{d}{dt}p(\omega)(t') < 0; and (b)\forall\omega' \in \Omega \setminus K : p(\omega')(t') \geq 0 \end{array} \right\}$$

From the definition of K, it follows that $t_\omega > t^*$. Define $t_1 := \inf\{t_\omega | \omega \in K\}$. Clearly, $t_1 > t^*$.

Now, pick any $t \in (t^*, t_1)$. Then for every $\omega \in K$, $p(\omega)(t) < 0$ and $\frac{d}{dt}p(\omega)(t) < 0$ while for every $\omega \in \Omega \setminus K$, $p(\omega)(t) \geq 0$.

Then,

$$\sum_{\omega \in K} \frac{d}{dt}p(\omega)(t) = [\sum_{\omega \in K} \sum_{i \in N : \omega \in PF^i(\omega^*)} \sum_{\omega' \in \Omega \setminus PF^i(\omega^*)} \frac{p(\omega')(t)e^i(\omega')}{\#PF^i(\omega^*)}] - [\sum_{\omega \in K} \sum_{i \in N : \omega \notin PF^i(\omega^*)} p(\omega)(t)e^i(\omega)]$$

$$= \sum_{i \in N} [[\sum_{\omega \in K : \omega \in PF^i(\omega^*)} \sum_{\omega' \in \Omega \setminus PF^i(\omega^*)} \frac{p(\omega')(t)e^i(\omega')}{\#PF^i(\omega^*)}] - [\sum_{\omega \in K : \omega \notin PF^i(\omega^*)} p(\omega)(t)e^i(\omega)]]$$

$$\geq \sum_{i \in N} [[\sum_{\omega \in K : \omega \in PF^i(\omega^*)} \sum_{\omega' \in K \setminus PF^i(\omega^*)} \frac{p(\omega')(t)e^i(\omega')}{\#PF^i(\omega^*)}] - [\sum_{\omega \in K \setminus PF^i(\omega^*)} p(\omega)(t)e^i(\omega)]]$$

$$= \sum_{i \in N} \sum_{\omega \in K \setminus PF^i(\omega^*)} [[\sum_{\omega' \in K \cap PF^i(\omega^*)} \frac{p(\omega')(t)e^i(\omega')}{\#PF^i(\omega^*)}] - [p(\omega)(t)e^i(\omega)]]$$

$$= \sum_{i \in N} \sum_{\omega \in K \setminus PF^i(\omega^*)} (-1)p(\omega)(t)e^i(\omega)[1 - \frac{\#[K \cap PF^i(\omega^*)]}{\#[PF^i(\omega^*)]}]$$

$$\geq 0$$

But we know from how t was selected that $\sum_{\omega \in K} \frac{d}{dt}p(\omega)(t) < 0$. A contradiction. Q.E.D.

Lemma 15. *For every* $t \in \mathbf{R}_+$, $\sum_{\omega\in\Omega} p(\omega)(t) = \sum_{\omega\in\Omega} p^0(\omega)$.

Proof of Lemma: It suffices to show that $\sum_{\omega\in\Omega} \frac{d}{dt}p(\omega)(t) = 0$. But under the present assumptions. this is an immediate consequence of Walras' Law.

Claim 20. *Pick any* $p^0 \in \mathbf{R}_+^{\#[\Omega]*\#[L]}$. *Then the solution* $\{p(t)\}_t$ *given in Theorem 3 satisfies the property that* $\inf_{t\in[0,\infty)} \sum_{\omega\in\Omega\setminus\{\omega^*\}} p(\omega)(t) = 0$.

Proof of Claim: From lemmas 1 and 2, it follows that $\inf_{t\in[0,\infty)} \sum_{\omega\in\Omega\setminus\{\omega^*\}} p(\omega)(t) \geq 0$. Suppose $\inf_{t\in[0,\infty)} \sum_{\omega\in\Omega\setminus\{\omega^*\}} p(\omega)(t) = \delta > 0$. Let

$$t^* > \frac{\sum_{\omega\in\Omega\setminus\{\omega^*\}} p(\omega)(0)}{\delta[\min_{\omega\in\Omega\setminus\{\omega^*\}} \sum_{i\in N:\omega\notin PF^i(\omega^*)} \frac{e^i(\omega)}{\#PF^i(\omega^*)}]}$$

Then,

$$p(\omega^*)(t^*) = p(\omega^*)(0) + \int_0^{t^*} \frac{d}{dt}p(\omega^*)(t)dt$$

$$= p(\omega^*)(0) + \int_0^{t^*} [\sum_{\omega\in\Omega\setminus\{\omega^*\}} p(\omega)(t)[\sum_{i\in N:\omega\notin PF^i(\omega^*)} \frac{e^i(\omega)}{\#PF^i(\omega^*)}]]dt$$

$$\geq p(\omega^*)(0) + \min_{\omega\in\Omega\setminus\{\omega^*\}}[\sum_{i\in N:\omega\notin PF^i(\omega^*)} \frac{e^i(\omega)}{\#PF^i(\omega^*)}] \int_0^{t^*} [\sum_{\omega\in\Omega\setminus\{\omega^*\}} p(\omega)(t)dt$$

$$\geq p(\omega^*)(0) + \min_{\omega\in\Omega\setminus\{\omega^*\}}[\sum_{i\in N:\omega\notin PF^i(\omega^*)} \frac{e^i(\omega)}{\#PF^i(\omega^*)}]t^*\delta$$

$$> p(\omega^*)(0) + \sum_{\omega\in\Omega\setminus\{\omega^*\}} p(\omega)(0)$$

$$= \sum_{\omega\in\Omega} p(\omega)(0)$$

.

But Lemma's 1 and 2 imply that $P(\omega^*)(t^*) \leq \sum_{\omega\in\Omega} p(\omega)(0)$. A contradiction. Q.E.D.

Proof of Theorem: From Lemma and Claim, it follows that

$$\sup_{t\in[0,\infty)} p(\omega^*)(t) = \sum_{\omega\in\Omega} p(\omega)(0) - \inf_{t\in[0,\infty)} \sum_{\omega\in\Omega\setminus\{\omega^*\}} p(\omega)(t) = \sum_{\omega\in\Omega} p(\omega)(0)$$

We claim that this in turn implies that $\lim_{t\to\infty} p(\omega^*)(t) = \sum_{\omega\in\Omega} p(\omega)(0)$. From Lemma, it follows that for every $t \in [0,\infty)$, $p(\omega^*)(t) \leq \sum_{\omega\in\Omega} p(\omega)(0)$. Suppose $p(\omega^*)(t)$ does not converge to $\sum_{\omega\in\Omega} p(\omega)(0)$. Then there exist $\epsilon > 0$ such that for every $t \in [0,\infty)$, there exist $t' \geq t$ such that $p(\omega^*)(t') \leq \sum_{\omega\in\Omega} p(\omega)(0) - \epsilon$. But it follows from Lemma that $p(\omega^*)(t)$ is non-decreasing. Hence for every $t \in [0,\infty)$, $p(\omega^*)(t) \leq \sum_{\omega\in\Omega} p(\omega)(0) - \epsilon$. But this contradicts $\sup_{t\in[0,\infty)} p(\omega^*)(t) = \sum_{\omega\in\Omega} p(\omega)(0)$. Hence $\lim_{t\to\infty} p(\omega^*)(t) = \sum_{\omega\in\Omega} p(\omega)(0)$.

Now, it follow from this and Lemma that for every $\omega \in \Omega \setminus \{\omega^*\}$, $\lim_{t\to\infty} p(\omega)(t) = 0$. Hence our Theorem is established. Q.E.D. *Proof of Theorem 3:* Consider any perfect state-ω competitive equilibrium. Then there exists a sequence of ϵ-perturbed economies and an associated sequence of state-ω competitive equilibria that converges to the perfect state-ω competitive equilibrium. Pick any such sequence $(p^*_k, \{\hat{x}^{i*}_{E,k}\}_{(i,E)\in M\times PF})^{\infty}_{k=1}$. Then the limit of the values of excess demands for every $E \in PF$ must equal zero. In particular,

$$lim_{k\to\infty} \sum_{i\in M} \sum_{\omega'\in PF(\omega)} p^*_k(\omega')(\hat{x}^{i*}_{PF(\omega'),k}(\omega') - e^i(\omega')) =$$

$$= \lim_{k\to\infty} \sum_{i\in M} r^i_{PF(\omega)}(p^*_k) \sum_{\omega''\in\Omega\setminus PF^i(\omega)} p^*_k(\omega'')e^i(\omega'') = 0$$

This can only happen if for every $\omega' \in \Omega \setminus PF(\omega)$, $lim_{k\to\infty} p^*_k(\omega') = 0$. Suppose now that there exists some consumer for which $b^{*i}(PF(\omega)) \neq 1$. Consider any $E \in PF \setminus PF(\omega)$ for which $b^*(PF(\omega)) > 0$. Then non-satiation implies that $\hat{x}^{i*}$ can not be optimal for consumer i which contradicts the definition of the perfect equilibrium. Hence $b^{*i}(PF(\omega)) = 1$. From this and the first fundamental Theorem of Welfare economics, efficiency then follows. Q.E.D.