A function may admit more than one local maximum and more than one local minimum. The absolute maximum of a function in a given region, however, is unique; it may occur at more than one point.

HOMEWORK Sec 3.8

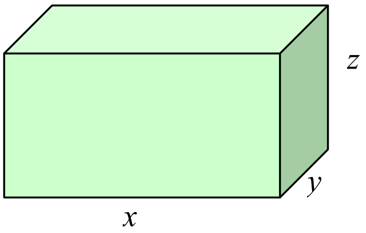
Solve the problems of section 14.8 in the book (p. 1008), numbers 9-20, 27-32.

3.9 Lagrange Multipliers

Given a function of two variables, $f(x, y)$, or three variables, $f(x, y,z)$, we want to solve the problem of finding absolute extrema of f subject to some constraint $g(x,y) = 0$ or $g(x,y,z) = 0$. Lagrange solved this problem in the eighteen century. Suppose we want to solve the problem of constructing a rectangular shoebox open on the top, having 32 ft^3, requiring the least amount of material. If x, y, z are the length, width, and height, the problem is translated into minimizing the area

$$\mathrm{A} = xy + 2xz + 2yz$$

subject to the condition (constraint) $xyz = 32$



Definition. We will say that $f(x,y)$ has a ***constraint absolute maximum*** (***minimum***) at (x_0,y_0) subject to the constraint $g(x,y) = 0$ if $f(x_0,y_0)$ is the largest (smallest) value of f on the constraint curve $g(x,y) = 0$. We will say that $f(x,y)$ has a ***constraint relative maximum*** (***minimum***) at (x_0,y_0) if $f(x_0,y_0)$ is the largest (smallest) value of f on some (small) segment of the constraint curve $g(x,y) = 0$ at both sides of (x_0,y_0).

In the problem of minimizing the area of the square box above, we have actually three variables, x, y, z. So

Definition. We will say that $f(x,y,z)$ has a ***constraint absolute maximum*** (***minimum***) at (x_0,y_0,z_0) subject to the constraint $g(x,y,z) = 0$ if $f(x_0,y_0,z_0)$ is the largest (smallest) value of f on the constraint curve $g(x,y,z) = 0$. We will say that $f(x,y,z)$ has a ***constraint relative maximum*** (***minimum***) at (x_0,y_0,z_0) if $f(x_0,y_0,z_0)$ is the largest (smallest) value of f on some (small) segment of the constraint curve $g(x,y,z) = 0$ at both sides of (x_0,y_0,z_0).

The definition can in fact be generalized without difficulty to any number of variables involved.

Theorem. Suppose that $f(x, y)$ and $g(x,y)$ have continuous first partial derivatives on some open set containing the constraint curve $g(x, y) = 0$, and assume that $\nabla g \neq 0$ at every point on the curve. If f has a constraint extremum then it occurs at a point (x_0, y_0) on the constraint curve at which the gradient vectors $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel, that is,

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Using the definition of gradient given above, the last equation becomes

$$f_x(x_0, y_0) i + f_y(x_0, y_0) j = \lambda \left(g_x(x_0, y_0) i + g_y(x_0, y_0) j \right)$$

The steps for solving a constraint extremum problem are:

1. Set the system of equations on the unknowns x, y, λ

$$\begin{aligned} f_x(x, y) &= \lambda g_x(x, y) \\ f_y(x, y) &= \lambda g_y(x, y) \\ g(x, y) &= 0 \end{aligned}$$

2. Solve the system

3. The nonlinear system above may have more than one solution (x,y). Plug all solutions into $f(x,y)$. The largest value is the absolute constraint maximum and the smallest value is the absolute constraint minimum.

The theorem generalizes to more than two variables in the obvious way. So, for three variables, we will have

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

Example 1. Find the maximum value and the minimum value of $f(x, y) = xy$ on the circumference $x^2 + y^2 = 1$.

1. Setting up and solving the system,

$$\begin{array}{lll} y = \lambda 2x & & \\ x = \lambda 2y & x = 2\lambda(\lambda 2x) & x(1 - 4\lambda^2) = 0 \\ x^2 + y^2 = 1 & x^2 + \lambda(\lambda 2x)^2 = 1 & x^2(1 + 4\lambda^2) = 1 \end{array}$$

The solutions are $\lambda = \pm \frac{1}{2}$ and $x = 0$. But the latter is inconsistent with the second equation, os it is not a solution. Substituting in the other equations, we find 4 solutions for this system:

$$\left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}} \right).$$

2. Plugging these 4 solutions into $f(x,y)$ we get

(x, y)	$(1/\sqrt{2}, 1/\sqrt{2})$	$(1/\sqrt{2}, -1/\sqrt{2})$	$(-1/\sqrt{2}, 1/\sqrt{2})$	$(-1/\sqrt{2}, -1/\sqrt{2})$
$f(x, y)$	1/2	-1/2	-1/2	1/2

Thus the absolute maximum of $f(x,y) = xy$ on the circle

$x^2 + y^2 = 1$ is 1/2 at the points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$, and the absolute minimum is -1/2 at the points $(-1/\sqrt{2}, 1/\sqrt{2})(1/\sqrt{2}, -1/\sqrt{2})$

Example. Let's solve the problem of minimizing the area A = $xy + 2xz + 2yz$ of the box subject to the constraint $xyz = 32$. We form the system

$$\begin{aligned} &A_x(x_0,y_0,z_0)=\lambda g_x(x_0,y_0,z_0) \\ &A_y(x_0,y_0,z_0)=\lambda g_y(x_0,y_0,z_0) \\ &A_z(x_0,y_0,z_0)=\lambda g_z(x_0,y_0,z_0) \\ &g(x,y,z)=0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} y + 2z &= \lambda yz \\ x + 2z &= \lambda xz \\ 2x + 2y &= \lambda xy \\ xyz &= 32 \end{aligned}$$

This is a tough system of 81th degree (admits up to 81 solutions) of four equations and four unknowns. Subtracting the first two equations we get $(y - x)(\lambda z - 1) = 0$, so there are two possibilities, $\lambda z = 1$ or $y = x$.

If $\lambda z = 1$, substituting in the first equation yields, $2z = 0$, which is inconsisten with the last equation. So the other solution means If $y = x$ and then the system is reduced to

$$\begin{cases} x + 2z = \lambda xz \\ 4x = \lambda x^2 \ \ or \ \ 4 = \lambda x \\ x^2 z = 32 \end{cases} \Rightarrow \begin{cases} x + 2z = 4z \\ x^2 z = 32 \end{cases} \Rightarrow \begin{cases} x = 2z \\ 4z^3 = 32 \end{cases} \Rightarrow \begin{cases} z = 2 \\ x = 4 \\ y = 4 \end{cases}$$

It is yet to be seen whether this is a maximum or a minimum. For this, take any other value of x, y, z satisfying the constraint, for instance $x = y = 1$, $z = 32$.

$A(4,4,2) = 48$ and $A(1,1,32) = 129$, thus we found the constraint absolute minimum

Lagrange multipliers lid to the problem of solving a system of equations, which often is non linear. You must practice somewhat in solving these types of systems (relax! a system such as the one in the last example will not be given in the exams).

HOMEWORK Sec 3.9

Solve the problems of section 14.9 in the book (p. 1018), numbers 5-12

Problems for Chapter 3

Problem 1. Calculate the following limits

a. $\lim\limits_{(x,y)\to(1,2)} \dfrac{x^3 - y^3}{x + y}$ b. $\lim\limits_{(x,y)\to(1,1)} \dfrac{x^4 - y^4}{x^2 - y^2}$

c. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x - y}{\sqrt{x} - \sqrt{y}}$ d. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$

Problem 2. Show that the following limits do not exist

a. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x - y}{x^2 + y^2}$ b. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{(x - y)^2}$

Problem 3. Given the following functions, find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}, \dfrac{\partial^2 f}{\partial x \partial y}$

a. $f(x,y) = x^2 y$ b. $f(x,y) = \sin(xy)$ c. $f(x,y) = x^2\cos(xy) - x/y$

Problem 4. Find the directional derivative of the following functions $f(x,y)$, at the given points P, in the direction of the indicated vectors **u**.

a. $f(x,y) = 3x^2y - 2x^4$, $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$, $P(-1, 2)$ b. $f(x,y) = 2x^2 - 2x^4y$, $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $P(1, 0)$

Problem 5. Find all relative extrema (maxima and minima) and saddle points of

a. $f(x,y) = x^2 - xy + y^3$. b. a. $f(x,y) = x^2y - 8x - y^2$.

Problem 6. Find the gradient of the functions in problem 5 at the point (-1, 1).

Problem 7. Given the following functions, find the direction angles of the directional derivative of maximum slope

a. $f(x,y) = 2x^2y - 2x^4 - y$, b. $f(x,y) = 2x^2 - 2x^3y + xy^2$,

Problem 8. Find the absolute maximum and absolute minimum of $f(x,y)$ subject to the given condition $g(x,y) = 0$

a. $f(x,y) = x^2 - y^2$, $g(x,y) = x^2 + y^2 = 25$ b. $f(x,y) = 4x^2 + y^2$, $g(x,y) = 2x^2 + y^2 = 1$